Projection and Quantifier Elimination Using Non-uniform Cylindrical Algebraic Decomposition

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ABSTRACT

Cylindrical Algebraic Decomposition (CAD) is an established tool in the computer algebra community for computing with semi-algebraic sets / Tarski formulas. The key property of CAD is that it provides a representation in which geometric projection and set complement (the analogues of the logical operations of quantifier elimination and negation for Tarski formulas) are trivial. However, constructing a CAD often requires an impractical amount of time and space. Non-uniform CAD (NuCAD) was introduced with the goal of providing a more practically efficient alternative to CAD for computing with semi-algebraic sets / Tarski formulas. As a first step towards that goal, previous work has shown that *Open* NuCADs do provide a much more efficient representation than *Open* CADs. However, it hasn't been shown that the key operation of projection can be computed efficiently in the NuCAD representation, because while set complement is trivial for NuCADs, as it is for CADs, projection, in contrast to the CAD case, is not. This paper provides another step towards the larger goal by showing how projection can be done efficiently in the *Open* NuCAD representation.

The importance of this contribution is not restricted to *Open* NuCADs, since the same approach to projection will carry over to the general case for NuCADs where, we hope, the practical benefits of the much smaller representation NuCAD provides will be even greater.

CCS CONCEPTS

• **Mathematics of computing** → *Solvers*;

KEYWORDS

cylindrical algebraic decomposition; polynomial inequalities; real constraint solving

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1 INTRODUCTION

Tarski formulas and semi-algebraic sets are fundamental objects of study in computer algebra.¹ Cylindrical Algebraic Decomposition (CAD) is an established tool for computing with semi-algebraic sets and Tarski formulas. CAD provides an *explicit* representation of semi-algebraic sets that can be constructed from Tarski formulas, which are *implicit* representations. The CAD representation supports many operations, but the most important are projection and set complement, which allow CAD to be used for quantifier elimination. Specifically, if F is a quantifier-free formula in variables $x_1, \dots, x_n$, then a CAD representation of $Q_{r+1}x_{r+1} \cdots Q_n x_n F$, where the Q_i are $\forall$ or $\exists$, can be constructed by first constructing a CAD of $\mathbb{R}^n$ representing the set defined by F , and second performing a combination of projections and set complements to arrive at a CAD of $\mathbb{R}^r$ representing the set defined by the quantified formula.

It is well-known that constructing a CAD from a Tarski formula in n variables has doubly-exponential worst-case running time and space [2]. In fact, even just the number of 1-level cells constructed is, in the worst case, doubly exponential in the number of variables. Moreover, practical experience has shown that the time and space inefficiencies of CAD often make even moderate sized problems impractical. Non-uniform CAD (NuCAD) was introduced with the goal of providing a more efficient alternative to CAD [1], using ideas from Jovanovic & de Moura's NLSat [3] (see [6] for an alternative approach to trying to leverage those ideas in CAD construction). Ultimately, meeting that goal requires not only that a NuCAD constructed from a given quantifier-free formula is more efficient than a CAD constructed from that formula, but also that NuCAD efficiently supports those crucial operations that CAD supports: most crucially, projection and set-complement. Since set-complement is trivial in both representations (simply swap labels on cells), the key issue is how to do projection efficiently for NuCADs, and that is what this paper addresses.

An "Open" CAD is a weak form of CAD that consists only of full-dimensional cells. It's not a true decomposition of $\mathbb{R}^n$, because there are points that are not in any cells. However, such points comprise a lower-dimensional subset of $\mathbb{R}^n$. Open CAD has appeared in, for example, [4, 5] where it is noted that projection and lifting are conceptually simpler, and faster than for the general case. For certain problems, e.g. satisfiability of strict inequalities and determining semi-definiteness, Open CAD suffices. What [1] actually defines is *Open* NuCAD, and an algorithm for constructing them from Tarski formulas. In this paper, we continue to work with Open

¹ A Tarski formula is a boolean combination of sign conditions on polynomials, possibly with quantifiers. A semi-algebraic set is a subset of $\mathbb{R}^n$ for which there is a quantifier-free Tarski formula F in $x_1, \dots, x_n$ such that $\alpha \in \mathbb{R}^n$ is in the set precisely when $F(\alpha)$ is true.

Figure 1: This figure shows the correspondence between NuCAD as a hierarchical decomposition of $\mathbb{R}^n$ and NuCAD as a data structure. Note that leaf cells have T/F (TRUE/FALSE) labels. The semi-algebraic set defined by a NuCAD is the union of the TRUE cells. Relative to its parent, each cell is labeled with either X , kL or kU . Globally, each cell is uniquely labeled by the concatenation of these relative labels along the path from the root node to the cell. Thus the highlighted cell has label $C2U2LXX$.

NuCAD because the initial development is simpler, and if there are efficiency gains in the “open” case, the gains to be had in the general case are likely to be still greater. The purpose of this paper is, in the context of *Open* CAD and NuCAD, to: 1) introduce an algorithm for computing projections of semi-algebraic sets in the NuCAD representation, 2) present experimental results demonstrating that the time required to compute the projection of a semi-algebraic set S from its NuCAD representation is small relative to the time required to construct the NuCAD for S in the first place, 3) present experimental results demonstrating that the number of cells in the result of computing the projection of semi-algebraic set S in the NuCAD representation is small compared to the number of cells in the result of computing the projection of S using partial CAD, and 4) present experimental results demonstrating that as problems get larger there are projections that can be computed using NuCAD that cannot be computed (in practice) with CAD.

