Multi-Scale Vorticle Fluids

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Fig. 1. Smoke trail that stretches from near the camera to the horizon, using a single simulation. The number of emitted vorticles is specified in screen space.

We present a multi-scale method for simulating incompressible gases in 3-dimensions with resolution variation suitable for perspective cameras and regions of importance. The dynamics is derived from the vorticity equation. Lagrangian particles are created, modified and deleted in a manner that handles advection with buoyancy and viscosity. Boundaries and deformable object collisions are modeled with the source and doublet panel method. Our acceleration structure is based on the FMM (Fast Multipole Method), but with a varying size to account for non-uniform sampling. Because the dynamics of our method is voxel free, we can freely specify the voxel resolution of the output density and velocity while keeping the main shapes and timing.

CCS Concepts: • **Computing methodologies** → **Physical simulation**;

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1 INTRODUCTION

Modeling the visual detail of a moving gas is a laborious task. The ability to control sampling varies per method. Foster and Metaxas [1997] solve in a uniformly voxelized grid the Navier-Stokes Equation [Aris 2012]. Stam [1999] further proposes a popular unconditionally stable model. De Witt et al. [2012] represent the velocity using the Laplacian eigenvectors as a basis for incompressible flow. The Navier-Stokes Equation can also be solved in a Lagrangian frame of reference with SPH (Smoothed Particle Hydrodynamics) [Gingold and Monaghan 1977], or with a Vortex Method, obtained by taking the curl of the Navier-Stokes Equation [Cottet and Koumoutsakos 2000]. Vortex Methods can store data on points [Park and Kim 2005], curves [Angelidis et al. 2006b; Weissmann and Pinkall 2010] or surfaces [Brochu et al. 2012], and can be solved with an integral, with a grid [Couet et al. 1981], or both [Zhang and Bridson 2014]. Some original approaches don't use Navier-Stokes: Elcott et

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al. [2007] use Kelvin's theorem, and Chern et al. [2016] solve the Schrödinger equation in a grid. To model different characteristics at different resolutions, multiple approaches can be combined and applied to a selective range of scales: Selle et al. [2005] advect vorticity carried by points and Pfaff et al. [2012] advect vorticity carried by sheets. Kim et al. [2008] advect procedurally generated detail. Pfaff et al. [2009] and Kim et al. [2012] show that the region of changing scale is located near boundaries, varying density, and temperature. In the methods that compute pressure explicitly, the boundary condition is met as part of computing the pressure. For the boundary condition of vortex methods, a harmonic field is needed, and the source and doublet panel method is the method of choice. Panel methods are well researched in Aerodynamics, and we refer the reader to an introduction by Erickson [1990]. Both the source and the doublet term are required. Using only the source term does not prevent the flow from passing through cup-shaped colliders, as opposed to spheroid colliders for which the doublet contribution is very small. This is the case of the single layer potential of Zhang and Bridson [2014]. Using only the doublet term does not account for colliders with changing volume, since the doublet term is not capable of generating or consuming volume. Park and Kim [2005] use vorticle panels, which also do not handle cup-shaped colliders. Also, vorticle panels cannot generate or consume volume by themselves, and induce a rotational field.

Our method is based on the FMM with a modification to handle multiple scales. The FMM was developed by Geengard and Rokhlin [1987] to solve the N-body problem in linear complexity. Reviewing the literature on FMM is beyond our scope, and we refer the reader to an introduction by Ying [2012]. Unlike hybrid methods, our method handles all ranges of scales in one model. Unlike SPH where small and large scales cannot overlap, our method's samples can overlap for all scales. Unlike grid-based methods that have a uniform sampling for the finest resolution, our method handles a sparse structure for all scales. Our main contributions are:

- An unconditionally stable method for vorticity stretching.
- A new model for buoyancy.
- A hierarchical and sparse method for free flow advection.
- A hierarchical and sparse method for boundaries.

2 DYNAMIC MODEL

To obtain the equation of dynamics, we apply the curl operator to the following form of the Navier-Stokes equation of a Newtonian fluid:

$$\rho \frac{d\vec{u}}{dt} = \mu \nabla^2 \vec{u} + \rho \vec{F} - \nabla p \quad (1)$$

If we assume that μ is constant, divide both sides by ρ , replace $\frac{\mu}{\rho}$ with ν , and use the identity $\nabla \rho / \rho = \nabla \log(\rho)$, then we obtain the following equation, where the vorticity $\vec{\omega}$ is the curl of the velocity $\vec{u}$:

$$\frac{d\vec{\omega}}{dt} = (\vec{\omega} \cdot \nabla) \vec{u} + \nu \nabla^2 \vec{\omega} + \nabla \log(\rho) \times (\vec{F} - \frac{d\vec{u}}{dt}) + \nabla \times \vec{F} \quad (2)$$

This is the complete equation, it includes boundaries and all forces. Eq. (2) means that the vorticity $\vec{\omega}$ evolves over time by advecting particles carrying $\vec{\omega}$, and by stretching $\vec{\omega}$ according to the velocity $\vec{u}$, with kinematic viscosity ν , buoyancy and boundary interaction specified by density ρ and external forces $\vec{F}$:

$$\vec{F}(\mathbf{p}) = \begin{cases} \vec{g} + \vec{e} & \text{if } \mathbf{p} \text{ outside solid objects} \\ \vec{f} & \text{otherwise} \end{cases} \quad (3)$$

where $\vec{g}$ is the constant for gravity, $\vec{e}$ is a set of user defined external forces, and $\vec{f}$ is the acceleration at the objects' boundaries, suitable for deformable objects. Solving $\vec{\omega}$ with Eq. (2) does not involve the pressure term. Instead, the velocity $\vec{u}$ is obtained from $\vec{\omega}$ in two parts: an advected field $\vec{v}$ derived from $\vec{\omega}$, and a harmonic field $\vec{h}$ derived from $\vec{v}$ at the boundaries:

$$\vec{u} = \vec{v} + \vec{h} \quad (4)$$

The fields $\vec{v}$ and $\vec{h}$ satisfy the following identities:

$$\nabla \cdot \vec{v} = 0 \quad \nabla \cdot \vec{h} = 0 \quad \nabla \times \vec{h} = 0 \quad (5)$$

Computing $\vec{v}$ from $\vec{\omega}$ is explained in Section 3, and computing $\vec{h}$ from $\vec{v}$ at the boundaries is explained in Section 4. The other sections focus on the dynamics in the right-hand side of Eq. (2): the stretching term is described in Section 5, the viscosity term is described in Section 6, and the buoyancy/shedding term is described in Section 7 and 8. In Section 9 we present a density advection scheme, and finally in Section 10 we lay out an implementation roadmap. Our nomenclature is given in Appendix A.

3 ADVECTED FIELD

The advected field $\vec{v}$ is the purely rotational and dynamic portion of $\vec{u}$ that can be computed directly from $\vec{\omega}$. It is defined with the Biot-Savart law:

$$\vec{v}(\mathbf{p}) = \frac{1}{4\pi} \int_{\mathbf{x} \in \mathbb{R}^3} \frac{\vec{\omega}(\mathbf{x}) \times (\mathbf{p} - \mathbf{x})}{\|\mathbf{p} - \mathbf{x}\|^3} d\mathbf{x} \quad (6)$$

Eq. (6) means that $\vec{v}$ is induced by a continuum of weighted rotations singular at $\mathbf{p} = \mathbf{x}$. To remove the singularity, we use a discrete overlapping partitioning, where each element i is a vorticle (vortex particle), also known as vortex blob [Beale and Majda 1982], defined by an integrable vorticity field $\vec{\omega}_i$ centered at $\mathbf{p}_i$:

$$\vec{v}(\mathbf{p}) = \sum_i \frac{1}{4\pi} \int_{\mathbf{x} \in \mathbb{R}^3} \frac{\vec{\omega}_i(\mathbf{x} - \mathbf{p}_i) \times (\mathbf{p} - \mathbf{x})}{\|\mathbf{p} - \mathbf{x}\|^3} d\mathbf{x} = \sum_i \vec{v}_i(\mathbf{p} - \mathbf{p}_i) \quad (7)$$

Our main motivation to use particles instead of a topologically connected structure, such as filaments, is to avoid the rapid entanglement occurring around objects and other filaments. A vorticle i is associated with a vorticity field $\vec{\omega}_i$, a velocity field $\vec{v}_i$ and a stream field $\vec{\psi}_i$. We chose the following explicit expressions, using $\xi_i(l) = 1/\sqrt{r_i^2 + l^2}$, size r_i and rotation strength $\vec{w}_i$:

$$\begin{aligned} \vec{\psi}_i(\vec{x}) &= \xi_i(\|\vec{x}\|) \vec{w}_i \\ \vec{v}_i(\vec{x}) &= \xi_i(\|\vec{x}\|)^3 \vec{w}_i \times \vec{x} \\ \vec{\omega}_i(\vec{x}) &= \xi_i(\|\vec{x}\|)^3 (2\vec{w}_i - 3\xi_i(\|\vec{x}\|)^2 \vec{x} \times (\vec{w}_i \times \vec{x})) \end{aligned} \quad (8)$$

The fields, shown in Figure 2, satisfy the following identities:

$$\nabla \times \vec{\psi}_i = \vec{v}_i \quad \nabla \times \vec{v}_i = \vec{\omega}_i \quad \nabla \cdot \vec{v}_i = 0 \quad \nabla \cdot \vec{\omega}_i = 0$$

We chose these functions because they are non-singular and smooth across $\vec{x} = 0$ since $r_i > 0$, and because the vorticle's rotation strength $\vec{w}_i$ is asymptotic to its vorticity $\vec{\omega}_i$ when decreasing r_i . Also, a vorticle's mean energy E_i has a closed form:

$$E_i = \frac{1}{2} \int \|\vec{v}_i(\vec{x})\|^2 d\vec{x} = \frac{\pi^2 \|\vec{w}_i\|^2}{4r_i} \quad (9)$$

This is convenient for splitting a vorticle emitter's energy E among N emitted vortices:

$$\vec{w}_i = \frac{2}{\pi} \sqrt{\frac{Er_i}{N}} X \quad (10)$$

where the variable X is a random vector on the unit sphere, or a normalized curl of an input field. A set of vortices induces a stream field $\vec{\psi}$, an advected field $\vec{v}$ and a vorticity field $\vec{\omega}$:

$$\begin{aligned} \vec{\psi}(\mathbf{p}) &= \sum \vec{\psi}_i(\mathbf{p} - \mathbf{p}_i) \\ \vec{v}(\mathbf{p}) &= \sum \vec{v}_i(\mathbf{p} - \mathbf{p}_i) \\ \vec{\omega}(\mathbf{p}) &= \sum \vec{\omega}_i(\mathbf{p} - \mathbf{p}_i) \end{aligned} \quad (11)$$

Therefore the following identities hold true by construction:

$$\nabla \times \vec{\psi} = \vec{v} \quad \nabla \times \vec{v} = \vec{\omega} \quad \nabla \cdot \vec{v} = 0 \quad \nabla \cdot \vec{\omega} = 0$$

The cost of evaluating $\vec{v}$ at every vorticle center $\mathbf{p}_i$ directly with Eq. (11) scales quadratically, and is prohibitive for large numbers of vortices. The FMM (Fast Multipole Method) of Geengard and Rokhlin [1987] provides a way to evaluate $\vec{v}(\mathbf{p})$ using only the near cells of $\mathbf{p}$ in an octree. We call near vortices the vortices in the cell and in the 1-cell ring of $\mathbf{p}$, far vortices the other ones, and split the advected fields into near and far fields:

$$\vec{v} = \vec{v}_{\text{near}} + \vec{v}_{\text{far}} \quad (12)$$

Note that our octree stores vortices of different sizes, and there are potentially near and far vortices at multiple levels.

Fig. 2. Fields of a vorticle: stream (left), velocity (middle) and vorticity (right), where $\vec{w}_i$ is up.

