Permanent V. Determinant: An Exponential Lower Bound Assuming Symmetry*

[Extended Abstract]

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ABSTRACT

Grenet's determinantal representation for the permanent is optimal among determinantal representations that are equivariant with respect to left multiplication by permutation and diagonal matrices (roughly half the symmetry group of the permanent). In particular, if any optimal determinantal representation of the permanent must be polynomially related to one with such symmetry, then Valiant's conjecture on permanent v. determinant is true.

Categories and Subject Descriptors

F.1.m [Theory]: Miscellaneous

Keywords

Geometric Complexity Theory, determinant, permanent, MSC 68Q15 (20G05)

1. INTRODUCTION

The determinant

$$\det_n(x) := \sum_{\sigma \in \mathfrak{S}_n} \text{sgn}(\sigma) x_{\sigma(1)}^1 x_{\sigma(2)}^2 \cdots x_{\sigma(n)}^n \quad (1)$$

is a homogeneous polynomial of degree n on the space of $n \times n$ matrices $\text{Mat}_n(\mathbb{C})$. Here $\mathfrak{S}_n$ denotes the group of permutations on n elements and $\text{sgn}(\sigma)$ denotes the sign of the permutation σ .

Despite its formula with $n!$ terms, $\det_n$ can be evaluated quickly, e.g., using Gaussian elimination, which exploits the large symmetry group of the determinant, e.g., $\det_n(x) = \det_n(AB^{-1})$ for any $n \times n$ matrices A, B with determinant equal to one.

*A full version of this paper is available as [14].

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ITCS'16, January 14 - 16, 2016, Cambridge, MA, USA

ACM 978-1-4503-4057-1/16/01 ...\$15.00.

DOI: <http://dx.doi.org/10.1145/2840728.2840735>

We will work exclusively over the complex numbers and with homogeneous polynomials, the latter restriction only for convenience. L. Valiant showed that given a homogeneous polynomial $P(y)$ in M variables, there exists an n and an affine linear map

$$\tilde{A} : \mathbb{C}^M \rightarrow \text{Mat}_n(\mathbb{C}) \\ y \mapsto \Lambda + A(y)$$

where Λ is a constant matrix and $y \mapsto A(y)$ is a linear map such that $P = \det_n \circ \tilde{A}$. Such $\tilde{A}$ is called a *determinantal representation* of P . When $M = m^2$ and P is the permanent polynomial

$$\text{perm}_m(y) := \sum_{\sigma \in \mathfrak{S}_m} y_{\sigma(1)}^1 y_{\sigma(2)}^2 \cdots y_{\sigma(m)}^m, \quad (2)$$

L. Valiant showed that one can take $n = O(2^m)$. As an algebraic analog of the $\mathbf{P} \neq \mathbf{NP}$ conjecture, he also conjectured that one cannot do much better:

CONJECTURE 1 (VALIANT [21]). *Let $n(m)$ be a function of m such that there exist affine linear maps*

$$\tilde{A}_m : \mathbb{C}^{m^2} \rightarrow \text{Mat}_{n(m)}(\mathbb{C})$$

satisfying

$$\text{perm}_m = \det_{n(m)} \circ \tilde{A}_m. \quad (3)$$

Then $n(m)$ grows faster than any polynomial.

To measure progress towards Conjecture 1, define the determinantal complexity of the permanent $\text{dc}(\text{perm}_m)$ to be the smallest $n(m)$ such that there exists $\tilde{A}_m$ satisfying (3). The conjecture is that $\text{dc}(\text{perm}_m)$ grows faster than any polynomial in m . Lower bounds on $\text{dc}(\text{perm}_m)$ are: $\text{dc}(\text{perm}_m) > m$ (Marcus and Minc [15]), $\text{dc}(\text{perm}_m) > 1.06m$ (Von zur Gathen [23]), $\text{dc}(\text{perm}_m) > \sqrt{2}m - O(\sqrt{m})$ (Meshulam, reported in [23], and Cai [4]), with the current world record $\text{dc}(\text{perm}_m) \geq \frac{m^2}{2}$ [16] by Mignon and the second author. (Over $\mathbb{R}$, Yabe recently showed that $\text{dc}_{\mathbb{R}}(\text{perm}_m) \geq m^2 - 2m + 2$ [24], and in [5] Cai, Chen and Li extended the $\frac{m^2}{2}$ bound to arbitrary fields other than characteristic two.)

Inspired by *Geometric Complexity Theory* (GCT) [18], we focus on the *symmetries* of $\det_n$ and perm_m . Let $\text{GL}(\mathbb{C}^M) = \text{GL}_M$ denote the group of invertible linear maps $\mathbb{C}^M \rightarrow \mathbb{C}^M$. For $P \in S^m \mathbb{C}^{M*}$, a homogeneous polynomial of degree m on $\mathbb{C}^M$, let

$$G_P := \{g \in \text{GL}_M \mid P(g^{-1}y) = P(y) \quad \forall y \in \mathbb{C}^M\}$$

denote the *symmetry group* of P . For example $G_{\det_n} \simeq \text{S}(\text{GL}_n \times \text{GL}_n) / \mu_n \rtimes \mathbb{Z}_2$ [9], where $\text{S}(\text{GL}_n \times \text{GL}_n) = \{(A, B) \in \text{GL}_n \times \text{GL}_n \mid \det(A) = \det(B)\}$ and μ_n denotes the n -th roots of unity. The $\text{S}(\text{GL}_n \times \text{GL}_n)$ invariance comes from $\det(AB^{-1}) = (\det A \det B^{-1}) \det(x)$ and the $\mathbb{Z}_2$ is because $\det_n(x) = \det_n(x^T)$ where x^T is the transpose of the matrix x .

