Fluid-Model-Based Car Routing for Modern Ridesharing Systems

Anton Braverman
Cornell University
ab2329@cornell.edu

Xin Liu
Arizona State University
xliu272@asu.edu

J.G. Dai
Cornell University
jim.dai@cornell.edu

Lei Ying
Arizona State University
lei.ying.2@asu.edu

ABSTRACT

This paper considers a closed queueing network model of ridesharing systems such as Didi Chuxing, Lyft, and Uber. We focus on empty-car routing, a mechanism by which we control car flow in the network to optimize system-wide utility functions, e.g. the availability of empty cars when a passenger arrives. We establish both process-level and steady-state convergence of the queueing network to a fluid limit in a large market regime where demand for rides and supply of cars tend to infinity, and use this limit to study a fluid-based optimization problem. We prove that the optimal network utility obtained from the fluid-based optimization is an upper bound on the utility in the finite car system for any routing policy, both static and dynamic, under which the closed queueing network has a stationary distribution. This upper bound is achieved asymptotically under the fluid-based optimal routing policy. Simulation results with real-word data released by Didi Chuxing demonstrate that the utility under the fluid-based optimal routing policy converges to the upper bound with a rate of $1/\sqrt{N}$.

1 INTRODUCTION

Because of the proliferation of ridesharing services, modeling and control of these systems have become important research topics over the last few years. This work focuses on empty-car routing mechanisms for controlling car flows in the network to optimize system-wide utility functions [2].

We consider a system with $r > 0$ regions and $N > 0$ cars. The regions can be interpreted as geographic regions in a city and cars drive around between regions transporting passengers. At time $t = 0$, all cars start off idling empty in some region, waiting for a passenger. Passengers arrive to region i according to a Poisson process with rate $N\lambda_i > 0$, and arrivals to different regions are independent. When a passenger arrives to region i , if there is an empty car available there, then the passenger occupies that car and travels to region j with probability P_{ij} . We allow $P_{ii} > 0$ to represent trips within a region. Travel times from region i to j have mean $1/\mu_{ij}$ and are assumed to be i.i.d. exponential random

variables, although our main results hold without this assumption. If no empty car is available, the passenger abandons the system and finds an alternative form of transportation to the destination. Once the passenger arrives at region j , the car becomes empty. The empty car can either decide to stay in region j (and wait for a new passenger) with probability Q_{jj} , or drive empty to a different region k and wait for a passenger there with probability Q_{jk} . In general, the routing matrix (also called the routing policy) $Q = (Q_{ij})$ is allowed to be *state-dependent*, i.e. Q may depend on the current distribution of cars in the system. We assume N, λ, μ , and $P = (P_{ij})$ are given, and seek to choose Q to optimize a system-wide utility function, e.g., to maximize system-wide availability of empty cars. To model these dynamics, we use a closed queueing network that belongs to the class of BCMP networks [1].

2 THE RIDESHARING OPTIMIZATION PROBLEM

In this section we introduce the fluid-based optimization and state our main results, Theorems 2.1 and 2.2. Let $E_{ij}^{(N)}(t)$ and $F_{ij}^{(N)}(t)$ be the number of empty and full cars en route from region i to region j , respectively, where $E_{ii}^{(N)}(t)$ is interpreted the number of empty cars waiting in region i for a new passenger. Furthermore, define $\tilde{E}_{ij}^{(N)}(t) = \frac{1}{N}E_{ij}^{(N)}(t)$, $\tilde{F}_{ij}^{(N)}(t) = \frac{1}{N}F_{ij}^{(N)}(t)$ and $A_i^{(N)} = \mathbb{P}(\tilde{E}_{ii}^{(N)}(\infty) > 0)$ to be the availability at region i . Let $\tilde{E}^{(N)}(t)$, $\tilde{F}^{(N)}(t)$ and $A^{(N)}$ be their corresponding matrix and vector versions.

We now consider the fluid-based optimization problem to be fully specified from (1) to (8) below. In the optimization problem, $c_i > 0$ are rewards for picking up a passenger at region i , and $\tilde{c}_{ij} > 0$ are costs of sending empty cars from i to j . The variables in the optimization problem are $q, \tilde{e}, \tilde{f}, \tilde{a}$. Let $q = (q_{ij})$ be an $r \times r$ matrix representing a static empty-car routing policy Q . Let $\tilde{e}, \tilde{f}$, and $\tilde{a}$ represent $\mathbb{E}[\tilde{E}^{(N)}(\infty)], \mathbb{E}[\tilde{F}^{(N)}(\infty)]$ and $A^{(N)}$ in the ridesharing network, respectively.

$$\max_{q, \tilde{e}, \tilde{f}, \tilde{a}} \sum_{i=1}^r c_i \lambda_i \tilde{a}_i - \sum_{i=1}^r \sum_{j=1, j \neq i}^r \tilde{c}_{ij} \tilde{e}_{ij} \quad (1)$$

$$\text{subject to } \lambda_i P_{ij} \tilde{a}_i = \mu_{ij} \tilde{f}_{ij}, \quad (2)$$

$$\mu_{ij} \tilde{e}_{ij} = q_{ij} \sum_{k=1}^r \mu_{ki} \tilde{f}_{ki}, \quad (3)$$