3.1 Near Advected Field

The near advected field could be evaluated by summing the velocity fields of the near vortices, in the cell and 1-cell ring across all levels:

$$\vec{v}_{\text{near}}(\mathbf{p}) = \sum_{i \in \text{near}(\mathbf{p})} \vec{v}_i(\mathbf{p} - \mathbf{p}_i) \quad (13)$$

The near field, however, can have a very large mean energy, especially near user defined vorticle emitters. Also, if r_i approaches 0, then $\vec{v}_i(\mathbf{p}_i)$ becomes infinite. For the simulation to be tolerant of any input, we must use an advection method that handles the most

extremely coiled trajectory in a single time step. For that, we replace the velocity with a displacement constructed from the union of twists, similar to the method of Angelidis et al. [2006a]:

$$\begin{aligned}\vec{k}_0 &= \Delta_t \sum_{i \in \text{near}(\mathbf{p})} \xi_i(\mathbf{p} - \mathbf{p}_i) \vec{\psi}_i(\mathbf{p} - \mathbf{p}_i) \\ \vec{k}_1 &= \Delta_t \sum_{i \in \text{near}(\mathbf{p})} \vec{v}_i(\mathbf{p} - \mathbf{p}_i) \\ \vec{v}_{\text{near}}(\mathbf{p}) &= \frac{1}{\Delta_t} \left(\vec{k}_1 + \vec{k}_0 \times \left(\frac{\|\vec{k}_0\| - \sin(\|\vec{k}_0\|)}{\|\vec{k}_0\|^3} \vec{k}_0 \times \vec{k}_1 + \frac{1 - \cos(\|\vec{k}_0\|)}{\|\vec{k}_0\|^2} \vec{k}_1 \right) \right)\end{aligned}\quad (14)$$

This method underestimates the length of the trajectory, but converges when decreasing the time step Δ_t and guarantees stability. We observe that strong vortices induce extremely coiled pathlines in a single time step similar to the pathlines integrated with tiny steps, and also that a small number of vortices placed on a ring moves along a straight and stable trajectory.

3.2 Far Advected Field

In this section we define the far advected field's near expansion using the near coefficients $\vec{\kappa}_{lmn}$ of the deepest cell containing $\mathbf{p}$:

$$\vec{v}_{\text{far}}(\mathbf{p}) = \sum_{l=0}^{l_{\max}} \sum_{m=-l}^l \sum_{n=1}^{n_{\max}} \nabla(Y_{lm}(\mathbf{p}) \|\mathbf{p}\|^n) \times \vec{\kappa}_{lmn} \quad (15)$$

The function $Y_{lm}(\mathbf{p})$ denotes the real spherical harmonics:

$$K_{lm} = \sqrt{\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}} \quad (16)$$

$$Y_{lm}(\mathbf{p}) = K_{lm} P_{l|m|}(\cos(\phi)) \begin{cases} \sqrt{2} \cos(m\theta) & \text{if } m > 0 \\ 1 & \text{if } m = 0 \\ \sqrt{2} \sin(-m\theta) & \text{if } m < 0 \end{cases}$$

The function P_{lm} denotes the associated Legendre polynomials, with Cartesian coordinates mapping to spherical coordinates as $(\theta, \phi, r) = (\text{atan2}(p_y, p_x), \text{acos}(\frac{p_z}{\|\mathbf{p}\|}), \|\mathbf{p}\|)$. The first few basis functions are given in Section B. To compute the near coefficients $\vec{\kappa}_{lmn}$, we use the three main steps of the FMM, illustrated in Figure 3. First we build an octree where the near coefficients $\vec{\kappa}_{lmn}$ of all cells are initialized to 0. Level 0 denotes the octree's root, and we call far interacting cells the cells that are children of the parent's neighbors but not direct neighbors. To make this algorithm multiscale, we insert vortices in the internal cells of size $2^{\lceil \frac{\log r_i}{\log 2} \rceil}$. The steps are:

- (1) We compute the far coefficients $\vec{\mu}_{lmn}$ of all cells with vortices. This is explained in Section 3.2.1 and shown in Figure 4. We also allocate the far interacting cells all the way to the root, so that steps (2) and (3) propagate valid coefficients to the deepest cell, to cover all positions with the octree.
- (2) We traverse the tree from the leaves to level 0, translating and accumulating the children's far coefficients $\vec{\mu}_{lmn}$ into the parent's far coefficient $\vec{\mu}'_{l'm'n'}$. The translation of far coefficients is explained in Section 3.2.2 and shown in Figure 5.
- (3) We traverse the tree from level 2 to the leaves: first we translate the parent's near coefficient as described in Section 3.2.4 and shown in Figure 7, second we transform the far interacting cells' far coefficient as given in Section 3.2.3 and shown in Figure 6. Those coefficients are summed into $\vec{\kappa}_{lmn}$.

After step (3), the coefficients at the leaves represent all far vortices, and are then used in Eq. (15). The reader may go to Section 4 to skip over the detail of computing $\vec{\kappa}_{lmn}$.

Fig. 3. A cell's far coefficient is computed by translating the 8 children's far coefficient (left). A cell's near coefficient is computed by translating the parent's near coefficient (middle), and transforming the 189 far interacting cells' far coefficient (right).

3.2.1 Vorticle Far Coefficients. In this section we explain how to compute the far coefficients $\vec{\mu}_{lmn}$ of the far decomposition of a field of vortices around the origin:

$$\vec{v}_{\text{far}}(\mathbf{p}) = \sum_{l=0}^{l_{\max}} \sum_{m=-l}^l \sum_{n=1}^{n_{\max}} \nabla \left(Y_{lm}(\mathbf{p}) \frac{1}{\|\mathbf{p}\|^n} \right) \times \vec{\mu}_{lmn} \quad (17)$$

Unlike the expansion of Sheel [2011], the novelty of our expansion is that it includes the vortices' size. To compute the far coefficients $\vec{\mu}_{lmn}$, we start with a far expansion of a single vorticle's stream function:

$$\vec{\psi}_i(\mathbf{p} - \mathbf{p}_i) = \vec{w}_i \xi_i(\|\mathbf{p}\|) \sum_k P_k \left(\frac{\mathbf{p} \cdot \mathbf{p}_i}{\|\mathbf{p}\| \|\mathbf{p}_i\|}, \tau_i(\|\mathbf{p}\|) \right) \frac{\|\mathbf{p}_i\|^k}{\|\mathbf{p}\|^k} \quad (18)$$

This is achieved by identifying the terms of a Taylor expansion near $\epsilon = 0$ of $\xi_i(\|\mathbf{p} - \mathbf{p}_i\|) = \xi_i(\|\mathbf{p}\|)(1 + \tau_i(\|\mathbf{p}\|)\epsilon)^{-1/2}$, where $\tau_i(l) = \frac{1}{1+r_i^2/l^2}$, and where $P_k(z, \tau)$ is a modified Legendre polynomial, that is, a Legendre polynomial where the i^{th} non-zero coefficient of the largest power of z is multiplied by τ_i^{k-i} . The first few are:

$$\begin{aligned}P_0(\alpha, \tau) &= 1 \\ P_1(\alpha, \tau) &= \alpha\tau \\ P_2(\alpha, \tau) &= -\frac{\tau}{2} + \frac{3\alpha^2\tau^2}{2}\end{aligned}\quad (19)$$

We project this modified Legendre decomposition on the Legendre polynomials:

$$\vec{\psi}_i(\mathbf{p} - \mathbf{p}_i) = \vec{w}_i \sum_l P_l \left(\frac{\mathbf{p} \cdot \mathbf{p}_i}{\|\mathbf{p}\| \|\mathbf{p}_i\|} \right) B_l(\|\mathbf{p}\|, \|\mathbf{p}_i\|) \quad (20)$$

where:

$$\begin{aligned}B_l(r, d_i) &= \xi_i(r) \sum_{k=l}^{\infty} \beta_{kl}(\tau_i(r)) \frac{d_i^k}{r^k} \\ \beta_{kl}(\tau) &= \int_{-1}^1 \frac{2l+1}{2} P_l(\alpha) P_k(\alpha, \tau) d\alpha\end{aligned}\quad (21)$$

We then perform a series expansion in $1/r$ to obtain the radial coefficients b_{ln} :

$$B_l = \sum_{n=1}^{\infty} b_{ln} \frac{1}{r^n} \quad (22)$$

Finally, we project the Legendre polynomial on the spherical harmonics to obtain the spherical coefficients y_{lm} :

$$y_{lm} = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} P_l \left(\frac{\mathbf{p}_r \theta \phi \cdot \mathbf{p}_i}{\|\mathbf{p}_i\|} \right) Y_{lm}(\theta, \phi) d\theta d\phi \quad (23)$$

Thus the far coefficients $\vec{\mu}_{lmn}$ of one vorticle is assembled from the radial coefficient, the spherical coefficient, and the vorticle's rotation strength $\vec{w}_i$:

$$\vec{\mu}_{lmn} = y_{lm} b_{ln} \vec{w}_i \quad (24)$$

The first few far coefficients are given in Appendix C. The far field of multiple vortices is obtained simply by summing the coefficients of each vorticle. In Figure 4, we compare the far field of multiple vortices individually summed to the far field of multiple vortices projected on the far field basis function. This reduction of complexity to a basis of constant size makes this method scalable. For evaluating $\vec{v}$ outside of the octree, we use the far expansion of Eq. (17) at the root.

Fig. 4. Individual vortices inside the box (left) are transformed into far coefficients relative to the box's center (right) for evaluations outside of the 26 neighboring cells shown with a dashed line.

3.2.2 Far Coefficients Translation. In Figure 5, we compare a far field before and after translation to its parent node. To compute the parent's far coefficients $\vec{\mu}_{l'm'n'}$ from child coefficients $\vec{\mu}_{lmn}$ relative to offset location $\mathbf{c}$, we do a series expansion of $\vec{\psi}_{\text{far}}(\mathbf{p} + \mathbf{c})$ in $1/r$ and project on the translated basis $Y_{l'm'}$. The first few are given in Appendix D. Since the relative spatial configuration between parent and children nodes are repeated in the octree, we precomputed the translation coefficients, the translation becoming a mere sum of weighted vectors. This is also the case for all other coefficients. We show in Figure 5 a translated projected far field.

3.2.3 Far to Near Coefficients Transform. In Figure 6, we compare a far field to a near field after transform. For far interacting cells, we transform the far expansion coefficients $\vec{\mu}_{lmn}$ into the near expansion coefficients $\vec{\kappa}_{l'm'n'}$. To compute the near coefficients $\vec{\kappa}_{l'm'n'}$ of the translated far coefficient $\vec{\mu}_{lmn}$ to a new location $\mathbf{c}$, we do a series expansion of $\vec{\psi}_{\text{far}}(\mathbf{p} + \mathbf{c})$ in r and project on the translated basis $Y_{l'm'}$ to obtain the coefficient of the translated field. Since some of the far coefficients are zero, they simplify, and the first few are given in Section E.

Fig. 5. The far field of a cell (left) is translated to the parent cell (right). The matching area is outlined in red. The 26 neighboring cells are shown with a dashed line.

Fig. 6. To evaluate the far field of a cell inside the dashed red line (left), it is transformed into a near field (right).

3.2.4 Near Coefficients Translation. We propagate the coefficients from parent nodes $\vec{\kappa}_{lmn}$ to the coefficient of children nodes $\vec{\kappa}_{l'm'n'}$, by translating the parent's coefficient to each child's center, and adding the coefficient. To compute the coefficients $\vec{\kappa}_{l'm'n'}$ of the translated coefficient $\vec{\kappa}_{lmn}$ to a new location $\mathbf{c}$, we do a series expansion of $\vec{\psi}_{\text{near}}(\mathbf{p} + \mathbf{c})$ in r and project on the translated basis $Y_{l'm'}$ to obtain the coefficient of the translated field. They simplify and the first few are given in Appendix F.

Fig. 7. The near field of a cell (left) is translated to a child cell (right).

4 HARMONIC FIELD

We compute the harmonic field $\vec{h}$ in Eq. (4) using the source and doublet panel method [Erickson 1990]. In contrast with the coupling of grid based solvers, which solve for both pressure and boundaries simultaneously in a volume domain, the source and doublet

panel method solves the boundaries' degrees of freedom on a surface domain, with a solution smoothly defined between the boundaries and infinity. Given a manifold surface defined by n triangle panels $\delta\Omega_j$ with normal $\vec{n}_j$ pointing outside, field $\vec{h}$ is defined as:

$$\vec{h} = \sum_j \sigma_j \nabla S_j + \mu_j \nabla D_j \quad (25)$$

Each triangle induces a source panel field S_j and a doublet panel field D_j , defined by surface integrals:

$$\begin{aligned} S_j(\mathbf{p}) &= \iint_{\delta\Omega_j} \frac{1}{\|\mathbf{p}-\mathbf{x}\|} d\mathbf{x} \\ D_j(\mathbf{p}) &= - \iint_{\delta\Omega_j} \vec{n}_j \cdot \frac{\mathbf{p}-\mathbf{x}}{\|\mathbf{p}-\mathbf{x}\|^3} d\mathbf{x} \end{aligned} \quad (26)$$

The gradients of the source and the doublet are both irrotational and incompressible, and therefore $\nabla \cdot \vec{h} = 0$ and $\nabla \times \vec{h} = 0$. To find σ_j and μ_j such that field $\vec{h}$ cancels field $\vec{v}$ across all boundaries, we solve the linear system of the direct approach by placing a control point $\mathbf{p}_i$ at the center of each panel. This produces a linear system of $2n$ equations with $2n$ variables:

$$\begin{cases} 0 &= \sum_j \sigma_j S_j(\mathbf{p}_i) + \mu_j D_j(\mathbf{p}_i) \\ \vec{n}_i \cdot (\vec{s}(\mathbf{p}_i) - \vec{v}(\mathbf{p}_i)) &= \sum_j \sigma_j \vec{n}_i \cdot \nabla S_j(\mathbf{p}_i) + \mu_j \vec{n}_i \cdot \nabla D_j(\mathbf{p}_i) \end{cases} \quad (27)$$

where $\vec{s}(\mathbf{p}_i)$ is the velocity at the center of panel i . If the boundary isn't moving, then $\vec{s}(\mathbf{p}_i) = 0$. We use analytical integrals for S_j and ∇S_j , D_j and ∇D_j , provided in Appendix I and J. For numerical evaluation, we split the harmonic field in near and far fields:

$$\vec{h} = \vec{h}_{\text{near}} + \vec{h}_{\text{far}} \quad (28)$$

The panels are stored in an octree structure similar to the one used for the advected field. Note that this must be a second octree, since vortices sample space between colliders, whereas the panels sample the surface of the colliders, and there is generally no cell correlation between the two.