As observed in [18], the permanent (resp. determinant) is *characterized by its symmetries* in the sense that any polynomial $P \in S^m \mathbb{C}^{m^2*}$ with a symmetry group G_P such that $G_P \supseteq G_{\text{perm}_m}$ (resp. $G_P \supseteq G_{\det_m}$) is a scalar multiple of the permanent (resp. determinant). This property is the cornerstone of GCT. The program outlined in [18, 19] is an approach to Valiant's conjecture based on the functions on GL_{n^2} that respect the symmetry group $G_{\det_n}$, i.e., are invariant under the action of $G_{\det_n}$.

Guided by the principles of GCT, we ask:

What are the $\tilde{A}$ that respect the symmetry group of the permanent?

To make this question precise:

Definition 1. Let $\tilde{A} : \mathbb{C}^M \rightarrow \text{Mat}_n(\mathbb{C})$ be a determinantal representation of $P \in S^m \mathbb{C}^{M*}$. Define

$$G_A = \{g \in G_{\det_n} \mid g \cdot \Lambda = \Lambda \text{ and } g \cdot A(\mathbb{C}^M) = A(\mathbb{C}^M)\},$$

the *symmetry group of the determinantal representation $\tilde{A}$ of P* .

The group G_A comes with a representation $\rho_A : G_A \rightarrow \text{GL}(A(\mathbb{C}^M))$ obtained by restricting the action to $A(\mathbb{C}^M)$. We assume that P cannot be expressed using $M-1$ variables, i.e., that $P \notin S^m \mathbb{C}^{M-1}$ for any hyperplane $\mathbb{C}^{M-1} \subset \mathbb{C}^{M*}$. Then $A : \mathbb{C}^M \rightarrow A(\mathbb{C}^M)$ is bijective. Let $A^{-1} : A(\mathbb{C}^M) \rightarrow \mathbb{C}^M$ denote its inverse. Set

$$\begin{aligned} \bar{\rho}_A : G_A &\rightarrow \text{GL}_M \\ g &\mapsto A^{-1} \circ \rho_A(g) \circ A. \end{aligned} \quad (4)$$

Then $\bar{\rho}_A(G_A) \subseteq G_P$.

Definition 2. We say $\tilde{A}$ is an *equivariant representation* of P if (4) surjects onto G_P .

If G is a subgroup of G_P , we say that $\tilde{A}$ is *G -equivariant* if G is contained in the image of $\bar{\rho}_A$.

In particular, in an equivariant presentation for the permanent there exists a subgroup of $\text{S}(\text{GL}_n \times \text{GL}_n) / \mu_n \rtimes \mathbb{Z}_2$ that surjects onto $[(\mathfrak{S}_m \rtimes T_m) \times (\mathfrak{S}_m \rtimes T_m)] \rtimes \mathbb{Z}_2$.

Example 1. Let $Q = \sum_{j=1}^M z_j^2 \in S^2 \mathbb{C}^{M*}$ be a nondegenerate quadric. Then $G_Q = O(M)$ where $O(M) = \{B \in \text{GL}_M \mid B^{-1} = B^T\}$ is the orthogonal group, as for such B , $B \cdot Q = \sum_{i,j,k} B_{i,j} B_{k,j} z_i z_k = \sum_{i,j} \delta_{ij} z_i z_j = Q$. Consider the determinantal representation

$$Q = \det_{M+1} \begin{pmatrix} 0 & -z_1 & \cdots & -z_M \\ z_1 & 1 & & \\ \vdots & & \ddots & \\ z_M & & & 1 \end{pmatrix}. \quad (5)$$

For $(\lambda, B) \in G_Q$, define an action on $Z \in \text{Mat}_{M+1}(\mathbb{C})$ by

$$Z \mapsto \begin{pmatrix} \lambda & 0 \\ 0 & B \end{pmatrix} Z \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & B \end{pmatrix}^{-1}.$$

Write

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_M \end{pmatrix} \quad \text{so} \quad \tilde{A} = \begin{pmatrix} 0 & -X^T \\ X & \text{Id}_M \end{pmatrix}.$$

The relation $B^{-1} = B^T$ implies

$$\begin{pmatrix} \lambda & 0 \\ 0 & B \end{pmatrix} \cdot \begin{pmatrix} 0 & -X^T \\ X & \text{Id}_M \end{pmatrix} \cdot \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & B \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -(\lambda B X)^T \\ \lambda B X & \text{Id}_M \end{pmatrix}.$$

Taking $\det_{M+1}$ on both sides gives

$$\lambda^2 Q(X) = (\lambda, B) \cdot Q(X).$$

Thus $\tilde{A}$ is an equivariant representation of Q .

Definition 3. For $P \in S^m \mathbb{C}^{M*}$, define the *equivariant determinantal complexity* of P , denoted $\text{edc}(P)$, to be the smallest n such that there is an equivariant determinantal representation of P .

Of course $\text{edc}(P) \geq \text{dc}(P)$. We do not know if $\text{edc}(P)$ is finite in general. Our main result is that $\text{edc}(\text{perm}_m)$ is exponential in m .