2 BACKGROUND

We assume the reader is already familiar with the usual CAD notions — like delineability, level of a polynomial, etc. Note that $\pi_k(\cdot)$ denotes projection down onto $\mathbb{R}^k$. This paper deals with open cylindrical cells which, except in the trivial case of a single cell, cannot truly decompose $\mathbb{R}^n$. Instead, we say that a set of open regions defines a *weak decomposition* of $\mathbb{R}^n$ if the regions are pairwise disjoint, and the union of their closures contains $\mathbb{R}^n$. Open CADs and Open NuCADs are weak decompositions of real space. By labeling cells as TRUE or FALSE, these decompositions provide representations of semi-algebraic sets that are correct up to lower-dimensional components. Specifically, all cells are correctly marked as completely in or out of the semi-algebraic set, but the boundaries between cells (which are of lower dimension) are not part of the decomposition, so whether we consider boundaries to be in the set or out, we may be miscategorizing some points in lower-dimensional components.

In this paper we would like to represent the projection of a semi-algebraic set as an Open CAD or NuCAD. To this end we define the notion of a *weak projection*, and we will show in this paper how to construct a NuCAD of r -space that is a weak projection of the semi-algebraic set defined by a NuCAD of n -space.

Definition 2.1. If S is an open semi-algebraic subset of $\mathbb{R}^n$, and $r \leq n$ a positive integer, we say that an open set $W \subseteq \mathbb{R}^r$ is a *weak projection* of S onto $\mathbb{R}^r$ if $W \subseteq \pi_r(S)$ and there are no open subsets of $\pi_r(S) - W$.

It’s worth noting that in this paper, the difference between the true projection and the weak projection onto r -space we compute using NuCADs will be a subset of some variety defined by a non-zero polynomial in r variables.

Here we recall the basic definitions of Open NuCAD (see [1]), which are also illustrated in Figure 1.

Definition 2.2. An *Open Cylindrical Cell* is a subset of $\mathbb{R}^n$ of the form $\{(\alpha_1, \dots, \alpha_n) \in B \times \mathbb{R} \mid f(\alpha_1, \dots, \alpha_{n-1}) < \alpha_n < g(\alpha_1, \dots, \alpha_{n-1})\}$ or $\{(\alpha_1, \dots, \alpha_n) \in B \times \mathbb{R} \mid f(\alpha_1, \dots, \alpha_{n-1}) < \alpha_n\}$ or $\{(\alpha_1, \dots, \alpha_n) \in B \times \mathbb{R} \mid \alpha_n < g(\alpha_1, \dots, \alpha_{n-1})\}$ where B is an open cylindrical cell in $\mathbb{R}^{n-1}$ and the graphs of f and g over B are disjoint sections of polynomials, and $()$ is considered an open cylindrical cell in $\mathbb{R}^0$.

As a data structure, an Open NuCAD Cell represents a cylindrical cell as an array C of bounds descriptions: so $C[i]$ is a description of the cell’s upper and lower bounds for x_i . Additionally, a cell data structure stores a *sample point* from inside the cell, and potentially carries a *truth label*, a *label* describing its position within a NuCAD data structure, and a list of child cells within a NuCAD data structure.

Definition 2.3. An *Open Non-uniform Cylindrical Algebraic Decomposition* (Open NuCAD) of $\mathbb{R}^n$ is a collection C of open cylindrical cells, each of which is labeled with a unique string of the form $C(X|[0-9] + (L|U))^*$. The relation $E = \{(C_1, C_2) \mid C_1, C_2 \text{ are cells with labels } lab_1 \text{ and } lab_2 \text{ satisfying } lab_2 = lab_1(X|[0-9] + (L|U))\}$ defines a graph on the cells such that

- (1) the graph (C, E) is a tree, rooted at the cell consisting of all of $\mathbb{R}^n$, which is labeled C ,
- (2) the children of cell C_0 with label lab_0 have labels from taken from the set $\{lab_01L, \dots, lab_0nL, lab_01U, \dots, lab_0nU, lab_0X\}$ and if C_0 has children, then one of them is labeled lab_0X ,
- (3) if cell C_2 is the child of C_1 with label lab_1X , then $C_2 \subset C_1$ and for each $i \in \{1, \dots, n\}$, in the cylinder over $\pi_{i-1}(C_2)$ the

section that defines the lower (resp. upper) boundary of C_2 in x_i is either identical to or disjoint from the section that defines the lower (resp. upper) boundary of C_1 in x_i

- (4) if cell C_X is the child of C_0 with label $lab_0 X$, then for all $i \in \{1, \dots, n\}$

$$(\pi_{i-1}(C_X) \times \mathbb{R}) \cap \pi_i(C_0) - \pi_i(C_X) \quad (1)$$

consists of zero, one or two open cells: the region with x_i -coordinates below $\pi_i(C_X)$ if it is non-empty, which is denoted B_L , and the region with x_i -coordinates above $\pi_i(C_X)$ if it is non-empty, which is denoted B_U . There is a cell with label $lab_0 iL$ if and only if B_L is non-empty and, if it exists, that cell is $(B_L \times \mathbb{R}^{n-i}) \cap C_0$. There is a cell with label $lab_0 iU$ if and only if B_U is non-empty and, if it exists, it is $(B_U \times \mathbb{R}^{n-i}) \cap C_0$.