4.1 Near Harmonic Field

The near harmonic field is evaluated directly by summing the harmonic fields of the near panels, in the cell and 1-cell ring across all levels, in a manner similar to Eq. (13):

$$\vec{h}_{\text{near}}(\mathbf{p}) = \sum_{j \in \text{near}(\mathbf{p})} \sigma_j \nabla S_j(\mathbf{p}) + \mu_j \nabla D_j(\mathbf{p}) \quad (29)$$

Fig. 8. Left: initial position of a boundary object (red) moving into points (green lattice). Right: the harmonic field warps space in an incompressible and irrotational manner, with a slip boundary.

4.2 Far Harmonic Field

The far harmonic field is similar to the far advected field, but instead of the curl of a field with vector coefficients, it is the gradient of a field with scalar coefficients. We compute the near harmonic coefficients κ_{lmn} using the same FMM algorithm, but only with 1-dimensional coefficients, and with samples on the surface:

$$\vec{h}_{\text{far}}(\mathbf{p}) = \sum_{l=0}^{l_{\text{max}}} \sum_{m=-l}^l \sum_{n=1}^{n_{\text{max}}} \kappa_{lmn} \nabla(Y_{lm}(\mathbf{p}) \|\mathbf{p}\|^n) \quad (30)$$

Eq. (30) is similar to Eq. (15). To derive the far coefficients, we approximate each source and doublet panel of area a_j with a point:

$$\vec{h}_{\text{far}} = \nabla \left(\sum_j a_j \sigma_j \frac{1}{\|\mathbf{p}-\mathbf{x}_j\|} + \frac{a_j \mu_j}{2\epsilon} \left(\frac{1}{\|\mathbf{p}-(\mathbf{x}_j-\epsilon\vec{n}_j)\|} - \frac{1}{\|\mathbf{p}-(\mathbf{x}_j+\epsilon\vec{n}_j)\|} \right) \right) \quad (31)$$

where the source panel is converted to a point with area a_i and the coefficients y_{lm} and b_{ln} are a special case of Eq. (24) with $r_j = 0$. For the source located at $\mathbf{x}_j$:

$$\mu_{lmn} = a_j \sigma_j y_{lm} b_{ln} \quad (32)$$

The doublet produces two sources, located at $\mathbf{x}_j - \epsilon\vec{n}_j$ and at $\mathbf{x}_j + \epsilon\vec{n}_j$, with far coefficients:

$$\begin{aligned} \mu_{lmn} &= \frac{a_j \mu_j}{2\epsilon} y_{lm} b_{ln} \\ \mu_{lmn} &= -\frac{a_j \mu_j}{2\epsilon} y_{lm} b_{ln} \end{aligned} \quad (33)$$

The coefficients y_{lm} and b_{ln} are also the ones of Eq. (24). If the surface is undersampled, the discretization error can manifest itself visually as weak currents leaking in or out of boundaries, usually far from the control points and near panel edges. This issue can be reduced for inward leakage by imposing a no-through condition per point, and for outward leakage by using the maximum principle: the magnitude of the harmonic field in the entire space is bound by the harmonic field's value on the surface. Near the surface we assume that $\vec{n}_i \cdot \vec{u}(\mathbf{p}) \leq \arg\max_i (\vec{n}_i \cdot (\vec{s}(\mathbf{p}_i) - \vec{v}(\mathbf{p}_i)))$

5 STRETCHING

While the previous sections focus on computing $(\nabla \times)^{-1} \vec{\omega}$, the following sections focus on the dynamic terms in the right-hand side of Eq. (2). The term $(\vec{\omega} \cdot \nabla) \vec{v}$ in Eq. (2) models the stretching of vorticity, and is responsible for changing the vorticle orientation as well as introducing increasingly high frequency velocities by transferring large scale eddies to smaller scale eddies [Frisch and Kolmogorov 1995]. This term does not have an explicit analogue in Eq. (1), and comes from the expansion into partial derivatives of the acceleration's curl. It is the change of vorticity by the velocity gradient. We measure stretching assuming that $\vec{w}_i$ and $\vec{\omega}(\mathbf{p}_i)$ have the same direction, which is true for an isolated vorticle, or when $r_i \rightarrow 0$. As the approach taken by Zhang and Bridson [2014], we displace two points on either side of $\mathbf{p}_i$. Reducing this measurement to two points is a valid approach because the flow is incompressible, and we can safely assume that a stretch along $\vec{w}_i$ is accompanied by a corresponding squash on the plane perpendicular to $\vec{w}_i$. After one

time step, $\vec{\omega}_i(\mathbf{p}_i)$ becomes $\vec{\omega}'_i(\mathbf{p}'_i)$:

$$\begin{aligned}\vec{\omega}_i(\mathbf{p}_i) &= 2r_i^{-3/2} \vec{\mathbf{w}}_i \\ \vec{\omega}'_i(\mathbf{p}'_i) &= 2r_i^{-3/2} (\vec{\mathbf{w}}_i + \frac{\Delta_t \|\vec{\mathbf{w}}_i\|}{r_i} (\vec{\mathbf{u}}(\mathbf{q}_1) - \vec{\mathbf{u}}(\mathbf{q}_0))) \\ \mathbf{q}_0 &= \mathbf{p}_i - \frac{r_i \vec{\mathbf{w}}_i}{2\|\vec{\mathbf{w}}_i\|} \\ \mathbf{q}_1 &= \mathbf{p}_i + \frac{r_i \vec{\mathbf{w}}_i}{2\|\vec{\mathbf{w}}_i\|}\end{aligned}\quad (34)$$

To apply this measure of stretching to a vorticity, we developed closed-form formulas for the ellipsoidal fields of a vorticity under uniform stretching:

$$\begin{aligned}\vec{v}_i(\vec{x}, s) &= s \xi_i (\|\sigma(\vec{x}, s)\|)^3 \vec{\mathbf{w}}_i \times \vec{x} \\ \vec{\omega}_i(\vec{x}, s) &= s \xi_i (\|\sigma(\vec{x}, s)\|)^3 (2\vec{\mathbf{w}}_i \\ &\quad - 3\xi_i (\|\sigma(\vec{x}, s)\|)^2 \sigma(\vec{x}, s^2) \times (\vec{\mathbf{w}}_i \times \vec{x}))\end{aligned}\quad (35)$$

where $\sigma(\vec{x}, s) = \frac{1}{\|\vec{\mathbf{w}}_i\|^2} (s^{-4/5} (\vec{x} \cdot \vec{\mathbf{w}}_i) \vec{\mathbf{w}}_i + s^{7/10} (\vec{\mathbf{w}}_i \times \vec{x}) \times \vec{\mathbf{w}}_i)$. When the stretching factor increases, it stretches $\vec{\omega}_i$ by a factor s along $\vec{\mathbf{w}}_i$, and squashes $\vec{\omega}_i$ by $\sqrt{1/s}$ along any direction perpendicular to $\vec{\mathbf{w}}_i$, in accordance with the deformation induced by an incompressible flow:

$$\begin{aligned}\frac{\vec{\omega}_i(\vec{x}, s) \cdot \vec{\mathbf{w}}_i}{\vec{\omega}_i(\sigma(\vec{x}, s)) \cdot \vec{\mathbf{w}}_i} &= s \\ \frac{\vec{\omega}_i(\vec{x}, s) \cdot (\mathbf{p} \times \vec{\mathbf{w}}_i)}{\vec{\omega}_i(\sigma(\vec{x}, s)) \cdot (\mathbf{p} \times \vec{\mathbf{w}}_i)} &= \sqrt{1/s}\end{aligned}\quad (36)$$

We apply the rotation part of stretching to $\vec{\mathbf{w}}'_i$ and the scale part to s'_i , and obtain a stretched vorticity:

$$r'_i = r_i \quad (37)$$

$$\vec{\mathbf{w}}'_i = \|\vec{\mathbf{w}}_i\| \frac{\vec{\omega}'_i(\mathbf{p}_i)}{\|\vec{\omega}'_i(\mathbf{p}'_i)\|} \quad (38)$$

$$s'_i = \frac{\|\vec{\omega}'_i(\mathbf{p}'_i)\|}{\|\vec{\omega}_i(\mathbf{p}_i)\|} \quad (39)$$

We then unstretch the stretchable vorticity in a manner that preserves both E_i and the enstrophy Ω_i of the stretched vorticity. Preserving enstrophy is a local way to reduce as much as possible the disruption of this step over the vorticity dynamics.

$$\Omega_i = \frac{1}{2} \int \vec{\omega}_i^2 d\mathbf{x} = \frac{3\pi^2 (1 + 4s_i'^3)}{64r_i'^3 s_i'^{8/5}} \|\vec{\mathbf{w}}'_i\|^2 \quad (40)$$

The stretching factor is restored to 1, and the resulting vorticity is unstretched:

$$\begin{aligned}r''_i &= r_i s_i'^{4/5} \left(\frac{5}{1+4s_i'^3} \right)^{1/2} \\ \vec{\mathbf{w}}''_i &= \sqrt{\frac{r''_i}{r_i}} \|\vec{\mathbf{w}}_i\| \frac{\vec{\omega}'_i(\mathbf{p}'_i)}{\|\vec{\omega}'_i(\mathbf{p}'_i)\|}\end{aligned}\quad (41)$$

This is a remarkable new insight: when a vorticity's stretching factor is greater than 1, its strength must be reduced so that the mean vorticity and enstrophy can be preserved. We also specify limit resolutions with a lower threshold $r_{\min}$ and upper threshold $r_{\max}$ on the vorticity size r_i . This limit resolution loses enstrophy, but does not lose energy because of Eq. (9). Thus Eq. (34), Eq. (39) and Eq. (41) provide the way to apply stable energy preserving stretching to a vorticity. In a real fluid, turbulence approaching the fluid's Kolmogorov length mix with an effective viscosity. When a vorticity is

squeezed below $r_{\min}$, we turn the stretching into a higher viscosity coefficient for that vorticity, with the diffusion per vorticity of the next section. This is necessary to remove vortices with small contribution.

Fig. 9. A vorticity (a) is stretched into vorticity (b): the mean energy is preserved while the enstrophy changes. Vorticity (b) is then resampled in an unstretched vorticity (c) with the same mean energy and enstrophy as vorticity (b).

6 VISCOSITY

The term $\nu \nabla^2 \vec{\omega}$ in Eq. (2) models viscosity. It is the analogue of the viscosity term of Eq. (1), with the difference that ν is varying if the density ρ is varying. Since vortices have a size and aren't singular at their center, we can compute a well behaved vorticity derivative that is consistent with the dynamics:

$$\nabla^2 \vec{\omega}_i(\vec{x}) = 15(\|\vec{x}\|^2 \xi_i^2 - 1) \xi_i^5 (2\vec{\mathbf{w}}_i - 7\xi_i^2 \vec{x} \times \vec{\mathbf{w}}_i \times \vec{x}) \quad (42)$$

This could conceivably be solved in a multiscale manner with a near and far diffusion in an octree, along the lines of Section 3 or Section 4. We propose however to solve more simply $\frac{d\vec{\omega}_i}{dt} = \nu \nabla^2 \vec{\omega}_i$, by updating the vorticity enstrophy to match that of $\vec{\omega}_i + \Delta_t \nu \nabla^2 \vec{\omega}_i$. Since vortices have no mass, this model disregards the mass of the advected quantity. Its advantage however is in providing users with a viscosity that is spatially tweakable per vorticity, and bound to the dynamics. The linear approximation is only valid when the diffusion $\Delta_t \nu$ is small enough. Therefore we also clamp the diffusion to enforce monotonicity. After one time step, the vorticity strength becomes:

$$\vec{\mathbf{w}}_i = \begin{cases} \sqrt{1 + \frac{\Delta_t \nu}{r_i^2} \left(\frac{11025 \Delta_t \nu}{256 r_i^2} - \frac{35}{4} \right)} \vec{\mathbf{w}}_i & \text{if } \Delta_t \nu < \frac{32}{315} r_i^2 \\ \frac{\sqrt{5}}{3} \vec{\mathbf{w}}_i & \text{otherwise} \end{cases} \quad (43)$$

Another limitation of this viscous model is that it does not carry the diffusion of direction from vorticity to vorticity.