2. RESULTS

2.1 Main Theorem

THEOREM 1. *Let $m \geq 3$. Then $\text{edc}(\text{perm}_m) = \binom{2m}{m} - 1 \sim 4^m$.*

There are several instances in complexity theory where an optimal algorithm partially respects symmetry, e.g. Strassen's algorithm for 2×2 matrix multiplication respects the $\mathbb{Z}_3$ -symmetry of the matrix multiplication operator (see [13, §4.2]), but not the $\text{GL}_2^{\times 3}$ symmetry.

For the purposes of Valiant's conjecture, we ask the weaker question:

QUESTION 1. *Does there exist a polynomial $e(d)$ such that $\text{edc}(\text{perm}_m) \leq e(\text{dc}(\text{perm}_m))$?*

Theorem 1 implies:

COROLLARY 1. *If the answer to Question 1 is affirmative, then Conjecture 1 is true.*

We have no opinion as to what the answer to Question 1 should be, but as it provides a new potential path to proving Valiant's conjecture, it merits further investigation. Note that Question 1 is a *flip* in the terminology of [17], since a positive answer is an existence result. It fits into the more general question: *When an object has symmetry, does it admit an optimal expression that preserves its symmetry?*

Example 2. Let $T \in W^{\otimes d}$ be a symmetric tensor, i.e. $T \in S^d W \subset W^{\otimes d}$. Say T can be written as a sum of r rank one tensors, then P. Comon conjectures [6] that it can be written as a sum of r rank one symmetric tensors.

Example 3. The optimal Waring decomposition of $x_1 \cdots x_n$, dating back at least to [8] and proved to be optimal in [20] is

$$x_1 \cdots x_n = \frac{1}{2^{n-1}n!} \sum_{\epsilon \in \{-1,1\}^n} \left(\sum_{j=1}^n \epsilon_j x_j \right)^n \prod_{i=1}^n \epsilon_i, \quad (6)$$

a sum with 2^{n-1} terms. This decomposition has an $\mathfrak{S}_{n-1}$ -symmetry but not an $\mathfrak{S}_n$ -symmetry, nor does it preserve the action of the torus T^{SL_n} of diagonal matrices with determinant one. One can obtain an $\mathfrak{S}_n$ -invariant expression by doubling the size:

$$x_1 \cdots x_n = \frac{1}{2^n n!} \sum_{\epsilon \in \{-1,1\}^n} \left(\sum_{j=1}^n \epsilon_j x_j \right)^n \prod_{i=1}^n \epsilon_i, \quad (7)$$

because

$$\begin{aligned} & (-x_1 + \epsilon_2 x_2 + \cdots + \epsilon_n x_n)^n (-1) \epsilon_2 \cdots \epsilon_n \\ &= (-1)^n (x_1 + (-\epsilon_2) x_2 + \cdots + (-\epsilon_n) x_n)^n (-1) \epsilon_2 \cdots \epsilon_n \\ &= (x_1 + (-\epsilon_2) x_2 + \cdots + (-\epsilon_n) x_n)^n (-\epsilon_2) \cdots (-\epsilon_n). \end{aligned}$$

The optimal Waring decomposition of the permanent is not known, but it is known to be of size greater than $\binom{n}{\lfloor n/2 \rfloor}^2 \sim 4^n / \sqrt{n}$. The Ryser-Glynn formula [10] is

$$\text{perm}_n(x) = 2^{-n+1} \sum_{\epsilon_1=1} \prod_{1 \leq i \leq n} \sum_{1 \leq j \leq n} \epsilon_i \epsilon_j x_{i,j}, \quad (8)$$

the outer sum taken over n -tuples $(\epsilon_1 = 1, \epsilon_2, \dots, \epsilon_n)$. This $\mathfrak{S}_{n-1} \times \mathfrak{S}_{n-1}$ -invariant formula can also be made $\mathfrak{S}_n \times \mathfrak{S}_n$ -invariant by enlarging it by a factor of 4, to get a $\mathfrak{S}_n \times \mathfrak{S}_n$ homogeneous depth three formula that is within a factor of four of the best known. Then expanding each monomial above, using Equation (7), one gets a $\mathfrak{S}_n \times \mathfrak{S}_n$ -Waring expression within a factor of $O(\sqrt{n})$ of the lower bound.

Example 4. Examples regarding equivariant representations of $\mathfrak{S}_N$ -invariant functions from the Boolean world give inconclusive indications regarding Question 1.

The MOD_m -degree of a Boolean function $f(x_1, \dots, x_N)$ is the smallest degree of any polynomial $P \in \mathbb{Z}[x_1, \dots, x_N]$

such that $f(x) = 0$ if and only if $P(x) = 0$ for all $x \in \{0,1\}^N$. The known upper bound for the MOD_m -degree of the Boolean OR function ($OR(x_1, \dots, x_N) = 1$ if any $x_j = 1$ and is zero if all $x_j = 0$) is attained by symmetric polynomials [3]. Moreover in [3] it is also shown that this bound cannot be improved with symmetric polynomials, and it is far from the known lower bound.

The boolean majority function $MAJ(x_1, \dots, x_N)$ takes on 1 if at least half the $x_j = 1$ and zero otherwise. The best monotone Boolean formula for MAJ [22] is polynomial in N and attained using random functions, and it is expected that the only symmetric monotone formula for majority is the trivial one, disjunction of all $\frac{n}{2}$ -size subsets (or its dual), which is of exponential size.