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$$\lambda_i \bar{a}_i = \sum_{k=1, k \neq i}^r \mu_{ki} \bar{e}_{ki} + q_{ii} \sum_{k=1}^r \mu_{ki} \bar{f}_{ki}, \quad (4)$$

$$(1 - \bar{a}_i) \bar{e}_{ii} = 0, \quad (5)$$

$$\sum_{i=1}^r \sum_{j=1}^r \bar{f}_{ij} + \sum_{i=1}^r \sum_{j=1}^r \bar{e}_{ij} = 1, \quad (6)$$

$$0 \leq \bar{e}_{ij} \leq 1, 0 \leq \bar{f}_{ij} \leq 1, 0 \leq \bar{a}_i \leq 1, \quad (7)$$

$$\sum_{j=1}^r q_{ij} = 1, q_{ij} \geq 0. \quad (8)$$

The interpretation of (1)–(8) is intuitive. Objective (1) captures the tradeoff between revenue generation, and the cost of empty-car routing. Constraints (2)–(4) are flow-balance constraints. Constraint (5) states that for region i , either $\bar{a}_i = 1$ or $\bar{e}_{ii} = 0$, i.e. either availability is at 100% or the long-run fraction of empty cars at the station is zero.

The following are our main results. Theorems 2.1 establishes the connection between the fluid-based optimization problem and $(\tilde{E}^{(N)}, \tilde{F}^{(N)})$. Theorems 2.2 shows that asymptotically, the optimal static policy from the fluid-based optimization outperforms all state dependent policies.

THEOREM 2.1. Let $q, \hat{e}, \hat{f}, \hat{a}$ be a feasible solution to the optimization problem in (1)–(8). Set $Q = q$. Assume $P_{ij} > 0$ for all $1 \leq i, j \leq r$ and $q_{ii} > 0$ for all $1 \leq i \leq r$. Then

$$\tilde{F}^{(N)}(\infty) \Rightarrow \hat{f}, \quad (9)$$

$$\tilde{E}_{ij}^{(N)}(\infty) \Rightarrow \hat{e}_{ij}, 0 \leq i \neq j \leq 1, \quad (10)$$

$$\tilde{E}_{ii}^{(N)}(\infty) \Rightarrow 0, \text{ for } i \text{ such that } \hat{a}_i < 1, \quad (11)$$

$$\sum_{i: \hat{a}_i = 1} \tilde{E}_{ii}^{(N)}(\infty) \Rightarrow \sum_{i: \hat{a}_i = 1} \hat{e}_{ii}, \quad (12)$$

and

$$\mathbb{P}(E_{ii}^{(N)}(\infty) > 0) \rightarrow \hat{a}_i, 0 \leq i \leq 1, \quad (13)$$

as $N \rightarrow \infty$.

THEOREM 2.2. (a) Suppose $(\tilde{E}^{(N)}, \tilde{F}^{(N)})$ is irreducible under P and Q , where Q is a state-dependent empty-car routing policy. Let $(q^*, \bar{e}^*, \bar{f}^*, \bar{a}^*)$ be an optimal solution of the optimization problem in (1)–(8). Then

$$\begin{aligned} & \sum_{i=1}^r c_i \lambda_i A_i^{(N)} - \sum_{i=1}^r \sum_{j=1, j \neq i}^r \tilde{c}_{ij} \mathbb{E}[\tilde{E}_{ij}^{(N)}(\infty)] \\ & \leq \sum_{i=1}^r c_i \lambda_i \bar{a}_i^* - \sum_{i=1}^r \sum_{j=1, j \neq i}^r \tilde{c}_{ij} \bar{e}_{ij}^*, N > 0. \end{aligned}$$

(b) Let $(\tilde{E}^{(N)*}, \tilde{F}^{(N)*})$ denote the CTMC under the static routing policy q^* . If $P_{ij} > 0$ for all $1 \leq i, j \leq r$ and $q_{ii}^* > 0$ for all $1 \leq i \leq r$, then

$$\begin{aligned} & \lim_{N \rightarrow \infty} \sum_{i=1}^r c_i \lambda_i A_i^{(N)*} - \sum_{i=1}^r \sum_{j=1, j \neq i}^r \tilde{c}_{ij} \mathbb{E}[\tilde{E}_{ij}^{(N)*}(\infty)] \\ & = \sum_{i=1}^r c_i \lambda_i \bar{a}_i^* - \sum_{i=1}^r \sum_{j=1, j \neq i}^r \tilde{c}_{ij} \bar{e}_{ij}^*. \end{aligned}$$

3 EVALUATION

In this section, we evaluate the convergence of the utility under the fluid-based policy to the fluid-based optimal solution, as N increases, on a nine-region network as shown in Figure 1, which was extracted from a data set released in the Di-Tech Challenge by the Didi Research Institute http://research.xiaojukeji.com/index_en.html. The nodes represent the regions, and the edge-weights are the estimated average travel times.

Figure 1: The nine-region network extracted from the DiDi dataset

We chose the utility function $U(\bar{a}) = \frac{\sum_{i=1}^r \lambda_i \bar{a}_i}{\sum_{i=1}^r \lambda_i}$, which is the probability that a passenger requesting a ride at any region is fulfilled. The utility function $U(\bar{a})$ can be retrieved by setting $c_i = \frac{1}{\sum_{i=1}^r \lambda_i}$ and $\tilde{c}_{ij} = 0$ in (1). Let Q^* be the optimal routing matrix from (2)–(8) with $U(\bar{a})$ and $a^* = U(\bar{a}^*)$ be the associated optimal system-wide availability. Let $a^{(N)} = U(A^{(N)})$ be the system-wide availability for the finite sized system with N cars under the optimal routing policy Q^* . Figure 2 shows that the convergence appears to be happening at a rate of $1/\sqrt{N}$.

Figure 2: Convergence of the system-wide availability to its fluid-based optimal availability in the nine-region network.

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