7 BUOYANCY

When $\mathbf{p}$ is outside of solid objects, the term $\nabla \log(\rho) \times (\vec{\mathbf{F}} - \frac{d\vec{\mathbf{u}}}{dt}) + \nabla \times \vec{\mathbf{F}}$ in Eq. (2) becomes $\nabla \log(\rho) \times (\vec{\mathbf{g}} + \vec{\mathbf{e}} - \frac{d\vec{\mathbf{u}}}{dt}) + \nabla \times \vec{\mathbf{e}}$, and models the vorticity induced by buoyancy. This component requires a varying density in order to have any effect, and can be omitted at first if the reader is looking to build a simpler system. New vortices are produced from the density field ρ releasing potential energy, as shown in Figure 10. We define ρ with a set of particles carrying density

and an ambient density $\rho_A > 0$, so the total density field ρ is strictly greater than 0, as required by Eq. (2):

$$\rho = \rho_A + \sum \rho_j \quad (44)$$

The falloff ρ_j of a density particle j is centered at $\mathbf{x}_j$, and defined per Appendix G in local coordinates $\mathbf{q}_j = \mathbf{p} - \mathbf{x}_j$ as:

$$\rho_j = m_j \frac{\exp\left((1 + \kappa_1 \frac{\mathbf{q}_j \cdot \mathbf{q}_j}{2r_j^2})^{-\kappa_2}\right) - 1}{e - 1} \quad (45)$$

where m_j is a multiplier of the particle density field, and κ_i is given in Eq. (55). The newly induced vortices are located along a ring of diameter r_j perpendicular to $\vec{\mathbf{g}} + \vec{\mathbf{e}} - \frac{d\vec{\mathbf{u}}}{dt}$. Instead of advecting an additional filament representation, we discretize the ring with n new vortices, equidistant for simplicity, and where $n = 2$ in practice. Let us define orthogonal unit vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ such that $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ has the direction of $\vec{\mathbf{g}} + \vec{\mathbf{e}} - \frac{d\vec{\mathbf{u}}}{dt}$. Given n randomly selected samples α_j , the new vortices are in the density particle's local coordinates:

$$\begin{aligned} \mathbf{x}_{\alpha_j} &= \mathbf{x}_j + \frac{r_j}{2} (\cos(\alpha_j) \vec{\mathbf{a}} + \sin(\alpha_j) \vec{\mathbf{b}}) \\ \vec{\mathbf{w}}_{\alpha_j} &= r_j \sqrt{r_j} \kappa_0 \frac{\pi}{n} \|\vec{\mathbf{g}} + \vec{\mathbf{e}} - \frac{d\vec{\mathbf{u}}}{dt}\| \frac{\rho_j(\mathbf{x}_{\alpha_j}) + \frac{m_j}{\rho(\mathbf{x}_{\alpha_j})}}{\rho(\mathbf{x}_{\alpha_j})} \\ &\quad (-\sin(\alpha_j) \vec{\mathbf{a}} + \cos(\alpha_j) \vec{\mathbf{b}}) \\ r_{\alpha_j} &= r_j \end{aligned} \quad (46)$$

To evaluate the acceleration $\frac{d\vec{\mathbf{u}}}{dt}$, we sample the velocity both after and before the advection of the particle, before emission of new vortices in order to avoid the temporal jolt of the new vortices.

Fig. 10. A density particle emits a ring of vortices (red). We sample the ring with vortices of strengths that match the mass of the density particles, as visualized on points (green lattice).

8 VORTICITY SHEDDING

When $\mathbf{p}$ is at the boundary of a solid object, the term $\nabla \log(\rho) \times (\vec{\mathbf{F}} - \frac{d\vec{\mathbf{u}}}{dt}) + \nabla \times \vec{\mathbf{F}}$ in Eq. (2) measures the change of vorticity at the moving object's boundary. This component is absent in inviscid fluids, and always present in real fluids. The viscosity coefficient for vorticity shedding can be specified independently from the viscosity of Section 6.

The surface vorticity spreads into the flow by a diffusion proportional to viscosity coefficient ν . We show in Appendix H that the surface vorticity that satisfies our boundary conditions is $(\vec{\mathbf{f}} - \vec{\mathbf{e}} - \frac{d\vec{\mathbf{u}}}{dt}) \times \vec{\mathbf{n}}$. The vorticity shedding is the solution to a differential equation with boundary condition:

$$\begin{cases} \frac{d\vec{\omega}}{dt} &= \nu \nabla^2 \vec{\omega} \\ \vec{\omega} &= \Delta_t (\vec{\mathbf{f}} - \vec{\mathbf{e}} - \frac{d\vec{\mathbf{u}}}{dt}) \times \vec{\mathbf{n}} \end{cases} \quad \text{p} \in \delta\Omega \quad (47)$$

We solve Eq. (47) by shedding vortices. We divide the surface into n panels of area a_i and centroid $\mathbf{x}_i$, and emit per panel a vorticle that roughly approximates the heat kernel during a time step Δ_t :

$$\begin{aligned} \vec{\mathbf{w}}_i &= a_i \frac{r_i^{1/2}}{4\pi} (\vec{\mathbf{f}} - \vec{\mathbf{e}} - \frac{d\vec{\mathbf{u}}}{dt}) \times \vec{\mathbf{n}} \\ r_i &= \sqrt{2} \sqrt{\Delta_t \nu} \end{aligned} \quad (48)$$

We use $\|(\vec{\mathbf{f}} - \vec{\mathbf{e}} - \frac{d\vec{\mathbf{u}}}{dt}) \times \vec{\mathbf{n}}\|$ normalized over the surface, as a probability distribution function to create samples at the locations that most affect the flow. Note that $\vec{\mathbf{f}}$ is the surface acceleration, as opposed to the velocity. To compute the acceleration, we store on the surface Eq. (6) at the previous time, and use $\frac{d\vec{\mathbf{u}}}{dt} \simeq \frac{\vec{\mathbf{u}}(t) - \vec{\mathbf{u}}(t - \Delta_t)}{\Delta_t}$. We show in Figure 11 that this produces the expected behavior. Furthermore, users can modify separately shedding induced by surface acceleration $\vec{\mathbf{f}}$ and fluid acceleration $\frac{d\vec{\mathbf{u}}}{dt}$.

Fig. 11. Left: a constant flow moves at 10 ms^{-1} with $\nu = 0.1$ around a static pole of radius 1 (red): the surface sheds vortices (black arrows). Right: a slice of the vorticity induced by the vortices reveals the emergent behavior of a von Kármán vortex street.

Optimization. User emitters, buoyancy, and shedding introduce an increasing number of vortices in the simulation. To address this rise of complexity, we use a fusing step. Using the existing octree data structure, we merge similar vortices in each octree cell. The extreme fusion case is one vorticle per cell. Our fusing method favors the vorticle of high energy, and equals the combined energy when $\vec{\mathbf{w}}_0 = \vec{\mathbf{w}}_1$ and $r_0 = r_1$:

$$\begin{aligned} \mathbf{p} &= \frac{\sqrt{E_0} \mathbf{p}_0 + \sqrt{E_1} \mathbf{p}_1}{\sqrt{E_0} + \sqrt{E_1}} & r &= \frac{\sqrt{E_0} r_0 + \sqrt{E_1} r_1}{\sqrt{E_0} + \sqrt{E_1}} \\ \vec{\mathbf{w}} &= (r_0/r)^{3/2} \vec{\mathbf{w}}_0 + (r_1/r)^{3/2} \vec{\mathbf{w}}_1 \end{aligned}$$

9 MULTI-SCALE DENSITY ADVECTION

Modern renderers handle volumes in the form of voxel grids. It is therefore a necessity to output the density and velocity in that form. While a complete multi-scale grid representation is beyond the scope of this paper, we propose a simple method as a starting point for a full solution, compatible with available volume data structures [Museth 2013; Wrenninge 2016]. Given a single smoke density emitter, we split the domain into multiple grids: density is emitted in a grid with a resolution based on the position in camera space of the emitter at the times of emission. The velocity is then evaluated in the grid, and used for advecting the density. Since all densities share a common velocity field, the dynamics remains consistent, and multiple density resolutions overlap in an additive manner thanks to the renderer.

10 IMPLEMENTATION OVERVIEW

We implemented our solver in Houdini (SideFX), summarized in Algorithm (1).

Algorithm 1 Multi-Scale Vorticle Smoke

- 1: Add emitted vortices, e.g. by splitting the emitter's energy among vortices using Eq. (10).
- 2: Build the octree of vortices: store each vorticle in the cell of size $2^{\lceil \frac{\log r_i}{\log 2} \rceil}$, allocate the far interacting cells, and compute the advected field near coefficients per Section 3.2.
- 3: Evaluate the advected field $\vec{v}$ at every panel location, using Eq. (12).
- 4: Compute the source and doublet panel coefficients σ_i and μ_i by solving Eq. (27).
- 5: Build the octree of panels: store each panel in the cell of size $2^{\lceil \frac{\log r_i}{\log 2} \rceil}$, allocate the far interacting cells, and compute the harmonic field near coefficients per Section 4.2.
- 6: Allocate velocity voxels and compute velocity field $\vec{u}$ at every voxel.
- 7: Insert new density in the density voxels.
- 8: Advect the density using the velocity voxels.
- 9: Compute the velocity field $\vec{u}$ at every vorticle. Note that we are not using the velocity voxels in this step.
- 10: Advect the vortices.
- 11: Stretch vortices per Section 5.
- 12: Diffuse vortices per Section 6.

11 CONCLUSION

Our method solves Navier-Stokes' differential equation using spherical harmonics across multiple resolutions. There is no constraint on placing vortices of arbitrary size, the method is very stable with mixed scales. The divergence-free basis functions produce a field with no divergence by construction. The sampling resolution of density, dynamics and boundary condition are decoupled from each other, and each can be sampled adaptively. Our panel method could be further improved by filtering the advected field with a filter size proportional to the panel size. We advect vortices that carry a volume of vorticity, as opposed to regularizing a point vorticity. This leads to stable and energy preserving stretching. Our stretching model could be further improved by splitting stretched vortices along the stretching directions, which would be closer to the physical phenomenon that spreads turbulence along stretched features. It would also make sense to vary the multipole expansion across scale for better control of the error. Additional stability is achieved by integrating twists instead of velocity. Also, we propose a model for buoyancy, which plays a key role in gas motion, for producing more lively smokes and pyroclastic effects. Our method provides the full set of features expected from a gas simulation, and is a key tool for making Visual Effects.