We note that in point 3 of the definition, the requirement that C_2 is a *strict* subset of C_1 is a change from the definition given in [1]. This means that a cell is either a leaf, or it has multiple children. There are several operations we will assume are defined for an Open Cylindrical Cell C within a NuCAD data structure:

- `getSamplePoint()` : returns the sample point of C .
- `getChild( $\omega$ )` : returns the child with label ω relative to C , or `FAIL` if no child has that label.
- `getXLLevel()` : returns the *split-level* (see Definition 3.1) of the X -child of C , or 0 if C is not a leaf. Note that this is the maximum of the split-levels of C 's children, and note that there must be an L -child or U -child with that split-level, in addition to the X -child.
- `locateLeaf( $\alpha$ )` : given point $\alpha \in \pi_i(C)$, returns a leaf cell C' from the subtree rooted at C such that $\alpha \times \mathbb{R}^{n-i} \cap C' \neq \emptyset$. If α lies on a boundary separating cells in the subtree rooted at C , “`FAIL`” plus some extra information is returned.
- `moveSamplePoint( $\alpha$ )` : given point $\alpha \in \pi_i(C)$, moves the sample point of C to lie on $\alpha \times \mathbb{R}^{n-i}$.

We do not define `locateLeaf` and `moveSamplePoint` here, due to space considerations, but we note that they just involve a number of univariate real-root isolations for polynomials with rational coefficients. Additionally, there are two algorithms related to NuCAD construction that we need.

- $E \leftarrow \text{OCMerge}(C_1, C_2, k)$: given two cells whose sample points agree in the first k coordinates, construct E , a subcell of C_1 such that $\pi_k(E) \subseteq \pi_k(C_2)$. There is not space here to describe this algorithm, though we remark that if Q is the set of level 1 through k boundary polynomials of C_2 , the OC-Merge-Set algorithm, described in [1], applied to C_1 and Q suffices to do this.
- $Q \leftarrow \text{SplitByCell}(C_1, C_2, k)$: given leaf cell C_1 and cell $C_2 \subseteq C_1$, splits C_1 into subcells that become its children, such that what becomes the X -child of C_1 is $C_1 \cap (\pi_k(C_2) \times \mathbb{R}^{n-k})$. A queue of all the children added other than the X -child is returned as “ Q ”. There is not enough space to define this algorithm, though we note that lines 15–35 of the algorithm “Split” from [1] provides one.

3 TREE-BASED PROJECTION

The notion of *level* is fundamental to descriptions of CAD. The CAD data structure consists of a tree in which each vertex is associated with a (cylindrical) cell, and the *level* of a cell is its depth in that tree. In that data structure, a k -level cell is a subset of $\mathbb{R}^k$, and

Figure 2: Projection based on the NuCAD tree.

the requirement that cells be cylindrically arranged means that the projection of a k -level cell onto $\mathbb{R}^j$, where $j < k$, is the cell's j -level ancestor in the tree. In a NuCAD, by contrast, each cell in the NuCAD tree is a subset of $\mathbb{R}^n$, so the notion of level from CAD does not carry over. We next provide a useful definition of *level* both for an individual cell, and for a subtree in the NuCAD tree.

Definition 3.1. Let C be a cell in D , an Open NuCAD of $\mathbb{R}^n$. If C' is a cell in the subtree rooted at C , we say that the *level within C of C'* is i if C' is exactly $(S \times \mathbb{R}^{n-i}) \cap C$ for some $S \subseteq \mathbb{R}^i$. By extension, we say that the *level of the subtree rooted at C* is i if all cells in the subtree (excluding C itself) have level at most i , and at least one has level exactly equal to i , or 0 if C is a leaf. Finally, the *split-level* of cell C is its level within its parent cell, or 0 for the root node.

Projection onto a lower dimensional space of a semi-algebraic set represented by a CAD is trivial (given that you already have a CAD!) because it can be done solely based on the CAD tree structure — no *algebraic* computation required. While this is not the case for NuCADs, there are some projections that can be computed this way, i.e. solely with non-algebraic operations on the NuCAD tree. We refer to this as “easy projection”, and describe it in this section.

The fundamental idea behind “easy projection” can be seen illustrated in Figure 2. In the top NuCAD tree, we see that cell C_{2U} has two children of split-level 2: a $2L$ -child and an X -child. The structure of NuCADs is such that these two sibling cells must have the same projection onto $\mathbb{R}^1$. Since the x -child has only `FALSE` cells in its subtree (in fact, it is the only cell in its subtree, being a leaf), we could “grow” the cells from the $2L$ -child upwards to cover the

entire region of the X -child, thereby eliminating the X -child, and the projection of the subtree rooted at $C2U$ would be unaltered. Figure 2 shows this change in the transition from the top NuCAD to the middle NuCAD. Moreover, it shows that this change in the NuCAD decomposition is effected by a simple operation on the NuCAD tree. We call this a “GrowUp” operation.

The transition from the middle NuCAD to the bottom in Figure 2 illustrates the same idea. Cell C has three children of splitlevel 2: $2L$, X and $2U$. The subtrees rooted at the $2L$ and X children of C contain only FALSE cells, so we can “grow” the $2U$ -child down to cover both the $2L$ and X children, and the projection onto 1-space of the NuCAD still defines the same set. Moreover, once again, this change is effected by a simple operation on the NuCAD tree.