QUESTION 2. *Does every P that is determined by its symmetry group admit an equivariant determinantal representation? For those P that do, how much larger must such a determinantal representation be from the size of a minimal one?*

2.2 Grenet's formulas

The starting point of our investigations was the result in [2] that $\text{dc}(\text{perm}_3) = 7$, in particular Grenet's representation [11] for perm_3 :

$$\text{perm}_3(y) = \det_7 \begin{pmatrix} 0 & 0 & 0 & 0 & y_3^3 & y_2^3 & y_1^3 \\ y_1^1 & 1 & & & & & \\ y_2^1 & & 1 & & & & \\ y_3^1 & & & 1 & & & \\ & y_2^2 & y_1^2 & 0 & 1 & & \\ & y_3^2 & 0 & y_1^2 & & 1 & \\ & 0 & y_3^2 & y_2^2 & & & 1 \end{pmatrix}, \quad (9)$$

is optimal. We sought to understand (9) from a geometric perspective. A first observation is that it, and more generally Grenet's representation for perm_m as a determinant of size $2^m - 1$ is equivariant with respect to about half the symmetries of the permanent. In particular, the optimal expression for perm_3 is equivariant with respect to about half its symmetries.

For example, the action of $T_3 \subset SL_3$ on $\text{Mat}_3(\mathbb{C})$ given by

$$\begin{pmatrix} y_1^1 & y_2^1 & y_3^1 \\ y_1^2 & y_2^2 & y_3^2 \\ y_1^3 & y_2^3 & y_3^3 \end{pmatrix} \mapsto \begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix} \begin{pmatrix} y_1^1 & y_2^1 & y_3^1 \\ y_1^2 & y_2^2 & y_3^2 \\ y_1^3 & y_2^3 & y_3^3 \end{pmatrix}$$

with $t_1 t_2 t_3 = 1$, appears in $S(\text{GL}_7 \times \text{GL}_7)$ as

$$A \mapsto \rho(t) A \rho(t)^{-1}$$

where

$$\rho(t) = \begin{pmatrix} t_3 & & & & & & \\ & t_1 t_3 & & & & & \\ & & t_1 t_3 & & & & \\ & & & t_1 t_3 & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ & & & & & & 1 \end{pmatrix}$$

and there is a similar inclusion of $\mathfrak{S}_3$ into $\text{SL}_7 \times \text{SL}_7$.

To explain this observation, introduce the following notation. Write $\text{Mat}_m(\mathbb{C}) = \text{Hom}(F, E) = F^* \otimes E$, where $E, F = \mathbb{C}^m$. This distinction of the two copies of $\mathbb{C}^m$ clarifies the action of the group $\text{SL}(E) \times \text{SL}(F)$ on $\text{Hom}(F, E)$. This action is $(A, B).x = Ax B^{-1}$, for any $x \in \text{Hom}(F, E)$ and $(A, B) \in \text{SL}(E) \times \text{SL}(F)$. Let $T^{\text{SL}(E)} \subset \text{SL}(E)$ consist of the diagonal matrices and let $N(T^{\text{SL}(E)}) = T^{\text{SL}(E)} \rtimes \mathfrak{S}_m \subset \text{GL}(E)$ be its normalizer, where $\mathfrak{S}_m$ denotes the group of permutations on m elements. Similarly for $T^{\text{SL}(F)}$ and $N(T^{\text{SL}(F)})$. Then $G_{\text{perm}_m} \simeq [(N(T^{\text{SL}(E)}) \times N(T^{\text{SL}(F)}))/\mathbb{C}^*] \rtimes \mathbb{Z}_2$, where the embedding of $(N(T^{\text{SL}(E)}) \times N(T^{\text{SL}(F)}))/\mathbb{C}^*$ in $\text{GL}(\text{Hom}(F, E))$ is given by the action above and the term $\mathbb{Z}_2$ corresponds to transposition.

The following refinement of Theorem 1 asserts that to get an exponential lower bound it is sufficient to be equivariant with respect about half the symmetries of the permanent.

THEOREM 2. *Let $m \geq 3$. Let $\tilde{A}_m : \text{Mat}_m(\mathbb{C}) \rightarrow \text{Mat}_n(\mathbb{C})$ be a determinantal representation of perm_m that is equivariant with respect to $N(T^{\text{SL}(E)})$. Then $n \geq 2^m - 1$.*

Moreover, Grenet's determinantal representation of perm_m is equivariant with respect to $N(T^{\text{SL}(E)})$ and has size $2^m - 1$.

We now explain Grenet's expressions from a representation-theoretic perspective. Let $[m] := \{1, \dots, m\}$ and let $k \in [m]$. Note that $S^k E$ is an irreducible $\text{SL}(E)$ -module but it is not irreducible as an $N(T^{\text{SL}(E)})$ -module. For example, let $e_1, \dots, e_m$ be a basis of E , and let $(S^k E)_{\text{reg}}$ denote the span of $\prod_{i \in I} e_i$, for $I \subset [m]$ of cardinality k (the space spanned by the square-free monomials, also known as the space of *regular weights*): $(S^k E)_{\text{reg}}$ is an irreducible $N(T^{\text{SL}(E)})$ -submodule of $S^k E$. Moreover, there exists a unique $N(T^{\text{SL}(E)})$ -equivariant projection π_k from $S^k E$ to $(S^k E)_{\text{reg}}$.