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A NOMENCLATURE

t	time	Δt	time step
$\mathbf{p}$	position	$\mathbf{p}_i$	vorticle center
$\vec{u}$	velocity field	$\vec{h}$	harmonic field
$\vec{\psi}$	stream field	$\vec{\psi}_i$	vorticle stream field
$\vec{v}$	advected field	$\vec{v}_i$	vorticle velocity field
$\vec{\omega}$	vorticity field	$\vec{\omega}_i$	vorticle vorticity field
$\vec{w}_i$	vorticle strength	r_i	vorticle size
s_i	vorticle stretching factor	E_i	vorticle mean energy
Ω_i	vorticle enstrophy	ρ	density field
ρ_A	ambient density field	ρ_j	particle density field
m_j	particle density multiplier	$\vec{s}$	solid objects velocity
$\vec{f}$	solid objects acceleration	ν	kinematic viscosity
$\vec{g}$	gravity	$\vec{e}$	user external forces
Y_{lm}	real spherical harmonics	P_{lm}	Legendre polynomial
$\vec{\mu}_{lmn}$	far vector coefficient	$\vec{\kappa}_{lmn}$	near vector coefficient
μ_{lmn}	far scalar coefficient	κ_{lmn}	near scalar coefficient
y_{lm}	spherical coefficient	b_{ln}	radial coefficient
$\delta\Omega$	boundary	$\delta\Omega_j$	triangle panel
σ_j	source coefficient	μ_j	doublet coefficient
S_j	source field	D_j	doublet field

B FIELD BASIS FUNCTIONS

The first few basis functions are of the form $\|\mathbf{p}\|^n (n\vec{a}_{lm} + \vec{b}_{lm})$, and are given in Cartesian coordinates:

$$\begin{aligned}
\nabla(Y_{0,0}(\mathbf{p})\|\mathbf{p}\|^n) &= \frac{\|\mathbf{p}\|^n}{\|\mathbf{p}\|^2} \sqrt{\frac{1}{4\pi}} (n\mathbf{p}) \\
\nabla(Y_{1,-1}(\mathbf{p})\|\mathbf{p}\|^n) &= \frac{\|\mathbf{p}\|^n}{\|\mathbf{p}\|^3} \sqrt{\frac{3}{4\pi}} (-np_y\mathbf{p} + p_y\mathbf{p} - \|\mathbf{p}\|^2\vec{y}) \\
\nabla(Y_{1,0}(\mathbf{p})\|\mathbf{p}\|^n) &= \frac{\|\mathbf{p}\|^n}{\|\mathbf{p}\|^3} \sqrt{\frac{3}{4\pi}} (np_z\mathbf{p} - p_z\mathbf{p} + \|\mathbf{p}\|^2\vec{z}) \\
\nabla(Y_{1,1}(\mathbf{p})\|\mathbf{p}\|^n) &= \frac{\|\mathbf{p}\|^n}{\|\mathbf{p}\|^3} \sqrt{\frac{3}{4\pi}} (-np_x\mathbf{p} + p_x\mathbf{p} - \|\mathbf{p}\|^2\vec{x}) \\
\nabla(Y_{2,-2}(\mathbf{p})\|\mathbf{p}\|^n) &= \frac{\|\mathbf{p}\|^n}{\|\mathbf{p}\|^4} \sqrt{\frac{15}{4\pi}} (np_xp_y\mathbf{p} + \|\mathbf{p}\|^2(p_y\vec{x} + p_x\vec{y}) - 2p_xp_y\mathbf{p}) \\
\nabla(Y_{2,-1}(\mathbf{p})\|\mathbf{p}\|^n) &= \frac{\|\mathbf{p}\|^n}{\|\mathbf{p}\|^4} \sqrt{\frac{15}{4\pi}} (-np_y p_z\mathbf{p} + 2p_y p_z\mathbf{p} - \|\mathbf{p}\|^2(p_z\vec{y} + p_y\vec{z})) \\
\nabla(Y_{2,0}(\mathbf{p})\|\mathbf{p}\|^n) &= \frac{\|\mathbf{p}\|^n}{\|\mathbf{p}\|^4} \sqrt{\frac{5}{16\pi}} (-n(3p_z^2 - \|\mathbf{p}\|^2)\mathbf{p} + 6p_z(p_z\mathbf{p} - \|\mathbf{p}\|^2\vec{z})) \\
\nabla(Y_{2,1}(\mathbf{p})\|\mathbf{p}\|^n) &= \frac{\|\mathbf{p}\|^n}{\|\mathbf{p}\|^4} \sqrt{\frac{15}{4\pi}} (np_z p_x\mathbf{p} + \|\mathbf{p}\|^2(p_x\vec{z} + p_z\vec{x}) - 2p_z p_x\mathbf{p}) \\
\nabla(Y_{2,2}(\mathbf{p})\|\mathbf{p}\|^n) &= \frac{\|\mathbf{p}\|^n}{\|\mathbf{p}\|^4} \sqrt{\frac{15}{4\pi}} (n\frac{p_x^2 - p_y^2}{2}\mathbf{p} + \|\mathbf{p}\|^2(p_x\vec{x} - p_y\vec{y}) + (p_y^2 - p_x^2)\mathbf{p})
\end{aligned}$$

C FIRST VORTICLE TO FAR COEFFICIENT

The far radial coefficients b_{ln} multiply the radial part of the basis:

$$B_l = \sum_{n=1}^{\infty} b_{ln} \frac{1}{r^n} \quad (49)$$

For a cell centered at $\mathbf{c}$, and $d_i = \|\mathbf{p}_i - \mathbf{c}\|$, the first few far radial coefficients b_{ln} are:

$$\begin{aligned}
B_0(r, d_i) &= \frac{1}{r} - \frac{r_i^2}{2r^3} + \frac{-4d_i^2 r_i^2 + 3r_i^4}{8r^5} + O(\frac{1}{r^7}) \\
B_1(r, d_i) &= \frac{d_i}{r^2} - \frac{3d_i r_i^2}{2r^4} - \frac{3d_i(4d_i^2 r_i^2 - 5r_i^4)}{8r^6} + O(\frac{1}{r^7}) \\
B_2(r, d_i) &= \frac{d_i^2}{r^3} - \frac{5d_i^2 r_i^2}{2r^5} + O(\frac{1}{r^7})
\end{aligned} \quad (50)$$

The first few spherical coefficients are:

$$\begin{aligned}
 y_{0,0} &= \sqrt{4\pi} \\
 y_{1,-1} &= -\sqrt{\frac{4\pi}{3}} \frac{p_{iy}}{\|\mathbf{p}_i\|} \\
 y_{1,0} &= \sqrt{\frac{4\pi}{3}} \frac{p_{iz}}{\|\mathbf{p}_i\|} \\
 y_{1,1} &= -\sqrt{\frac{4\pi}{3}} \frac{p_{ix}}{\|\mathbf{p}_i\|} \\
 y_{2,-2} &= \sqrt{\frac{12\pi}{5}} \frac{p_{ix}p_{iy}}{\|\mathbf{p}_i\|^2} \\
 y_{2,-1} &= -\sqrt{\frac{12\pi}{5}} \frac{p_{iy}p_{iz}}{\|\mathbf{p}_i\|^2} \\
 y_{2,0} &= -\sqrt{\frac{\pi}{5}} (1 - 3 \frac{p_{iz}^2}{\|\mathbf{p}_i\|^2}) \\
 y_{2,1} &= -\sqrt{\frac{12\pi}{5}} \frac{p_{ix}p_{iz}}{\|\mathbf{p}_i\|^2} \\
 y_{2,2} &= \sqrt{\frac{3\pi}{5}} \frac{(p_{ix}^2 - p_{iy}^2)}{\|\mathbf{p}_i\|^2}
 \end{aligned} \tag{51}$$

D FIRST FAR COEFFICIENT TRANSLATION

The first few coefficients:

$$\begin{aligned}
 \tilde{\mu}'_{0,0,1} &= \tilde{\mu}_{0,0,1} \\
 \tilde{\mu}'_{0,0,2} &= 0 \\
 \tilde{\mu}'_{0,0,3} &= \tilde{\mu}_{0,0,3} \\
 \tilde{\mu}'_{1,-1,1} &= 0 \\
 \tilde{\mu}'_{1,-1,2} &= \tilde{\mu}_{1,-1,2} + \sqrt{\frac{3}{5}} c_y \tilde{\mu}_{0,0,1} \\
 \tilde{\mu}'_{1,-1,3} &= 0 \\
 \tilde{\mu}'_{1,0,1} &= 0 \\
 \tilde{\mu}'_{1,0,2} &= \tilde{\mu}_{1,0,2} - \sqrt{\frac{3}{5}} c_z \tilde{\mu}_{0,0,1} \\
 \tilde{\mu}'_{1,0,3} &= 0 \\
 \tilde{\mu}'_{1,1,1} &= 0 \\
 \tilde{\mu}'_{1,1,2} &= \tilde{\mu}_{1,1,2} + \sqrt{\frac{3}{5}} c_x \tilde{\mu}_{0,0,1} \\
 \tilde{\mu}'_{1,1,3} &= 0 \\
 \tilde{\mu}'_{2,-2,1} &= 0 \\
 \tilde{\mu}'_{2,-2,2} &= 0 \\
 \tilde{\mu}'_{2,-2,3} &= \tilde{\mu}_{2,-2,3} + \sqrt{\frac{3}{5}} c_x c_y \tilde{\mu}_{0,0,1} + \sqrt{\frac{9}{5}} c_x \tilde{\mu}_{1,-1,2} \\
 &\quad + \sqrt{\frac{9}{5}} c_y \tilde{\mu}_{1,1,2} \\
 \tilde{\mu}'_{2,-1,1} &= 0 \\
 \tilde{\mu}'_{2,-1,2} &= 0 \\
 \tilde{\mu}'_{2,-1,3} &= \tilde{\mu}_{2,-1,3} - \sqrt{\frac{3}{5}} c_y c_z \tilde{\mu}_{0,0,1} - \sqrt{\frac{9}{5}} c_z \tilde{\mu}_{1,-1,2} \\
 &\quad + \sqrt{\frac{9}{5}} c_y \tilde{\mu}_{1,0,2} \\
 \tilde{\mu}'_{2,0,1} &= 0 \\
 \tilde{\mu}'_{2,0,2} &= 0 \\
 \tilde{\mu}'_{2,0,3} &= \tilde{\mu}_{2,0,3} - \sqrt{\frac{1}{20}} (c_x^2 + c_y^2 - 2c_z^2) \tilde{\mu}_{0,0,1} \\
 &\quad - \sqrt{\frac{3}{5}} c_y \tilde{\mu}_{1,-1,2} - \sqrt{\frac{12}{5}} c_z \tilde{\mu}_{1,0,2} \\
 &\quad - \sqrt{\frac{3}{5}} c_x \tilde{\mu}_{1,1,2} \\
 \tilde{\mu}'_{2,1,1} &= 0 \\
 \tilde{\mu}'_{2,1,2} &= 0 \\
 \tilde{\mu}'_{2,1,3} &= \tilde{\mu}_{2,1,3} - \sqrt{\frac{3}{5}} c_x c_z \tilde{\mu}_{0,0,1} + \\
 &\quad \sqrt{\frac{9}{5}} c_x \tilde{\mu}_{1,0,2} - \sqrt{\frac{9}{5}} c_z \tilde{\mu}_{1,1,2} \\
 \tilde{\mu}'_{2,2,1} &= 0 \\
 \tilde{\mu}'_{2,2,2} &= 0 \\
 \tilde{\mu}'_{2,2,3} &= \tilde{\mu}_{2,2,3} + \sqrt{\frac{3}{20}} (c_x^2 - c_y^2) \tilde{\mu}_{0,0,1} \\
 &\quad - \sqrt{\frac{9}{5}} c_y \tilde{\mu}_{1,-1,2} + \sqrt{\frac{9}{5}} c_x \tilde{\mu}_{1,1,2}
 \end{aligned}$$

E FIRST FAR TO NEAR COEFFICIENT TRANSFORMS

The coefficients transforms simplify, and are precomputed and stored before running the simulation. This optimization is worth it since there is at most 189 far interacting cells.