These ideas lead to three algorithms, GrowUp, GrowDown and Grow, that expand one cell and its subtree to “absorb” its one or more of its siblings.

- $\text{GrowUp}(C, f)$: grow f -child of C to absorb the sibling directly “above” it.
- $\text{GrowDown}(C, f)$: same as GrowUp , but in the other direction.
- $\text{Grow}(C, f)$: grow f -child of C to absorb *both* sibling cells of its split-level.

We give an explicit description of GrowUp as Algorithm 1. The others follow the same ideas. The following theorem regarding GrowUp is stated, though its proof is left out for space considerations.

THEOREM 3.2. *Let C be a cell, C_X its X -child, and r the split-level of C_X . If $\text{GrowUp}(C, L)$ is called, and either C_X has every cell marked FALSE or the rL -child (call it C_L) has every cell within it marked TRUE, then for any $k < r$, the projection onto $\mathbb{R}^k$ of C prior to the call of GrowUp is the same as it is after the call. Similarly, If $\text{GrowUp}(C, X)$ is called, and either C_X has every cell marked TRUE or the rU -child (call it C_U) has every cell within it marked FALSE, then for any $k < r$, the projection onto $\mathbb{R}^k$ of C prior to the call of GrowUp is the same as it is after the call.*

Finally, we define Algorithm 2, EasyProjection, which projects cells based only on the structure of the NuCAD graph. It is incomplete, in the sense that it is not necessarily able to project away all cells of level greater than the target level k . A precise characterization of when and why this happens appears later.

THEOREM 3.3. *Algorithm 2 meets its specifications.*

PROOF. We give a sketch of the key points of this proof here. First, note that the “FAIL” case (lines 1,2) is there to avoid having to constantly check whether cell C has a U or an L child for a particular level. A quick look at how it is used within the algorithm should show you that such results are essentially ignored. Also note that when C is a leaf, the value $(0, t)$ returned in C is what the specification requires. This will allow us to proceed by induction on the number of nodes in the subtree rooted at C , since we have taken care of the base case here.

We next consider the case in which the initial value of r from line 6 is less than or equal to k . In this case, by induction, r_X is the level of the subtree rooted at C , and t_X is TRUE (resp. FALSE) exactly when all cells in that subtree are TRUE (resp. FALSE), and UNDET otherwise. We will skip over the while-loop, and after lines 26–32 we will have $r_{\max} = r_X$ and $t = t_X$. Lines 33–38 will call

Algorithm 1: $\text{GrowUp}(C, f)$

In: cell C and $f \in \{L, X\}$. If C is a non-leaf, let l_C be the label of C , and r the split-level of cell $l_C X$. For $f = X$, the level of the subtree rooted at cell $l_C X$ must be less than r , and for $f = L$, the level of the subtree rooted at $l_C rL$ must be less than r .
Eff: C is modified. If $f = L$, then cells C_L and C_X are replaced with a single cell covering $C_L \cup C_X$. If $f = X$, then cells C_X and C_U are replaced with a single cell covering $C_X \cup C_U$.

```

1 if  $C$  is a leaf then
2   return
3  $C_X \leftarrow C.\text{getChild}(X)$ 
4  $r \leftarrow C.\text{getXLevel}()$ 
5 if  $f = X$  then
6    $C_U \leftarrow C.\text{getChild}(rU)$ 
7   foreach cell  $c$  in the subtree rooted at  $C_X$  do
8      $\lfloor$  set  $c$ 's  $r$ -level upper bound to that of  $C_U$ 
9   remove cell  $C_U$  from the list of children of  $C$ 
10 else
11    $C_L \leftarrow C.\text{getChild}(rL)$ 
12   foreach cell  $c$  in the subtree rooted at  $C_L$  do
13      $\lfloor$  set  $c$ 's  $r$ -level upper bound to that of  $C_X$ 
14   remove cell  $C_X$  from the list of children of  $C$ 
15   set the label of  $C_L$  to  $l_C X$ , where  $l_C$  is the label of  $C$ 
16 if  $C$  has only one child then
17   replace the child list of  $C$  with the child list of its unique child

```

EasyProjection recursively on the other children of C , updating $r_{\max}$ on line 36 if a higher-level descendant cell is encountered, and setting $t = \text{UNDET}$ if anything other than the value of t_X is encountered. Thus, on line 39, the returned result $(r_{\max}, t)$ meets the specification.

Finally, we consider the general situation, where the initial split-level of C_X is greater than k . Thus we actually enter the first while loop. We exit that loop in one of two distinct cases. The first case is $r = r_{\text{prev}}$ (which implies $r > k$). This happens when, after a pass through the loop body, the split-level of the X -child has not been reduced, which means there is an L or U child of that same split-level. What we require is that $r_{\max}$ is the maximum of the split-level of these cells and any cells in their subtrees, and t is TRUE (resp. FALSE) if all the leaves of the subtrees rooted at these cells are TRUE (resp. FALSE), and UNDET otherwise. If we have left the loop this way, the only children of C on which EasyProjection has not been called have level less than r . The remaining lines, 31–39, simply call EasyProjection on those remaining children, updating $r_{\max}$ and t appropriately.

The second case is when $r \neq r_{\text{prev}}$ but $r \leq k$. In this case, we only need $r_{\max}$ to be the maximum of the split-level of C_X and all the cells in its subtree, and t to be TRUE (resp. FALSE) if all the leaves of only the subtree rooted at C_X are TRUE (resp. FALSE), and UNDET otherwise. In this case, the remaining lines of the algorithm (i.e.