For $v \in E$, define $s_k(v) : (S^k E)_{\text{reg}} \rightarrow (S^{k+1} E)_{\text{reg}}$ to be multiplication by v followed by π_{k+1} . Alternatively, $(S^{k+1} E)_{\text{reg}}$ is an $N(T^{\text{SL}(E)})$ -submodule of $E \otimes (S^k E)_{\text{reg}}$, and $s_k : E \rightarrow (S^k E)_{\text{reg}}^* \otimes (S^{k+1} E)_{\text{reg}}$ is the unique $N(T^{\text{SL}(E)})$ -equivariant inclusion. Let $\text{Id}_W : W \rightarrow W$ denote the identity map on the vector space W . Fix a basis $f_1, \dots, f_m$ of F^* .

PROPOSITION 1. *The following is Grenet's determinantal representation of perm_m . Let $\mathbb{C}^n = \bigoplus_{k=0}^{m-1} (S^k E)_{\text{reg}}$, so $n = 2^m - 1$, and identify $S^0 E \simeq (S^m E)_{\text{reg}}$. Set*

$$\Lambda_0 = \sum_{k=1}^{m-1} \text{Id}_{(S^k E)_{\text{reg}}}$$

and define

$$\tilde{A} = \Lambda_0 + \sum_{k=0}^{m-1} s_k \otimes f_{k+1}. \quad (10)$$

Then $(-1)^{m+1} \text{perm}_m = \det_n \circ \tilde{A}$. To obtain the permanent exactly, replace $\text{Id}_{(S^1 E)_{\text{reg}}}$ by $(-1)^{m+1} \text{Id}_{(S^1 E)_{\text{reg}}}$ in the formula for Λ_0 .

In bases respecting the block decomposition induced from the direct sum, the linear part, other than the last term which lies in the upper right block, lies just below the diagonal blocks, and all blocks other than the upper right block and the diagonal and sub-diagonal blocks, are zero.

Moreover $N(T^{\text{SL}(E)}) \subseteq \bar{\rho}_A(G_A)$.

Remark 1. In the spirit of the GCT program, Proposition 1 gives an essentially calculation free proof of Grenet's formulas, as the permanent is the unique polynomial up to scale invariant under $N(T^{\text{SL}(E)})$. One need only check the value e.g. at the identity matrix.

2.3 An equivariant representation of the permanent

We now give a minimal equivariant determinantal representation of perm_m . By Theorem 1, its size is $\binom{2m}{m} - 1$. For $e \otimes f \in E \otimes F^*$, let $S_k(e \otimes f) : (S^k E)_{\text{reg}} \otimes (S^k F^*)_{\text{reg}} \rightarrow (S^{k+1} E)_{\text{reg}} \otimes (S^{k+1} F^*)_{\text{reg}}$ be multiplication by e on the first factor and f on the second followed by projection into

$$(S^{k+1} E)_{\text{reg}} \otimes (S^{k+1} F^*)_{\text{reg}}.$$

Equivalently,

$$S_k : (E \otimes F^*) \rightarrow ((S^k E)_{\text{reg}} \otimes (S^k F^*)_{\text{reg}})^* \otimes (S^{k+1} E)_{\text{reg}} \otimes (S^{k+1} F^*)_{\text{reg}}$$

is the unique $N(T^{\text{SL}(E)}) \times N(T^{\text{SL}(F)})$ equivariant inclusion.

PROPOSITION 2. *The following is an equivariant determinantal representation of perm_m : Let*

$$\mathbb{C}^n = \bigoplus_{k=0}^{m-1} (S^k E)_{\text{reg}} \otimes (S^k F^*)_{\text{reg}},$$

so $n = \binom{2m}{m} - 1 \sim 4^m$. Fix a linear isomorphism $S^0 E \otimes S^0 F^* \simeq (S^m E)_{\text{reg}} \otimes (S^m F^*)_{\text{reg}}$. Set

$$\Lambda_0 = \sum_{k=1}^{m-1} \text{Id}_{(S^k E)_{\text{reg}} \otimes (S^k F^*)_{\text{reg}}}$$

and define

$$\tilde{A} = (m!)^{\frac{-1}{n-m}} \Lambda_0 + \sum_{k=0}^{m-1} S_k. \quad (11)$$

Then $(-1)^{m+1} \text{perm}_m = \det_n \circ \tilde{A}$. In bases respecting the block structure induced by the direct sum, except for S_{m-1} , which lies in the upper right hand block, the linear part lies just below the diagonal block.

2.4 Determinantal representations of quadrics

It will be instructive to examine other polynomials determined by their symmetry groups. Perhaps the simplest such is a nondegenerate quadratic form.

Let $Q = \sum_{j=1}^s x_j y_j \in S^2 \mathbb{C}^{2s*}$ be a non-degenerate quadratic form in $2s$ variables (such is equivalent to $\sum_{u=1}^{2s} z_u^2$ under a change of basis). The polynomial Q is characterized by its

symmetries. By [16], if $s \geq 3$, the smallest determinantal representation of Q is of size $s + 1$:

$$\tilde{A} = \begin{pmatrix} 0 & -x_1 & \cdots & -x_s \\ y^1 & 1 & & \\ \vdots & & \ddots & \\ y^s & & & 1 \end{pmatrix}. \quad (12)$$

This representation is equivariant with respect to $O(s) \subset G_Q = O(2s)$ (where the inclusion of $O(s)$ in $GL_{(s+1)^2}$ is first into GL_{s+1} acting on the last s basis vectors and then into $GL_{(s+1)^2}$ acting by conjugation) and there is no size $s + 1$ determinantal representation equivariant with respect to G_Q . However, Example 1 shows there is a size $2s + 1$ determinantal representation equivariant with respect to G_Q .