$$\begin{aligned}
 \tilde{\kappa}'_{0,0,0} &= \frac{1}{\|\mathbf{c}\|} \tilde{\mu}_{0,0,1} + \frac{1}{\|\mathbf{c}\|^3} \tilde{\mu}_{0,0,3} \\
 &\quad - \sqrt{3} \frac{c_y}{\|\mathbf{c}\|^3} \tilde{\mu}_{1,-1,2} + \sqrt{3} \frac{c_z}{\|\mathbf{c}\|^3} \tilde{\mu}_{1,0,2} \\
 &\quad - \sqrt{3} \frac{c_x}{\|\mathbf{c}\|^3} \tilde{\mu}_{1,1,2} - \sqrt{\frac{5}{4}} \frac{\|\mathbf{c}\|^2 - 3c_z^2}{\|\mathbf{c}\|^5} \tilde{\mu}_{2,0,3} \\
 &\quad + \sqrt{15} \frac{c_x c_y}{\|\mathbf{c}\|^3} \tilde{\mu}_{2,-2,3} - \sqrt{15} \frac{c_y c_z}{\|\mathbf{c}\|^3} \tilde{\mu}_{2,-1,3} \\
 &\quad - \sqrt{15} \frac{c_x c_z}{\|\mathbf{c}\|^3} \tilde{\mu}_{2,1,3} + \sqrt{\frac{15}{4}} \frac{c_x^2 - c_y^2}{\|\mathbf{c}\|^5} \tilde{\mu}_{2,2,3}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{\kappa}'_{0,0,1} &= 0 \\
 \tilde{\kappa}'_{0,0,2} &= \frac{1}{\|\mathbf{c}\|^5} \tilde{\mu}_{0,0,3} \\
 \tilde{\kappa}'_{0,0,3} &= 0 \\
 \tilde{\kappa}'_{1,-1,0} &= 0 \\
 \tilde{\kappa}'_{1,-1,1} &= \sqrt{\frac{1}{3}} \frac{c_y}{\|\mathbf{c}\|^3} \tilde{\mu}_{0,0,1} + \sqrt{3} \frac{c_y}{\|\mathbf{c}\|^5} \tilde{\mu}_{0,0,3} \\
 &\quad + \frac{\|\mathbf{c}\|^2 - 3c_z^2}{\|\mathbf{c}\|^5} \tilde{\mu}_{1,-1,2} + 3 \frac{c_y c_z}{\|\mathbf{c}\|^5} \tilde{\mu}_{1,0,2} \\
 &\quad - 3 \frac{c_x c_y}{\|\mathbf{c}\|^3} \tilde{\mu}_{1,1,2} - \sqrt{5} \frac{c_x (\|\mathbf{c}\|^2 - 5c_y^2)}{\|\mathbf{c}\|^7} \tilde{\mu}_{2,-2,3} \\
 &\quad + \sqrt{5} \frac{c_z (\|\mathbf{c}\|^2 - 5c_y^2)}{\|\mathbf{c}\|^7} \tilde{\mu}_{2,-1,3} - \sqrt{\frac{15}{4}} \frac{c_y (\|\mathbf{c}\|^2 - 5c_z^2)}{\|\mathbf{c}\|^7} \tilde{\mu}_{2,0,3} \\
 &\quad - \sqrt{125} \frac{c_x c_y c_z}{\|\mathbf{c}\|^7} \tilde{\mu}_{2,1,3} \\
 &\quad + \sqrt{\frac{3}{4}} \frac{c_y (2\|\mathbf{c}\|^2 + 5(c_x^2 - c_y^2))}{\|\mathbf{c}\|^7} \tilde{\mu}_{2,2,3} \\
 \tilde{\kappa}'_{1,-1,2} &= 0 \\
 \tilde{\kappa}'_{1,-1,3} &= \sqrt{3} \frac{c_y}{\|\mathbf{c}\|^7} \tilde{\mu}_{0,0,3} \\
 \tilde{\kappa}'_{1,0,0} &= 0 \\
 \tilde{\kappa}'_{1,0,1} &= -\sqrt{\frac{1}{3}} \frac{c_z}{\|\mathbf{c}\|^3} \tilde{\mu}_{0,0,1} - \sqrt{3} \frac{c_z}{\|\mathbf{c}\|^5} \tilde{\mu}_{0,0,3} \\
 &\quad + 3 \frac{c_y c_z}{\|\mathbf{c}\|^3} \tilde{\mu}_{1,-1,2} + \frac{\|\mathbf{c}\|^2 - 3c_z^2}{\|\mathbf{c}\|^5} \tilde{\mu}_{1,0,2} \\
 &\quad + 3 \frac{c_x c_z}{\|\mathbf{c}\|^3} \tilde{\mu}_{1,1,2} - \sqrt{125} \frac{c_x c_y c_z}{\|\mathbf{c}\|^7} \tilde{\mu}_{2,-2,3} \\
 &\quad - \sqrt{5} \frac{c_y (\|\mathbf{c}\|^2 - 5c_z^2)}{\|\mathbf{c}\|^7} \tilde{\mu}_{2,-1,3} + \sqrt{\frac{15}{4}} \frac{c_z (3\|\mathbf{c}\|^2 - 5c_z^2)}{\|\mathbf{c}\|^7} \tilde{\mu}_{2,0,3} \\
 &\quad - \sqrt{5} \frac{c_x (\|\mathbf{c}\|^2 - 5c_z^2)}{\|\mathbf{c}\|^7} \tilde{\mu}_{2,1,3} \\
 &\quad - \sqrt{\frac{125}{4}} \frac{c_z (c_x^2 - c_y^2)}{\|\mathbf{c}\|^7} \tilde{\mu}_{2,2,3} \\
 \tilde{\kappa}'_{1,0,2} &= 0 \\
 \tilde{\kappa}'_{1,0,3} &= -\sqrt{3} \frac{c_z}{\|\mathbf{c}\|^7} \tilde{\mu}_{0,0,3} \\
 \tilde{\kappa}'_{1,1,0} &= 0 \\
 \tilde{\kappa}'_{1,1,1} &= \sqrt{\frac{1}{3}} \frac{c_x}{\|\mathbf{c}\|^3} \tilde{\mu}_{0,0,1} + \sqrt{3} \frac{c_x}{\|\mathbf{c}\|^5} \tilde{\mu}_{0,0,3} \\
 &\quad - 3 \frac{c_x c_y}{\|\mathbf{c}\|^3} \tilde{\mu}_{1,-1,2} + 3 \frac{c_x c_z}{\|\mathbf{c}\|^3} \tilde{\mu}_{1,0,2} \\
 &\quad + \frac{\|\mathbf{c}\|^2 - 3c_x^2}{\|\mathbf{c}\|^5} \tilde{\mu}_{1,1,2} - \sqrt{5} \frac{c_y (\|\mathbf{c}\|^2 - 5c_x^2)}{\|\mathbf{c}\|^7} \tilde{\mu}_{2,-2,3} \\
 &\quad - \sqrt{125} \frac{c_x c_y c_z}{\|\mathbf{c}\|^7} \tilde{\mu}_{2,-1,3} \\
 &\quad - \sqrt{\frac{15}{4}} \frac{c_x (\|\mathbf{c}\|^2 - 5c_x^2)}{\|\mathbf{c}\|^7} \tilde{\mu}_{2,0,3} \\
 &\quad + \sqrt{5} \frac{c_z (\|\mathbf{c}\|^2 - 5c_x^2)}{\|\mathbf{c}\|^7} \tilde{\mu}_{2,1,3} \\
 &\quad + \sqrt{\frac{3}{4}} \frac{c_x (-2\|\mathbf{c}\|^2 + 5(c_x^2 - c_y^2))}{\|\mathbf{c}\|^7} \tilde{\mu}_{2,2,3} \\
 \tilde{\kappa}'_{1,1,2} &= 0 \\
 \tilde{\kappa}'_{1,1,3} &= \sqrt{3} \frac{c_x}{\|\mathbf{c}\|^7} \tilde{\mu}_{0,0,3} \\
 \tilde{\kappa}'_{2,-2,0} &= 0 \\
 \tilde{\kappa}'_{2,-2,1} &= 0 \\
 \tilde{\kappa}'_{2,-2,2} &= \sqrt{\frac{3}{5}} \frac{c_x c_y}{\|\mathbf{c}\|^3} \tilde{\mu}_{0,0,1} + \sqrt{15} \frac{c_x c_y}{\|\mathbf{c}\|^5} \tilde{\mu}_{0,0,3} \\
 &\quad + \sqrt{\frac{9}{5}} \frac{c_x (\|\mathbf{c}\|^2 - 5c_y^2)}{\|\mathbf{c}\|^7} \tilde{\mu}_{1,-1,2} + \sqrt{45} \frac{c_x c_y c_z}{\|\mathbf{c}\|^7} \tilde{\mu}_{1,0,2} \\
 &\quad + \sqrt{\frac{9}{5}} \frac{c_y (\|\mathbf{c}\|^2 - 5c_x^2)}{\|\mathbf{c}\|^7} \tilde{\mu}_{1,1,2} \\
 &\quad - 4\|\mathbf{c}\|^4 + 5c_z^2 \|\mathbf{c}\|^2 + 35c_x^2 c_y^2 \\
 &\quad + \frac{\|\mathbf{c}\|^9}{\|\mathbf{c}\|^9} \tilde{\mu}_{2,-2,3} \\
 &\quad + 5 \frac{c_x c_z (\|\mathbf{c}\|^2 - 7c_y^2)}{\|\mathbf{c}\|^9} \tilde{\mu}_{2,-1,3} \\
 &\quad - \sqrt{\frac{75}{4}} \frac{c_x c_y (\|\mathbf{c}\|^2 - 7c_z^2)}{\|\mathbf{c}\|^9} \tilde{\mu}_{2,0,3} \\
 &\quad + 5 \frac{c_y c_z (\|\mathbf{c}\|^2 - 7c_x^2)}{\|\mathbf{c}\|^9} \tilde{\mu}_{2,1,3} + \frac{35}{2} \frac{c_x c_y (c_x^2 - c_y^2)}{\|\mathbf{c}\|^9} \tilde{\mu}_{2,2,3} \\
 \tilde{\kappa}'_{2,-2,3} &= 0 \\
 \tilde{\kappa}'_{2,-1,0} &= 0 \\
 \tilde{\kappa}'_{2,-1,1} &= 0 \\
 \tilde{\kappa}'_{2,-1,2} &= -\sqrt{\frac{3}{5}} \frac{c_y c_z}{\|\mathbf{c}\|^3} \tilde{\mu}_{0,0,1} - \sqrt{15} \frac{c_y c_z}{\|\mathbf{c}\|^5} \tilde{\mu}_{0,0,3} \\
 &\quad - \sqrt{\frac{9}{5}} \frac{c_z (\|\mathbf{c}\|^2 - 5c_y^2)}{\|\mathbf{c}\|^7} \tilde{\mu}_{1,-1,2} \\
 &\quad + \sqrt{\frac{9}{5}} \frac{c_y (\|\mathbf{c}\|^2 - 5c_z^2)}{\|\mathbf{c}\|^7} \tilde{\mu}_{1,0,2} \\
 &\quad + \sqrt{45} \frac{c_x c_y c_z}{\|\mathbf{c}\|^7} \tilde{\mu}_{1,1,2} + 5 \frac{c_x c_z (\|\mathbf{c}\|^2 - 7c_y^2)}{\|\mathbf{c}\|^9} \tilde{\mu}_{2,-2,3} \\
 &\quad - 4\|\mathbf{c}\|^4 + 5c_x^2 \|\mathbf{c}\|^2 + 35c_y^2 c_z^2 \\
 &\quad + \frac{\|\mathbf{c}\|^9}{\|\mathbf{c}\|^9} \tilde{\mu}_{2,-1,3} \\
 &\quad + \sqrt{\frac{75}{4}} \frac{c_y c_z (3\|\mathbf{c}\|^2 - 7c_z^2)}{\|\mathbf{c}\|^9} \tilde{\mu}_{2,0,3} \\
 &\quad - 5 \frac{c_x c_y (\|\mathbf{c}\|^2 - 7c_z^2)}{\|\mathbf{c}\|^9} \tilde{\mu}_{2,1,3} \\
 &\quad + \frac{5}{2} \frac{c_y c_z (-2\|\mathbf{c}\|^2 - 7(c_x^2 - c_y^2))}{\|\mathbf{c}\|^9} \tilde{\mu}_{2,2,3}
 \end{aligned}$$