Algorithm 2: EasyProjection(C, k)

In: C, k , where C is a cell (or FAIL) and k is a level
Out: (r, t) where r is the level of the subtree rooted at C , and t is TRUE (resp. FALSE) if all leaf nodes of the subtree rooted at C are TRUE (resp. FALSE) and UNDET otherwise. If C is FAIL, $(-1, \text{FAIL})$ is returned.
Note: if t returned is TRUE or FALSE, $r \leq k$.
Eff: the subtree of C may be changed, but if S_{pre} is the projection onto $\mathbb{R}^k$ of the TRUE cells in the original C , and S_{post} the projection after, we have $S_{post} = S_{pre} - Z$, where Z is a subset of the variety defined by a non-zero polynomial in $x_1, \dots, x_k$.

```

1 if  $C = \text{FAIL}$  then
2   return  $(-1, \text{FAIL})$ 
3 if  $C$  is a leaf then
4   return  $(0, t)$ , where  $t$  is the truth label of  $C$ 
5  $C_X \leftarrow C.\text{getChild}(X)$ 
6  $r \leftarrow C.\text{getXLevel}()$ 
7  $(r_X, t_X) \leftarrow \text{EasyProjection}(C_X, k)$ 
8  $r_{prev} \leftarrow r + 1$ ;  $r_{max} \leftarrow \max\{r, r_X\}$ 
9 while  $r \neq r_{prev}$  and  $r > k$  do
10    $C_L \leftarrow C.\text{getChild}(r_L)$ 
11    $C_U \leftarrow C.\text{getChild}(r_U)$ 
12    $(r_L, t_L) \leftarrow \text{EasyProjection}(C_L, k)$ 
13    $(r_U, t_U) \leftarrow \text{EasyProjection}(C_U, k)$ 
14   if  $t_i = \text{TRUE}$  for some  $i \in \{L, X, U\}$  then
15      $\text{Grow}(C, i)$ 
16      $(r_L, t_L), (r_X, t_X), (r_U, t_U) \leftarrow (r_i, t_i)$ 
17   else
18     if  $t_L = \text{FALSE}$  then
19        $\text{GrowDown}(C, X)$ ;  $(r_L, t_L) \leftarrow (r_X, t_X)$ 
20     if  $t_U = \text{FALSE}$  then
21        $\text{GrowUp}(C, X)$ ;  $(r_U, t_U) \leftarrow (r_X, t_X)$ 
22     if  $t_X = \text{FALSE}$  and  $t_U = \text{UNDET}$  then
23        $\text{GrowDown}(C, U)$ ;  $(r_X, t_X) \leftarrow (r_U, t_U)$ 
24     else if  $t_X = \text{FALSE}$  and  $t_L = \text{UNDET}$  then
25        $\text{GrowUp}(C, L)$ ;  $(r_X, t_X) \leftarrow (r_L, t_L)$ 
26    $r_{max} \leftarrow \max\{r, r_L, r_X, r_U\}$ 
27    $t \leftarrow t_L$  if  $t_L = t_X = t_U$ , UNDET otherwise
28    $r_{prev} \leftarrow r$ 
29    $C_X \leftarrow C.\text{getChild}(X)$ 
30    $r \leftarrow \text{split-level of } C_X \text{ or } 0 \text{ if } C_X = \text{FAIL}$ 
31 if  $r = r_{prev}$  then
32    $r \leftarrow r - 1$ 
33 while  $r > 0$  do
34    $(r_L, t_L) \leftarrow \text{EasyProjection}(C.\text{getChild}(r_L), k)$ 
35    $(r_U, t_U) \leftarrow \text{EasyProjection}(C.\text{getChild}(r_U), k)$ 
36    $r_{max} \leftarrow \max\{r_{max}, r_L, r_U\}$ 
37    $t \leftarrow t_L$  if  $t_L = t = t_U$ , UNDET otherwise
38    $r \leftarrow r - 1$ 
39 return  $(r_{max}, t)$ 

```

31–39) operate just as in the case in which the first while-loop was never entered.

At this point, the full proof of correctness would require verifying that the first while-loop exits satisfying the conditions required of these two cases. Due to space constraints, we do not include that argument. Conceptually it is not difficult, but its presentation is complicated by the proliferation of cases in which a U or L child of a given level may or may not exist. $\square$

EasyProjection(C, k) may not completely project C onto level k , meaning that there may be cells in the tree with split-level greater than k . The following theorem characterizes the NuCAD tree after such a call to Easy Project.

THEOREM 3.4. *If EasyProjection(C, k) returns (r, t) where $r > k$, then there exists a cell C' in the subtree rooted at C (after the call to EasyProjection) with two distinct child cells of level $C'.\text{getXLevel}()$ that both have subtrees that a) are of level less than or equal to k , and b) contain both TRUE and FALSE cells.*

PROOF. We proceed by induction on the number of cells in the subtree rooted at C .

Case 1: after the call, the subtrees of all children of C have level at most k , but C has children of split-level greater than k . Let C_X be the X -child of C and, C_L its sibling of the same level (for simplicity assume C_X has no U -sibling). If either C_X or C_L has all TRUE or all FALSE cells, then EasyProjection(C, k) would have merged them in lines 14–25. So C_X and C_L must contain both TRUE and FALSE cells, and the theorem holds taking C itself as C' . This is the base case of our induction.