PROPOSITION 3. *Let $Q \in S^2\mathbb{C}^{M*}$ be a nondegenerate quadratic form, that is, a homogeneous polynomial of degree 2. Then*

$$\text{edc}(Q) = M + 1.$$

2.5 Determinantal representations of the determinant

Although it may appear strange at first, one can ask for determinantal representations of $\det_m$. In this case, to get an interesting lower bound, we add a regularity condition:

Definition 4. Let $P \in S^m\mathbb{C}^{M*}$. A determinantal representation $\tilde{A} : \mathbb{C}^M \rightarrow \text{Mat}_n(\mathbb{C})$ is said to be *regular* if $\tilde{A}(0)$ has rank $n - 1$.

Call the minimal size of a regular determinantal representation of P the *regular determinantal complexity* of P and denote it by $\text{rdc}(P)$. Let $\text{erdc}(P)$ denote the minimal size of a regular equivariant determinantal representation of P .

In [23], von zur Gathen showed that any determinantal representation of perm_m or a smooth quadric is regular. In contrast, the trivial determinantal representation of $\det_m$ is not regular; but this representation is equivariant so $\text{edc}(\det_m) = m$.

$$\text{THEOREM 3. } \text{erdc}(\det_m) = \binom{2m}{m} - 1 \sim 4^m.$$

As in the case of the permanent, we can get an exponential lower bound using only about half the symmetries of the determinant.

THEOREM 4. *Let $\tilde{A}_m : \text{Mat}_m(\mathbb{C}) \rightarrow \text{Mat}_n(\mathbb{C})$ be a regular determinantal representation of $\det_m$ that is equivariant with respect to $\text{SL}(E)$. Then $n \geq 2^m - 1$.*

Moreover, there exists a regular determinantal representation of $\det_m$ that is equivariant with respect to $\text{SL}(E)$ of size $2^m - 1$.

Remark 2. Normally when one obtains the same lower bound for the determinant as the permanent in some model it is discouraging for the model. However here there is an important difference due to the imposition of regularity for the determinant. We discuss this further below Question 3.

We now introduce notation to describe the regular determinantal representation of $\det_m$ that is equivariant with respect to $\text{SL}(E)$ of size $2^m - 1$ mentioned in Theorem 4.

Observe that $(S^k E)_{\text{reg}} \subset E^{\otimes k}$ is isomorphic to the skew-symmetric tensors $\Lambda^k E \subset E^{\otimes k}$ as a $T^{\text{SL}(E)}$ -module but not as an $\mathfrak{S}_m$ -module.

Write $\text{Mat}_m(\mathbb{C}) = E \otimes F^*$. Let $f_1, \dots, f_m$ be a basis of F^* . Let ex_k denote exterior multiplication in E :

$$\begin{aligned} ex_k : E &\rightarrow (\Lambda^k E)^* \otimes (\Lambda^{k+1} E) \\ v &\mapsto \{\omega \mapsto v \wedge \omega\}. \end{aligned}$$

PROPOSITION 4. *The following is a regular determinantal representation of $\det_m$ that is equivariant with respect to $\text{SL}(E)$. Let*

$$\mathbb{C}^n = \bigoplus_{j=0}^{m-1} \Lambda^j E,$$

so $n = 2^m - 1$ and

$$\text{End}(\mathbb{C}^n) = \bigoplus_{0 \leq i, j \leq m-1} \text{Hom}(\Lambda^j E, \Lambda^i E).$$

Fix an identification $\Lambda^m E \simeq \Lambda^0 E$. Set

$$\Lambda_0 = \sum_{k=1}^{m-1} \text{Id}_{\Lambda^k E},$$

and

$$\tilde{A} = \Lambda_0 + \sum_{k=0}^{m-1} ex_k \otimes f_{k+1}. \quad (13)$$

Then $\det_m = \det_n \circ \tilde{A}$ if $m \equiv 1, 2 \pmod{4}$ and $\det_m = -\det_n \circ \tilde{A}$ if $m \equiv 0, 3 \pmod{4}$. In bases respecting the direct sum, the linear part, other than the last term which lies in the upper right block, lies just below the diagonal blocks, and all blocks other than the upper right, the diagonal and sub-diagonal are zero.

Note the similarity with the expression (10). This guided the proofs in the permanent case. There is a functor, called the the Howe-Young duality functor [1], which enables one to transport arguments in the determinant case (which is more familiar) to the permanent cases. (This functor also enabled the computation of the linear strand of the minimal free resolution of the ideal generated by subpermanents in [7].)

When $m = 2$ this is

$$\begin{pmatrix} 0 & -y_2^2 & y_1^2 \\ y_1^1 & 1 & 0 \\ y_2^1 & 0 & 1 \end{pmatrix}$$

agreeing with our earlier calculation of a rank four quadric. Note the minus sign in front of y_2^2 because $ex(e_2)(e_1) = -e_1 \wedge e_2$.