$$\begin{aligned}
\vec{\kappa}'_{2,-1,3} &= 0 \\
\vec{\kappa}'_{2,0,0} &= 0 \\
\vec{\kappa}'_{2,0,1} &= 0 \\
\vec{\kappa}'_{2,0,2} &= -\sqrt{\frac{1}{20}} \frac{\|\mathbf{c}\|^2 - 3c_z^2}{\|\mathbf{c}\|^5} \vec{\mu}_{0,0,1} - \sqrt{\frac{3}{4}} \frac{\|\mathbf{c}\|^2 - 3c_z^2}{\|\mathbf{c}\|^7} \vec{\mu}_{0,0,3} \\
&\quad + \sqrt{\frac{27}{20}} \frac{c_y(\|\mathbf{c}\|^2 - 5c_z^2)}{\|\mathbf{c}\|^7} \vec{\mu}_{1,-1,2} + \sqrt{\frac{27}{20}} \frac{c_z(-3\|\mathbf{c}\|^2 + 5c_z^2)}{\|\mathbf{c}\|^7} \vec{\mu}_{1,0,2} \\
&\quad + \sqrt{\frac{27}{20}} \frac{c_x(\|\mathbf{c}\|^2 - 5c_z^2)}{\|\mathbf{c}\|^7} \vec{\mu}_{1,1,2} \\
&\quad - \sqrt{\frac{75}{4}} \frac{c_x c_y (\|\mathbf{c}\|^2 - 7c_z^2)}{\|\mathbf{c}\|^9} \vec{\mu}_{2,-2,3} \\
&\quad + \sqrt{\frac{75}{4}} \frac{c_y c_z (3\|\mathbf{c}\|^2 - 7c_z^2)}{\|\mathbf{c}\|^9} \vec{\mu}_{2,-1,3} \\
&\quad + \frac{3}{4} \frac{3\|\mathbf{c}\|^4 - 30\|\mathbf{c}\|^2 c_z^2 + 35c_z^4}{\|\mathbf{c}\|^9} \vec{\mu}_{2,0,3} \\
&\quad + \sqrt{\frac{75}{4}} \frac{c_x c_z (3\|\mathbf{c}\|^2 - 7c_z^2)}{\|\mathbf{c}\|^9} \vec{\mu}_{2,1,3} \\
&\quad - \sqrt{\frac{75}{16}} \frac{(c_x^2 - c_y^2)(\|\mathbf{c}\|^2 - 7c_z^2)}{\|\mathbf{c}\|^9} \vec{\mu}_{2,2,3} \\
\vec{\kappa}'_{2,0,3} &= 0 \\
\vec{\kappa}'_{2,1,0} &= 0 \\
\vec{\kappa}'_{2,1,1} &= 0 \\
\vec{\kappa}'_{2,1,2} &= -\sqrt{\frac{3}{5}} \frac{c_x c_z}{\|\mathbf{c}\|^5} \vec{\mu}_{0,0,1} - \sqrt{15} \frac{c_x c_z}{\|\mathbf{c}\|^7} \vec{\mu}_{0,0,3} \\
&\quad + \sqrt{45} \frac{c_x c_y c_z}{\|\mathbf{c}\|^7} \vec{\mu}_{1,-1,2} + \sqrt{\frac{9}{5}} \frac{c_x(\|\mathbf{c}\|^2 - 5c_z^2)}{\|\mathbf{c}\|^7} \vec{\mu}_{1,0,2} \\
&\quad - \sqrt{\frac{9}{5}} \frac{c_z(\|\mathbf{c}\|^2 - 5c_x^2)}{\|\mathbf{c}\|^7} \vec{\mu}_{1,1,2} + 5 \frac{c_y c_z (\|\mathbf{c}\|^2 - 7c_z^2)}{\|\mathbf{c}\|^9} \vec{\mu}_{2,-2,3} \\
&\quad - 5 \frac{c_x c_y (\|\mathbf{c}\|^2 - 7c_z^2)}{\|\mathbf{c}\|^9} \vec{\mu}_{2,-1,3} \\
&\quad + \sqrt{\frac{75}{4}} \frac{c_x c_z (3\|\mathbf{c}\|^2 - 7c_z^2)}{\|\mathbf{c}\|^9} \vec{\mu}_{2,0,3} \\
&\quad + \frac{-4\|\mathbf{c}\|^4 + 5(c_y^2 \|\mathbf{c}\|^2 + 7c_x^2 c_z^2)}{\|\mathbf{c}\|^9} \vec{\mu}_{2,1,3} \\
&\quad + \frac{5}{2} \frac{c_x c_z (2\|\mathbf{c}\|^2 - 7(c_x^2 - c_y^2))}{\|\mathbf{c}\|^9} \vec{\mu}_{2,2,3} \\
\vec{\kappa}'_{2,1,3} &= 0 \\
\vec{\kappa}'_{2,2,0} &= 0 \\
\vec{\kappa}'_{2,2,1} &= 0 \\
\vec{\kappa}'_{2,2,2} &= \sqrt{\frac{3}{20}} \frac{c_x^2 - c_y^2}{\|\mathbf{c}\|^5} \vec{\mu}_{0,0,1} + \sqrt{\frac{15}{4}} \frac{c_x^2 - c_y^2}{\|\mathbf{c}\|^7} \vec{\mu}_{0,0,3} \\
&\quad + \sqrt{\frac{9}{20}} \frac{c_y(-2\|\mathbf{c}\|^2 - 5(c_x^2 - c_y^2))}{\|\mathbf{c}\|^7} \vec{\mu}_{1,-1,2} \\
&\quad + \sqrt{\frac{45}{4}} \frac{(c_x^2 - c_y^2) c_z}{\|\mathbf{c}\|^7} \vec{\mu}_{1,0,2} \\
&\quad + \sqrt{\frac{9}{20}} \frac{c_x(2\|\mathbf{c}\|^2 - 5(c_x^2 - c_y^2))}{\|\mathbf{c}\|^7} \vec{\mu}_{1,1,2} \\
&\quad + \frac{35}{2} \frac{c_x c_y (c_x^2 - c_y^2)}{\|\mathbf{c}\|^9} \vec{\mu}_{2,-2,3} \\
&\quad + \frac{5}{2} \frac{c_y c_z (-2\|\mathbf{c}\|^2 - 7(c_x^2 - c_y^2))}{\|\mathbf{c}\|^9} \vec{\mu}_{2,-1,3} \\
&\quad - \sqrt{\frac{75}{16}} \frac{(c_x^2 - c_y^2)(\|\mathbf{c}\|^2 - 7c_z^2)}{\|\mathbf{c}\|^9} \vec{\mu}_{2,0,3} \\
&\quad + \frac{5}{2} \frac{c_x c_z (2\|\mathbf{c}\|^2 - 7(c_x^2 - c_y^2))}{\|\mathbf{c}\|^9} \vec{\mu}_{2,1,3} \\
&\quad + \frac{1}{4} \frac{19\|\mathbf{c}\|^4 - 50\|\mathbf{c}\|^2 c_z^2 - 140c_x^2 c_y^2 + 35(c_z^4)}{\|\mathbf{c}\|^9} \vec{\mu}_{2,2,3} \\
\vec{\kappa}'_{2,2,3} &= 0
\end{aligned}$$

F FIRST NEAR COEFFICIENT TRANSLATION

The coefficients simplify, and the first few are:

$$\begin{aligned}
\vec{\kappa}'_{0,0,0} &= \vec{\kappa}_{0,0,0} + \|\mathbf{c}\|^2 \vec{\kappa}_{0,0,2} - \sqrt{3} c_y \vec{\kappa}_{1,-1,1} \\
&\quad - \sqrt{3} c_y \|\mathbf{c}\|^2 \vec{\kappa}_{1,-1,3} + \sqrt{3} c_z \vec{\kappa}_{1,0,1} \\
&\quad + \sqrt{3} c_z \|\mathbf{c}\|^2 \vec{\kappa}_{1,0,3} - \sqrt{3} c_x \vec{\kappa}_{1,1,1} \\
&\quad - \sqrt{3} c_x \|\mathbf{c}\|^2 \vec{\kappa}_{1,1,3} + \sqrt{15} c_x c_y \vec{\kappa}_{2,-2,2} \\
&\quad - \sqrt{15} c_y c_z \vec{\kappa}_{2,-1,2} - \sqrt{\frac{3}{4}} (\|\mathbf{c}\|^2 - 3c_z^2) \vec{\kappa}_{2,0,2} \\
&\quad - \sqrt{15} c_x c_z \vec{\kappa}_{2,1,2} + \sqrt{\frac{15}{4}} (c_x^2 - c_y^2) \vec{\kappa}_{2,2,2} \\
\vec{\kappa}'_{0,0,1} &= 0 \\
\vec{\kappa}'_{0,0,2} &= \vec{\kappa}_{0,0,2} - \sqrt{\frac{25}{3}} c_y \vec{\kappa}_{1,-1,3} \\
&\quad + \sqrt{\frac{25}{3}} c_z \vec{\kappa}_{1,0,3} - \sqrt{\frac{25}{3}} c_x \vec{\kappa}_{1,1,3} \\
\vec{\kappa}'_{0,0,3} &= 0 \\
\vec{\kappa}'_{1,-1,0} &= 0
\end{aligned}$$

$$\begin{aligned}
\vec{\kappa}'_{1,-1,1} &= -\sqrt{\frac{4}{3}} c_y \vec{\kappa}_{0,0,2} + \vec{\kappa}_{1,-1,1} \\
&\quad + (\|\mathbf{c}\|^2 + 2c_y^2) \vec{\kappa}_{1,-1,3} - 2c_y c_z \vec{\kappa}_{1,0,3} \\
&\quad + 2c_x c_y \vec{\kappa}_{1,1,3} - \sqrt{5} c_x \vec{\kappa}_{2,-2,2} \\
&\quad + \sqrt{5} c_z \vec{\kappa}_{2,-1,2} + \sqrt{\frac{5}{3}} c_y \vec{\kappa}_{2,0,2} \\
&\quad + \sqrt{5} c_y \vec{\kappa}_{2,2,2} \\
\vec{\kappa}'_{1,-1,2} &= 0 \\
\vec{\kappa}'_{1,-1,3} &= \vec{\kappa}_{1,-1,3} \\
\vec{\kappa}'_{1,0,0} &= 0 \\
\vec{\kappa}'_{1,0,1} &= \sqrt{\frac{4}{3}} c_z \vec{\kappa}_{0,0,2} - 2c_y c_z \vec{\kappa}_{1,-1,3} \\
&\quad + \vec{\kappa}_{1,0,1} + (\|\mathbf{c}\|^2 + 2c_z^2) \vec{\kappa}_{1,0,3} \\
&\quad - 2c_x c_z \vec{\kappa}_{1,1,3} - \sqrt{5} c_y \vec{\kappa}_{2,-1,2} \\
&\quad + \sqrt{\frac{20}{3}} c_z \vec{\kappa}_{2,0,2} - \sqrt{5} c_x \vec{\kappa}_{2,1,2} \\
\vec{\kappa}'_{1,0,2} &= 0 \\
\vec{\kappa}'_{1,0,3} &= \vec{\kappa}_{1,0,3} \\
\vec{\kappa}'_{1,1,0} &= 0 \\
\vec{\kappa}'_{1,1,1} &= -\sqrt{\frac{4}{3}} c_x \vec{\kappa}_{0,0,2} + 2c_x c_y \vec{\kappa}_{1,-1,3} \\
&\quad - 2c_x c_z \vec{\kappa}_{1,0,3} + \vec{\kappa}_{1,1,1} \\
&\quad + (\|\mathbf{c}\|^2 + 2c_x^2) \vec{\kappa}_{1,1,3} - \sqrt{5} c_y \vec{\kappa}_{2,-2,2} \\
&\quad + \sqrt{\frac{5}{3}} c_x \vec{\kappa}_{2,0,2} + \sqrt{5} c_z \vec{\kappa}_{2,1,2} \\
&\quad - \sqrt{5} c_x \vec{\kappa}_{2,2,2} \\
\vec{\kappa}'_{1,1,2} &= 0 \\
\vec{\kappa}'_{1,1,3} &= \vec{\kappa}_{1,1,3} \\
\vec{\kappa}'_{2,-2,0} &= 0 \\
\vec{\kappa}'_{2,-2,1} &= 0 \\
\vec{\kappa}'_{2,-2,2} &= -\sqrt{\frac{4}{3}} c_x \vec{\kappa}_{1,-1,3} - \sqrt{\frac{4}{5}} c_y \vec{\kappa}_{1,1,3} + \vec{\kappa}_{2,-2,2} \\
\vec{\kappa}'_{2,-2,3} &= 0 \\
\vec{\kappa}'_{2,-1,0} &= 0 \\
\vec{\kappa}'_{2,-1,1} &= 0 \\
\vec{\kappa}'_{2,-1,2} &= \sqrt{\frac{4}{5}} c_z \vec{\kappa}_{1,-1,3} - \sqrt{\frac{4}{5}} c_y \vec{\kappa}_{1,0,3} + \vec{\kappa}_{2,-1,2} \\
\vec{\kappa}'_{2,-1,3} &= 0 \\
\vec{\kappa}'_{2,0,0} &= 0 \\
\vec{\kappa}'_{2,0,1} &= 0 \\
\vec{\kappa}'_{2,0,2} &= \sqrt{\frac{4}{15}} c_y \vec{\kappa}_{1,-1,3} + \sqrt{\frac{16}{15}} c_z \vec{\kappa}_{1,0,3} \\
&\quad + \sqrt{\frac{4}{15}} c_x \vec{\kappa}_{1,1,3} + \vec{\kappa}_{2,0,2} \\
\vec{\kappa}'_{2,0,3} &= 0 \\
\vec{\kappa}'_{2,1,0} &= 0 \\
\vec{\kappa}'_{2,1,1} &= 0 \\
\vec{\kappa}'_{2,1,2} &= -\sqrt{\frac{4}{5}} c_x \vec{\kappa}_{1,0,3} + \sqrt{\frac{4}{5}} c_z \vec{\kappa}_{1,1,3} + \vec{\kappa}_{2,1,2} \\
\vec{\kappa}'_{2,1,3} &= 0 \\
\vec{\kappa}'_{2,2,0} &= 0 \\
\vec{\kappa}'_{2,2,1} &= 0 \\
\vec{\kappa}'_{2,2,2} &= \sqrt{\frac{4}{5}} c_y \vec{\kappa}_{1,-1,3} - \sqrt{\frac{4}{5}} c_x \vec{\kappa}_{1,1,3} + \vec{\kappa}_{2,2,2} \\
\vec{\kappa}'_{2,2,3} &= 0
\end{aligned}$$

G BUOYANCY

The vorticity induced by buoyancy could be computed directly by sampling everywhere in $\mathcal{R}^3$ the buoyancy term with vortices. But this laborious integral can be avoided, since buoyancy vorticity is concentrated near places of varying density and where the density gradient and buoyant acceleration are perpendicular. First we associate a set of vortices with a single particle j that we will generalize to multiple density particles. Let us define a local coordinate system