Case 2: after the call, there is a child of C with a subtree of level greater than k . In this case, the subtree rooted at that child must contain both TRUE and FALSE cells. So there must have been a recursive call, EasyProjection(C^*, k) that returned (r^*, t^*) where $r^* > k$. In this case, induction says that the subtree rooted at C^* has a cell C' with two distinct child cells of level $C'.\text{getXLevel}()$ that both have subtrees that are of level less than or equal to k , and contain both TRUE and FALSE cells. Because C^* is in the subtree rooted at C , so is the cell C' , and the theorem is proved for this case.

Note that the induction is valid because when Case 2 makes use of induction, the tree C^* has fewer nodes. Thus, the process eventually terminates in Case 1. $\square$

Figure 3 provides an illustration of Theorem 3.4.

4 COMPLETE PROJECTION

Given NuCAD C and level k on which to project, EasyProjection(C, k) may fail to reduce the split-levels of all cells in C to k or below. If so, Theorem 3.4 tells us that there is a cell C such that two children of C with maximum split-level both have subtrees that contain a mix of TRUE and FALSE cells, and the split-level of these two child cells is greater than k , but the levels of the subtrees rooted at these two cells are less than or equal to k . So, to develop a complete projection algorithm, we need to be able to resolve exactly this situation. Algorithm 3, ProjectionResolve, does this. It takes two cells of split-level greater than k , both of which have subtrees of level at most k , and both of which have identical projections onto k -space. In our context, both cells will have the same split-level, though the algorithm is phrased more generally. It then refines the

Figure 3: On the left is the NuCAD produced from the formula $x^2 + y^2 > 1 \wedge y(y - 3x) > 0 \wedge y < \frac{3}{2} \wedge y > -\frac{3}{2} \wedge x < \frac{7}{8} \wedge x > -\frac{7}{8}$. On the right is the NuCAD resulting from calling EasyProjection with $k = 1$, and beneath it is the tree for that NuCAD. Note EasyProjection did not succeed in eliminating all cells of split-level greater than k . The cell C' guaranteed by Theorem 3.4 is cell C1U1L and, as the two highlighted child cells have subtrees that, as promised, are of level less than or equal to k , and contain both true and false cells.

subtree rooted at the first cell so that projection of its TRUE cells is (modulo a missing lower-dimensional component) equal to the union of the projections of the TRUE cells of both inputs. The second cell can afterwards be truncated to a leaf with truth value FALSE, and the projection onto k -space of all TRUE cells in the NuCAD will remain unchanged (modulo that missing lower-dimensional component).

THEOREM 4.1. *Algorithm 3 meets its specification.*

PROOF. We provide a proof of this theorem without covering cases in which locateLeaf or OCMerge return a FAIL result. Those FAIL results occur because we are dealing only with “open” NuCAD, which means situations in which sample points fall on the zero sets of certain polynomials have to be addressed by perturbing those sample points to move them off those zero sets. In [1], this is handled completely and rigorously proved correct. In this paper we handle it incompletely in the algorithm and ignore it in the correctness proof because we don’t have the space to describe the perturbation completely, or cover its correctness. However, the perturbation

process and the arguments of its correctness are essentially the same here as in [1].

Let S be the union of the cells marked TRUE in T_1 and T_2 , the subtrees rooted at C_1 and C_2 respectively, prior to the start of the algorithm. The essential invariant of this process is that, at the point of evaluating the continuation condition at line 2, the union of the projections onto k space of the cells in Q and the cells marked TRUE in the subtree rooted at C_1 is a *weak projection* of S . In particular, it is equal to $S - Z$, where Z is a subset of a variety defined by a non-zero polynomial in $x_1, \dots, x_k$.

The proposed invariant condition is certainly true when we encounter the loop for the first time, as C_1 hasn’t changed, and Q contains exactly the TRUE cells from T_2 , the subtree rooted at C_2 at the beginning of the algorithm. So we will show that if the condition is true at the beginning of a loop iteration, it is true at the end of that iteration.

Line 3 removes c from Q , and line 4 sets α to the projection of c ’s sample point onto r -space. Line 5 locates a leaf node in the subtree rooted at C_1 that contains a point in $\alpha \times \mathbb{R}^{n-r}$. As explained above, we will assume here that this step is successful, and move to Step 11. At this point, $\pi_r(c) \cap \pi_r(d)$ is an open subset containing α . OC-Merge is called on line 12 to compute e , a cylindrical cell in r -space that contains α and is contained within $\pi_r(c) \cap \pi_r(d)$. Once again, as explained above, we will assume that this operation does not fail.

At line 17 we make the call SplitByCell(c, e, k). This will refine cell c (i.e. it will no longer be a leaf node in C_2) so that its X -child’s projection onto r -space is exactly e . The other child cells are enqueued on Q so that the net change to the union of the cells on Q is that one open set has been removed (corresponding to the X -child) and some points appearing on the boundaries may have been lost (but these are a subset of the variety defined by the product of all the polynomials defining boundaries).

If d is marked TRUE, we are actually done, because the open set that was lost (the X -child) had a projection (cell e) that was a subset of the projection of d (which is a TRUE cell in C_1) anyway. If d is marked FALSE, then the algorithm splits d by cell e . The new X -child is marked TRUE, so e is picked up in the projection of C_1 , and since the other new children of d are marked FALSE, the projection of the leaves of C_1 is otherwise unchanged. Thus the condition holds. $\square$

Finally, we describe Algorithm 4, OpenNuCADProject. The correctness of OpenNuCADProject follows from previous theorems quite easily. Essentially, the only important observation is that, because ProjectionResolve(C_1, C_2, k) sets all leaves in the subtree rooted at C_2 as FALSE, the following call to EasyProjection is guaranteed to merge two cells, and thus reduce the total number of cells in C of split-level greater than k .