For example, ordering the bases of $\Lambda^2 \mathbb{C}^3$ by $e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3$, the matrix for $\det_3$ is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & y_3^3 & -y_2^3 & y_1^3 \\ y_1^1 & 1 & & & & & \\ y_2^1 & & 1 & & & & \\ y_3^1 & & & 1 & & & \\ & -y_2^2 & y_1^2 & 0 & 1 & & \\ & -y_3^2 & 0 & y_1^2 & & 1 & \\ & 0 & -y_3^2 & y_2^2 & & & 1 \end{pmatrix}.$$

We now give a regular equivariant determinantal representation of $\det_m$. Let EX_k denote the exterior multiplication

$$EX_k : E \otimes F^* \longrightarrow (\Lambda^k E \otimes \Lambda^k F^*)^* \otimes (\Lambda^{k+1} E \otimes \Lambda^{k+1} F^*) \\ e \otimes f \mapsto \{\omega \otimes \eta \mapsto e \wedge \omega \otimes f \wedge \eta\},$$

PROPOSITION 5. *The following is an equivariant regular determinantal representation of $\det_m$. Let*

$$\mathbb{C}^n = \bigoplus_{j=0}^{m-1} \Lambda^j E \otimes \Lambda^j F^*,$$

so $n = \binom{2m}{m} - 1 \sim 4^m$ and

$$\text{End}(\mathbb{C}^n) = \bigoplus_{0 \leq i, j \leq m} \text{Hom}(\Lambda^j E \otimes \Lambda^j F^*, \Lambda^i E \otimes \Lambda^i F^*).$$

Fix an identification $\Lambda^m E \otimes \Lambda^m F^* \simeq \Lambda^0 E \otimes \Lambda^0 F^*$. Set

$$\Lambda_0 = \sum_{k=1}^{m-1} \text{Id}_{\Lambda^k E \otimes \Lambda^k F^*}$$

and define

$$\tilde{A} = (m!)^{\frac{-1}{n-m}} \Lambda_0 + \sum_{k=0}^{m-1} EX_k. \quad (14)$$

Then $(-1)^{m+1} \det_m = \det_n \circ \tilde{A}$.

Comparing Theorems 1 and 3, Theorems 2 and 4, Propositions 1 and 4 and Propositions 2 and 5, one can see that $\det_m$ and perm_m have the same behavior relatively to equivariant regular determinantal representations. This prompts the question: What is the regular determinantal complexity of the determinant? In particular:

QUESTION 3. *Let $\text{rdc}(\det_m)$ be the smallest value of n such that there exist affine linear maps $\tilde{A}_m : \mathbb{C}^{m^2} \rightarrow \mathbb{C}^{n^2}$ such that*

$$\det_m = \det_n \circ \tilde{A}_m \text{ and } \text{rank } \tilde{A}(0) = n - 1. \quad (15)$$

What is the growth of $\text{rdc}(\det_m)$?

In [12] it is shown that $\text{rdc}(\det_m)$ grows at most like $O(m^4)$. If one could prove a lower bound on $\text{rdc}(\det_m)$ of $m^2 \log(m)$

or better, and transfer it to the permanent via the Howe-Young duality functor this would be a significant improvement of the state of the art for $\text{dc}(\text{perm}_m)$. Because of the symmetries of $\det_n$, such a bound might be easier than determining the growth of $\text{dc}(\text{perm}_m)$. However to prove unrestricted exponential lower bounds, one must deal directly with the permanent.

3. OVERVIEW OF THE PROOFS

Let $\Lambda_{n-1} \in \text{Mat}_n(\mathbb{C})$ be the matrix with 1 in the $n-1$ last diagonal entries and 0 elsewhere. Any determinantal representation $\tilde{A}$ of P of size n with $\text{rank}(\tilde{A}(0)) = n-1$ can be transformed (by multiplying on the left and right by constant invertible matrices) to a determinantal representation of P satisfying $\tilde{A}(0) = \Lambda_{n-1}$.

The following group plays a central role in the study of regular equivariant determinantal representations:

$$G_{\det_n, \Lambda_{n-1}} = \{g \in G_{\det_n} \mid g \cdot \Lambda_{n-1} = \Lambda_{n-1}\}.$$

Let $\mathbb{H} = \mathbb{C}^{n-1} \subset \mathbb{C}^n$ denote the image of Λ_{n-1} and $\ell_1 \in \mathbb{C}^n$ its kernel. Write ℓ_2 for ℓ_1 in the target $\mathbb{C}^n$. Then $\text{Mat}_n(\mathbb{C}) = (\ell_1 \oplus \mathbb{H})^* \otimes (\ell_2 \oplus \mathbb{H})$. Let $\text{transp} \in \text{GL}(\text{Mat}_n(\mathbb{C}))$ denote the transpose, and let w^T the row vector obtained by transposing the column vector w .

LEMMA 1. *The group $G_{\det_n, \Lambda_{n-1}}$ is*

$$\{M \mapsto \begin{pmatrix} \lambda & 0 \\ v & g \end{pmatrix} M \begin{pmatrix} \lambda^{-1} & w^T \\ 0 & g^{-1} \end{pmatrix} \mid g \in \text{GL}(\mathbb{H}), v, w \in \mathbb{H}, \lambda^{-1} = \det(g)\} \cdot \langle \text{transp} \rangle.$$

Let $P \in S^m \mathbb{C}^{M*}$ be either a quadric, a permanent or a determinant. Say a regular representation $\tilde{A}$ is equivariant with respect to some $G \subseteq G_P$. We may assume that $\tilde{A}(0) = \Lambda_{n-1}$.