Fig. 12. Left: overlapping plot of the largest component of the left (red) and right (blue) side of Eq. (54). Our density fits the closed form integral with a third decimal accuracy on all 3 vector components. Right: density field ρ_j for $r_j = 1$ and $m_j = 1$.

centered at $\mathbf{x}_j$ such that $\vec{z}$ is aligned with $\vec{g} + \vec{e} - \frac{d\vec{u}}{dt}$. For a density particle or radius r_j , we expect the vortices induced by buoyancy to be concentrated around a ring $\{\mathbf{x}_\alpha, \alpha \in [2\pi]\}$ of diameter r_j , perpendicular to vector unit $\vec{z}$, with an axis $\vec{n}_\alpha$. In local coordinates:

$$\begin{aligned} \mathbf{x}_\alpha &= \frac{r_j}{2} (\cos(\alpha), \sin(\alpha), 0) \\ \vec{n}_\alpha &= (-\sin(\alpha), \cos(\alpha), 0) \end{aligned} \quad (52)$$

For a canonical density particle, we use the following density field:

$$\hat{\rho}_j = \exp\left((1 + \kappa_1 \frac{\mathbf{x} \cdot \mathbf{x}}{2r_j^2})^{-\kappa_2}\right) \quad (53)$$

We fit parameters κ_0 , κ_1 and κ_2 using the following equation:

$$\nabla \log(\hat{\rho}_j) \times \vec{z} = r_j^2 \kappa_0 \nabla \times \left(\int_\alpha \xi_i(\mathbf{p} - \mathbf{x}_\alpha)^3 \vec{n}_\alpha \times (\mathbf{p} - \mathbf{x}_\alpha) d\alpha \right) \quad (54)$$

We obtain the following parameters. The left and right side of Eq. (54) are compared in Figure 12:

$$\kappa_0 = 0.493483 \quad \kappa_1 = 0.572636 \quad \kappa_2 = 3.423340 \quad (55)$$

To generalize to multiple particles, we remap $\hat{\rho}_j$ to values in $(0, m_j)$ so that densities can be added, and obtain the field of a single particle density $\rho_j = m_j \frac{\hat{\rho}_j^{-1}}{e-1}$. Since $\frac{\nabla \rho}{\rho} \times \vec{z} = \sum \frac{m_j}{e-1} \frac{\hat{\rho}_j}{\rho} \nabla \log(\hat{\rho}_j) \times \vec{z}$, we obtain a set of weights that modulate the rotations of the vortices induced by the density particle j , that accounts for acceleration and multiple particles:

$$\vec{w}_\alpha = r_j^2 \kappa_0 \frac{\pi}{n} \|\vec{g} + \vec{e} - \frac{d\vec{u}}{dt}\| \frac{m_j}{e-1} \frac{\hat{\rho}_j(\mathbf{x}_\alpha)}{\rho(\mathbf{x}_\alpha)} \vec{n}_\alpha \quad (56)$$

H BOUNDARY VORTICITY

The vorticity near the boundary $\delta\Omega$ of a moving object is given by $\nabla \log(\rho) \times (\vec{F} - \vec{a}) + \nabla \times \vec{F}$, where $\vec{a}$ is the acceleration near the boundary. To compute this term, we consider a coordinate system where the half space $z < 0$ is inside the solid object, $z = 0$ is on the object's surface, and $z > 0$ is outside. Using the Heavyside step function H , the nearby density is formally defined as $\rho = H(-z)\rho_0 + H(z)\rho_1$, the external forces term is defined as $\vec{F} = H(-z)\vec{f} + H(z)(\vec{g} + \vec{e})$, and the acceleration is $\vec{a} = H(-z)\vec{f} + H(z)\frac{d\vec{u}}{dt}$. Since we solve for vorticity, we can replace $\vec{F}$ with any $\vec{F} + \vec{G}$ to satisfy the boundary condition as long as $\nabla \times \vec{G} = 0$ outside of solid objects. First let us consider the following $\vec{G}$:

$$\vec{G} = H(z) \left(\frac{d\vec{u}}{dt} - \vec{g} - \frac{1}{2}\vec{e} \right) - \frac{1}{2}\vec{e} \quad (57)$$

Although $\frac{d\vec{u}}{dt}$ may have curl, we discretize the object surface in regions of constant $\frac{d\vec{u}}{dt}$. To prevent the space inside objects from participating to the non-rigid fluid motion, we set $\rho_0 \rightarrow \infty$, thus enforcing $\vec{f}$ inside objects. We take the limit of the integral of the collision term near the surface, using the following limit to the Heavyside step function $H(z) = \lim_{t \rightarrow \infty} \frac{1}{2} + \frac{1}{\pi} \tan^{-1}(zt)$, and obtain the surface vorticity:

$$\begin{aligned} & \lim_{\rho_0 \rightarrow \infty} \lim_{t \rightarrow \infty} \int_{-h}^h \nabla \log(\rho) \times (\vec{F} + \vec{G} - \frac{d\vec{u}}{dt}) + \\ & \nabla \times (\vec{F} + \vec{G}) dz \\ & = (\vec{f} - \vec{e} - \vec{a}) \times \vec{n} \end{aligned} \quad (58)$$

I SOURCE FIELDS

The source integral of Eq. (26) is given by van Oosterom [2011]. We give below the formula of the source panel field S , and its derivative ∇S , for a triangle defined by $\{\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2\}$, using the symbols of Eq. (60):

$$\begin{aligned} S(\mathbf{p}) &= \frac{1}{\|\vec{n}\|} \left(\frac{\vec{n} \cdot (\vec{d}_1 \times \vec{d}_2)}{e_0} \log\left(\frac{f_0}{g_0}\right) \right. \\ &+ \frac{\vec{n} \cdot (\vec{d}_2 \times \vec{d}_0)}{e_1} \log\left(\frac{f_1}{g_1}\right) \\ &+ \frac{\vec{n} \cdot (\vec{d}_0 \times \vec{d}_1)}{e_2} \log\left(\frac{f_2}{g_2}\right) \\ &\left. - 2k_1 \text{atan2}(k_1, k_0) \right) \\ \nabla S(\mathbf{p}) &= \frac{1}{\|\vec{n}\|} \left((\vec{n} \cdot (\vec{d}_1 \times \vec{d}_2)) \vec{h}_0 + \frac{(\vec{d}_1 - \vec{d}_2) \times \vec{n}}{e_0} \log\left(\frac{f_0}{g_0}\right) \right. \\ &+ (\vec{n} \cdot (\vec{d}_2 \times \vec{d}_0)) \vec{h}_1 + \frac{(\vec{d}_2 - \vec{d}_0) \times \vec{n}}{e_1} \log\left(\frac{f_1}{g_1}\right) \\ &+ (\vec{n} \cdot (\vec{d}_0 \times \vec{d}_1)) \vec{h}_2 + \frac{(\vec{d}_0 - \vec{d}_1) \times \vec{n}}{e_2} \log\left(\frac{f_2}{g_2}\right) \\ &\left. + D(\mathbf{p})\vec{n} + k_1 \nabla D(\mathbf{p}) \right) \end{aligned} \quad (59)$$

The symbols above are defined as:

$$\begin{aligned} \vec{n} &= (\mathbf{p}_1 - \mathbf{p}_0) \times (\mathbf{p}_2 - \mathbf{p}_0) \\ \vec{d}_i &= \mathbf{p}_i - \mathbf{p} \\ l_i &= \|\vec{d}_i\| \\ k_0 &= l_0 l_1 l_2 + l_0 \vec{d}_1 \cdot \vec{d}_2 + l_1 \vec{d}_2 \cdot \vec{d}_0 + l_2 \vec{d}_0 \cdot \vec{d}_1 \\ k_1 &= (\vec{d}_2 \times \vec{d}_1) \cdot \vec{d}_0 \\ e_i &= \|\mathbf{p}_j - \mathbf{p}_k\| \\ f_i &= l_j + l_k + e_i \\ g_i &= l_j + l_k - e_i \\ \vec{h}_i &= \frac{2}{f_i g_i} \left(\frac{\vec{d}_j}{l_j} + \frac{\vec{d}_k}{l_k} \right) \\ j &= (i + 1) \% 3 \\ k &= (i + 2) \% 3 \end{aligned} \quad (60)$$

The function $\nabla S(\mathbf{p})$ is numerically unstable when $\mathbf{p}$ is aligned with the edges. This corresponds to a situation where some triangles in the decomposition of Carley [2012], shown in Figure 13, are very thin. Using this decomposition, we can eliminate the thin triangles. The source field induced at a point $\mathbf{p}$ located on the z axis by a triangle with a right angle at $\mathbf{p}_0$ lying in the plane $z = 0$ with corner $\mathbf{p}_2$ located at the origin is given by:

$$\begin{aligned} \nabla \tilde{S}(\mathbf{p})_x &= \text{arctanh}(k_1/k_2) + k_1 k_7 \\ \nabla \tilde{S}(\mathbf{p})_y &= k_6 - \log(1 + k_5) - k_7 \\ \nabla \tilde{S}(\mathbf{p})_z &= \text{atan2}(k_1(k_2 - k_0), k_0 k_1^2 + k_2) \end{aligned} \quad (61)$$

The symbols above are defined as:

$$\begin{aligned}
 k_0 &= \|\mathbf{p}\|/\|\mathbf{p}_0\| & k_4 &= \sqrt{1+k_1^2} \\
 k_1 &= \|\mathbf{p}_1 - \mathbf{p}_0\|/\|\mathbf{p}_0\| & k_5 &= \sqrt{1+k_0^2} \\
 k_2 &= \sqrt{1+k_1^2+k_0^2} & k_6 &= \log(k_0) \\
 k_3 &= (k_1^2-1)(k_0^2-1)-2 & k_7 &= (k_6 - \log(k_2+k_4))/k_4
 \end{aligned} \tag{62}$$

Fig. 13. The integral over any triangle can be decomposed into 6 right-angle triangle integrals: (a) Point $\bar{\mathbf{p}}$ is the projection of the evaluation point $\mathbf{p}$ onto the plane of the triangle. (b) The triangle is split into three signed triangles with a corner at $\bar{\mathbf{p}}$. (c) Each triangle is split into two signed right angle triangles. (d) The integral of a piece is evaluated with $\bar{\mathbf{p}}$ as the origin and an axis aligned triangle.

J DOUBLET FIELDS

The doublet integral of Eq. (26) is given by van Oosterom and Strackee [1983]. We give below the formula of the doublet panel field D , and its derivative ∇D , for a triangle defined by $\{\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2\}$, using the symbols of Eq. (64):

$$\begin{aligned}
 D(\mathbf{p}) &= -2\text{atan2}(k_1, k_0) \\
 \nabla D(\mathbf{p}) &= -2 \frac{k_0 \vec{\mathbf{n}} - k_1 \vec{\mathbf{k}}_2}{k_0^2 + k_1^2}
 \end{aligned} \tag{63}$$

The symbols above are defined as:

$$\begin{aligned}
 \vec{\mathbf{n}} &= (\mathbf{p}_1 - \mathbf{p}_0) \times (\mathbf{p}_2 - \mathbf{p}_0) \\
 \vec{\mathbf{d}}_i &= \mathbf{p}_i - \mathbf{p} \\
 l_i &= \|\vec{\mathbf{d}}_i\| \\
 k_0 &= l_0 l_1 l_2 + l_0 \vec{\mathbf{d}}_1 \cdot \vec{\mathbf{d}}_2 + l_1 \vec{\mathbf{d}}_2 \cdot \vec{\mathbf{d}}_0 + l_2 \vec{\mathbf{d}}_0 \cdot \vec{\mathbf{d}}_1 \\
 k_1 &= (\vec{\mathbf{d}}_2 \times \vec{\mathbf{d}}_1) \cdot \vec{\mathbf{d}}_0 \\
 \vec{\mathbf{k}}_2 &= -\left(\frac{l_1 l_2 + \vec{\mathbf{d}}_1 \cdot \vec{\mathbf{d}}_2}{l_0} + l_1 + l_2\right) \vec{\mathbf{d}}_0 \\
 &\quad -\left(\frac{l_0 l_2 + \vec{\mathbf{d}}_2 \cdot \vec{\mathbf{d}}_0}{l_1} + l_2 + l_0\right) \vec{\mathbf{d}}_1 \\
 &\quad -\left(\frac{l_0 l_1 + \vec{\mathbf{d}}_0 \cdot \vec{\mathbf{d}}_1}{l_2} + l_0 + l_1\right) \vec{\mathbf{d}}_2
 \end{aligned} \tag{64}$$

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