5 EXPERIMENTAL RESULTS

This section describes the results of experiments² performed with a prototype implementation of Open NuCAD construction and the projection algorithm described here. These are available in Tarski, a system for computing with Tarski formulas and semi-algebraic

²All experiments were performed on an Intel(R) Core(TM) i7-3770 CPU 3.40GHz with 8GB RAM.

Algorithm 3: ProjectionResolve(C_1, C_2, k)

In: i -level cell C_1 from labeled NuCAD D , j -level cell C_2 from D and level $k \leq i, j$ such that *a*) the levels of the subtree rooted at C_1 (call it T_1) and the subtree rooted at C_2 (call it T_2) are at most k , and *b*) the k -level projections of C_1 and C_2 are identical.

Eff: The subtree of C_1 is replaced with T'_1 where, letting S_1 (resp. S_2 and S'_1) be the projection onto k -space of all TRUE leaf node in T_1 (resp. T_2 and T'_1), we have

$$S'_1 = (S_1 \cup S_2) - Z$$

where Z is a subset of the variety defined by a non-zero polynomial in $x_1, \dots, x_k$. Also, T_2 is replaced with T'_2 , in which all leaves are FALSE.

```

1   $Q \leftarrow$  new queue containing all TRUE leaf cells in  $T_2$ 
2  while  $Q$  not empty do
3       $c \leftarrow Q.dequeue()$ ; mark  $c$  as FALSE
4       $\alpha \leftarrow \pi_k(c.getSamplePoint())$ 
5       $d \leftarrow C_1.locateLeaf(\alpha)$ 
6      while  $d = (FAIL, d_f, k_f, dir_f)$  for some  $d_f, k_f, dir_f$  do
7           $p \leftarrow$  polynomial defining the  $dir_f$ -boundary of cell  $d_f$ 
              at level  $k_f$ 
8          perturb sample point of  $c$  based on  $p$ 
9           $\alpha \leftarrow \pi_k(c.getSamplePoint())$ 
10          $d \leftarrow C_1.locateLeaf(\alpha)$ 
11     moveSamplePoint( $d, \alpha, k$ )
12      $e \leftarrow OCMerge(c, d, k)$ 
13     if  $e = (FAIL, p_f, k_f)$  for some  $p_f, k_f$  then
14         perturb sample point of  $c$  based on  $p$ 
15          $Q.enqueue(c)$ 
16         continue
17      $Q' \leftarrow SplitByCell(c, e, k)$ 
18     mark each node in  $Q'$  TRUE, and enqueue on  $Q$  if  $d$ 
        marked FALSE then
19          $Q'' \leftarrow SplitByCell(d, e, k)$ 
20         mark  $d$ 's  $X$ -child TRUE & nodes in  $Q''$  FALSE

```

Algorithm 4: OpenNuCADProject(C, k)

In: C , the root cell of an Open NuCAD of $\mathbb{R}^n$, and $k < n$, the level to project down to.

Eff: C is modified. Let $S \subseteq \mathbb{R}^n$ be the union of all TRUE cells in the tree rooted at C prior to the algorithm's execution, S' the union of TRUE cells in the tree rooted at C after the algorithm's execution. Then S' is a weak projection onto k -space of S .

```

1   $(r, t) \leftarrow EasyProjection(C, k)$ 
2  while  $r > k$  do
3       $(C_1, C_2) \leftarrow$  cells of  $C$  as guaranteed in Theorem
4      ProjectionResolve( $C_1, C_2, k$ )
5       $(r, t) \leftarrow EasyProjection(C, k)$ 

```

m	$n = 3$ variables		$n = 4$ variables		
	$r = 1$	$r = 2$	$r = 1$	$r = 2$	$r = 3$
4	5.2%	4.1%	5.8%	10.0%	7.6%
5	5.4%	5.1%	4.5%	12.6%	6.2%
6	5.0%	5.9%	3.8%	16.6%	11.7%

Figure 4: Time taken to compute the projection onto r -space of an Open NuCAD of n -space defined by m inequalities, shown as a percentage of the time taken to compute the Open NuCAD of n -space in the first place. Percentage shown is the average of the 25 examples for each (m, n, r) triple.

sets³. Where referred to, Open CAD experiments are done with QEPCAD B⁴. The experiments used the “nonlinear set” of formulas⁵ from [1], which includes 25 formulas for each combination of m inequalities on n variables, where $m \in \{4, 5, 6\}$ and $n \in \{2, 3, 4\}$. Each inequality is randomly generated, degree two, and strict. There is good reason to be concerned about the meaning of experiments on “random” inputs like these, since the performance of many algorithms for computing with real polynomials behave differently (usually much worse!) on random input than input from actual applications. However, in this context it seems reasonable to do, since we are primarily interested in understanding how the NuCAD projection algorithm behaves as the parameters of number of variables, dimension to project onto and number of polynomials increase.