The first step consists in lifting G to G_A . More precisely, in each case we construct a reductive subgroup $\tilde{G}$ of G_A such that $\tilde{\rho}_A : \tilde{G} \rightarrow G$ is finite and surjective. In a first reading, it is relatively harmless to assume that $\tilde{G} \simeq G$. Then, using Malcev's theorem (see, e.g. [14]), after possibly conjugating $\tilde{A}$, we may assume that $\tilde{G}$ is contained in $(\text{GL}(\ell_2) \times \text{GL}(\mathbb{H})) \rtimes \mathbb{Z}_2$. Up to considering an index two subgroup of $\tilde{G}$ if necessary, we assume that $\tilde{G}$ is contained in $\text{GL}(\ell_2) \times \text{GL}(\mathbb{H})$.

The starting point is Schur's lemma, which restricts non-zero G -module maps between irreducible G -modules to isomorphic modules. In the case $G = G_P$ (respectively $G \subset G_P$), then $\mathbb{C}^M$ is an irreducible G -module (resp. we decompose it into a direct sum of irreducible G -modules). Write $\text{Mat}_n(\mathbb{C}) = (\ell_1 \oplus \mathbb{H})^* \otimes (\ell_2 \oplus \mathbb{H})$, where $\mathbb{H} \subset \mathbb{C}^n$ is a hyperplane and the ℓ_j are lines.

In the two permanent cases $\mathbb{C}^M$ is respectively $E^{\oplus m}$ as a $\text{GL}(E)$ -module and $E \otimes F$ as a $\text{GL}(E) \times \text{GL}(F)$ -module

Write

$$\text{Mat}_n(\mathbb{C}) = \begin{pmatrix} \ell_1^* \otimes \ell_2 & \mathbb{H}^* \otimes \ell_2 \\ \ell_1^* \otimes \mathbb{H} & \mathbb{H}^* \otimes \mathbb{H} \end{pmatrix}, \quad \Lambda_{n-1} = \begin{pmatrix} 0 & 0 \\ 0 & \text{Id}_{\mathbb{H}} \end{pmatrix}.$$

If $m \geq 2$ the $\ell_1^* \otimes \ell_2$ coefficient of $\tilde{A}$ has to be zero. Then, since $P \neq 0$, the projection of $A(\mathbb{C}^M)$ on the first column $\ell_1^* \otimes \mathbb{H} \simeq \mathbb{H}$ has to be non-zero. We thus have a G -submodule $\mathbb{H}_1 \subset \mathbb{H}$ isomorphic to an irreducible submodule of $\mathbb{C}^M$ (in the permanent cases, respectively $E = \mathbb{C}^m$ or $E \otimes F = \mathbb{C}^{m^2}$). A similar argument shows that there must be another irreducible G -submodule $\mathbb{H}_2 \subset \mathbb{H}$ such that an irreducible submodule of $\mathbb{C}^M$ appears in $\mathbb{H}_1^* \otimes \mathbb{H}_2$.

For example, in the $SL(E)$ -equivariant determinant case, $\mathbb{H}_1$ must be isomorphic to E , so $\mathbb{H}_2$ must be such that $E \subset E^* \otimes \mathbb{H}_1$. The only two $SL(E)$ -modules which work are $\mathbb{H}_1 = S^2 E$ or $\mathbb{H}_1 = \Lambda^2 E$. In either case, as long as $m > 3$, we must have $E \subset \mathbb{H}_1^* \otimes \mathbb{H}_3$ which implies $\mathbb{H}_3$ is an irreducible submodule of $E^{\otimes 3}$, the smallest of which is $\Lambda^3 E$. Continuing along the minimal path, one gets the sum of exterior powers as in Proposition 4.

In each case, we construct a sequence of irreducible sub- $\tilde{G}$ -modules $\mathbb{H}_k$ of $\mathbb{H}$ satisfying very restrictive conditions. This allows us to get our lower bounds.

To prove the representations A actually compute the polynomials we want, in the case $G = G_P$, we first check that G_P is contained in the image of $\bar{\rho}_A$. Since P is characterized by its symmetries, we deduce that $\det_n \circ \tilde{A}$ is a scalar multiple of P . We then specialize to evaluating on the diagonal matrices in $\text{Mat}_m(\mathbb{C})$ to determine this constant, proving in particular that it is non-zero.

4. ACKNOWLEDGMENTS

The seed for this article was planted during the Fall 2014 semester program *Algorithms and Complexity in Algebraic Geometry* at the Simons Institute for the Theory of Computing, UC Berkeley. Most of the work was done when the Landsberg was a guest of Ressayre and Pascal Koiran. Landsberg thanks his hosts as well as U. Lyon and ENS Lyon for their hospitality and support. We thank: Christian Ikenmeyer for useful discussions about equivariant determinantal presentations of the permanent and Grenet's algorithm, Josh Grochow, Christian Ikenmeyer and Shrawan Kumar for useful suggestions for improving the exposition, Jérôme Germoni for mentioning the existence of the spin symmetric group, Michael Forbes for suggesting Example 3, and Avi Wigderson for suggesting Example 4. Landsberg supported by NSF grant DMS-1405348. Ressayre supported by ANR Project (ANR-13-BS02-0001-01) and by Institut Universitaire de France.

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