The first set of experiments investigates how the time required to compute the projection compares to the time required to compute the original NuCAD. The results, shown in Figure 4, show that the cost of computing the projection is generally small compared to the cost of computing the original Open NuCAD.⁶

The next set of data explores how the number of cells in the projected NuCAD compares with the number of cells in the original. It is not clear *a priori* that the number of cells will decrease, since the projection process requires subdividing cells. However, the results (shown in Figure 5) demonstrate that this is indeed the case, at least for the kinds of inputs considered here.⁷

We next compare computing an Open NuCAD for the projection of a semi-algebraic set given by a Tarski formula, to computing an Open CAD for the formula. Open CAD computations for projections were done with QEPCAD B, using settings that force it to do purely “Open” CAD constructions, in both free and bound variable space. It uses “partial” CAD, allowing it to terminate stack constructions in bound variable space early when a satisfying cell is found. Both TARSKI and QEPCAD B are built on top of the SACLIB computer algebra system, which requires deciding up-front on the size of the garbage collected array (GCA). For TARSKI, all computations

³www.usna.edu/Users/cs/wcbrown/tarski

⁴www.usna.edu/CS/qepcadweb/B/QEPCAD.html

⁵www.usna.edu/Users/cs/wcbrown/research/ISSAC17

⁶There was only one instance in all the tests for which the time for both was measurable (i.e. more than the 4ms resolution of the timer) that the time for projection exceeded the time to construct: that was a $m = 6, n = 4, r = 3$ example for which projection took 183% of the construction time. This example is labeled F15 in the data-set available at the URL given in the prior footnote.

⁷The worst example, in which the number of cells in the projected NuCAD is 70.3% of that in the original, came on example $m = 6, n = 4, r = 3$, F15 — the only example for which computing the projection actually took longer than computing the original NuCAD.

	$n = 2$		$n = 3$		$n = 4$		
m	$r = 1$	$r = 1$	$r = 2$	$r = 1$	$r = 2$	$r = 3$	
4	32.3%	4.9%	32.9%	0.3%	5.5%	35.6%	
5	25.0%	6.0%	30.7%	0.3%	3.9%	33.2%	
6	24.7%	6.7%	28.6%	0.1%	3.9%	32.7%	

Figure 5: Number of cells in the projected NuCAD of r -space as a percentage of the number of cells in the original Open NuCAD of n -space. The percentage shown is the average of the 25 examples for each (m, n, r) triple.

m	n	r	NuCAD		CAD		#cells ratio
			#cells	ex	#cells	ex.	
4	2	1	2.72	25	10.98	25	0.24772
5	2	1	3.04	25	17.78	25	0.17098
6	2	1	3.04	25	24.86	25	0.12228
4	3	1	4.32	25	34.50	25	0.12522
4	3	2	34.76	25	373.10	25	0.09317
5	3	1	5.84	25	83.06	25	0.07031
5	3	2	42.80	25	1342.94	25	0.03187
6	3	1	5.40	25	189.46	25	0.02850
6	3	2	41.16	25	4201.14	25	0.00980
4	4	1	1.40	25	223.22	25	0.00627
4	4	2	72.72	25	7618.18	25	0.00955
4	4	3	826.88	25	86301.30	25	0.00958
5	4	1	3.56	25	1943.50	25	0.00183
5	4	2	128.12	25	131456.68	25	0.00097
5	4	3	1492.20	25	1259602.00	23	0.00118
6	4	1	4.44	25	13689.17	24	0.00032
6	4	2	322.32	25	1023992.00	8	0.00031
6	4	3	4108.88	25	N/A	0	N/A

Figure 6: Number of cells in the projected Open NuCAD of r -space compared to the number of cells in the Open CAD of r -space. Number shown is the average of the completed examples for each (m, n, r) triple. The “ex” columns shows how many of the 25 example computations actually ran to completion. Uncompleted example computations were not included in computing the cell count averages.

were done with a 10 million cell GCA. For QEPCAD B, 10 million was used for $n = 2$, 100 million for $n = 3$ and 1 billion for $n = 4$. Still, a number of QEPCAD B computations failed for lack of memory. Figure 6 compares the number of cells in the open decompositions of r -space ultimately produced. Figure 7 compares the time required on the various 4-variable problem sets. Because QEPCAD B, using partial CAD, does not have to construct a decomposition of n -space, it is at an advantage for $r = 1$. (Development of a “partial” version of NuCAD is an avenue for future research.) However, as m and r increase, the complexity of the original semi-algebraic set and its projection increase, and Open NuCAD becomes faster by several orders of magnitude, despite naive and unoptimized implementation in TARSKI and the fact that QEPCAD B uses partial CAD, and TARSKI has to compute a full Open NuCAD of $\mathbb{R}^n$.

Figure 7: This plot shows number of examples (of the 25 per data set) completed as a function of time spent, where examples are ordered from least to greatest execution time. Only 4-variable examples are considered. The dotted lines represent Open partial-CAD computations for different m and r values, where r is the dimension projected onto and m is the number of inequalities. The solid lines show computations for Open NuCADs of 4-space (since previous results show the time to project is typically small relative to the time required to construct the NuCAD). Solid lines show times to compute complete Open NuCADs of 4-space, time to then project is left out to improve readability.

6 CONCLUSION

This paper provided an algorithm for computing (weak) projections of semi-algebraic sets in the (Open) NuCAD representation. The experiments reported here indicate the potential of this approach. However, there is still much work to be done, including extending the theory of NuCAD beyond the “open” case, and along with that, extending the projection algorithm to general NuCADs.

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