Supplementary: Optimal Combination of Techniques in Multiple Importance Sampling

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Appendix A - Recapitulation of basic notions of random variable and estimators

We recall here the basic statistics and Monte Carlo estimation concepts that will be needed in this paper. The reader can consult the classic references of Monte Carlo [Rubinstein and Kroese 2008; Kalos and Whitlock 1986].

Random Variable. Given a random variable X over a domain Ω , distributed according to probability distribution function pdf $f(X)$ its mean or expected value $E[X]$ is the integral $\int_{\Omega} xf(x)dx$. The expected value of a function $g(X)$ of random variable X (which in turn is a random variable) is $E[g(X)] = \int_{\Omega} g(x)f(x)dx$. The expected value of a sum of random variables is the sum of expected values, and the product with a constant k is the constant times the expected value, i.e., $E[k \sum g_i(X_i)] = k \sum E[g_i(X_i)]$. We will use below the particular case when $k = 1/N$, and all random variables are identically distributed, then we have:

$$E[\frac{1}{N} \sum_{i=1}^N g(X_i)] = \frac{1}{N} \sum_{i=1}^N E[g(X_i)] = E[g(X)] \quad (46)$$

Variance. The variance $V[X]$ of distribution X is defined as

$$\begin{aligned} V[X] &= E[(X - E[X])^2] = E[X^2] - (E[X])^2 \\ &= \int_{\Omega} x^2 f(x)dx - (E[X])^2. \end{aligned} \quad (47)$$

In general, the variance $V[g(X)]$ of function $g(X)$ is defined as

$$\begin{aligned} V[g(X)] &= E[(g(X) - E[g(X)])^2] = E[g(X)^2] - (E[g(X)])^2 \\ &= \int_{\Omega} g(x)^2 f(x)dx - (E[g(X)])^2 \end{aligned} \quad (48)$$

The variance of a sum of independent random variables is the sum of variances, and the product with a constant k is the squared constant times the variance, i.e., $V[k \sum g_i(X_i)] = k^2 \sum V[g_i(X_i)]$. Thus, if all random variables are independent and identically distributed, then:

$$V[\frac{1}{N} \sum_{i=1}^N g(X_i)] = \frac{1}{N^2} \sum_{i=1}^N V[g(X_i)] = \frac{1}{N} V[g(X)] \quad (49)$$

An estimator of a quantity is a random variable such that its expected value is equal to that quantity. A *biased estimator* is when the expected value differs by a *bias* from this quantity. An *unbiased estimation* of the mean value $E[g(X)]$ is obtained by drawing N samples according to X , when the *bias* is zero:

$$E[g(X)] \approx \frac{1}{N} \sum_{i=1}^N g(x_i) \quad (50)$$

Observe that the estimator is just the average of the sum of N random variables identically distributed:

$$E[\widehat{g(X)}] = \frac{1}{N} \sum_{i=1}^N g(X_i). \quad (51)$$

Clearly $E[E[\widehat{g(X)}]] = E[g(X)]$. We use a "hat" symbol to differentiate the estimator from the quantity being estimated. We will use in the definition of successive estimators directly the samples instead of the random variables where they are sampled from. An *unbiased estimator* of the variance $V[g(X)]$ is as follows:

$$V[\widehat{g(X)}] = \frac{\sum_{i=1}^N (g(x_i) - (\sum_{i=1}^N g(x_i))/N)^2}{N-1} \quad (52)$$

But observe that an estimator for $E[g(X)^2]$ is

$$E[\widehat{g^2(X)}] = \frac{1}{N} \sum_{i=1}^N g^2(x_i) \quad (53)$$

Appendix B - BRDF properties

BRDF (i.e. bidirectional reflectance distribution function) is defined by Nicodemus[1977] by means of the reflectance equation:

$$f_r(x, \omega_o, \omega_i) = \frac{dL(x \rightarrow \omega_o)}{L(x \leftarrow \omega_i) \cdot \cos(\theta_i) d\omega_i} = \frac{dL(x \rightarrow \omega_o)}{L(x \leftarrow \omega_i) \cdot (\omega_i \cdot \vec{n}) d\omega_i} \quad (54)$$

In this definition it fulfills Helmholtz reciprocity given as:

$$f_r(x, \omega_o, \omega_i) = f_r(x, \omega_i, \omega_o)$$

Observe that $f_r(x, \omega_o, \omega_i)$ is not in principle a probability density function (as it does not integrate to 1 with incoming directions ω_i , but see below), it is unitless (i.e. unit is $[\text{sr}^{-1}]$, but this is still unitless as steradian is unitless). To obtain a probability density function (pdf) from the BRDF, it should be redefined using the reflectance equation. We can define it in the following way, for fixed ω_o and integrating over incoming directions ω_i .

$$\text{pdf}(x, \omega_o, \omega_i) = \frac{1}{a(x, \omega_o)} \cdot f_r(x, \omega_i, \omega_o) \cdot (\omega_i \cdot \vec{n}), \quad (55)$$

where albedo $a(x, \omega_o)$ for fixed outgoing direction ω_o is defined as:

$$a(x, \omega_o) = \int_{\Omega} f_r(x, \omega_i, \omega_o) \cdot (\omega_i \cdot \vec{n}) d\omega_i \quad (56)$$

Appendix C - One-dimensional examples

Suppose we want to solve the integral $I = \int_0^{\pi} x(x^2 - \frac{x}{\pi}) \sin(x) dx = 10.2884$ by MIS sampling on functions x , $(x^2 - \frac{x}{\pi})$, and $\sin(x)$ respectively. We first find the normalization constants: $\int_0^{\pi} x dx = \pi^2/2$, $\int_0^{\pi} (x^2 - \frac{x}{\pi}) dx = 8.764$, $\int_0^{\pi} \sin(x) dx = 2$. Then we find the three variances when doing importance sampling with all three pdfs respectively:

$$V_1 = \pi^2/2 \int_0^{\pi} x(x^2 - \frac{x}{\pi})^2 \sin^2(x) dx - I^2 = 29.7928$$

$$V_2 = 8.764 \int_0^{\pi} x^2(x^2 - \frac{x}{\pi})^2 \sin^2(x) dx - I^2 = 23.4828$$

$$V_3 = 2 \int_0^{\pi} x^2(x^2 - \frac{x}{\pi})^2 \sin(x) dx - I^2 = 123.896$$

Thus, according to eq. 44 and applying eq. 25, with all $n_i = 1$, we get, with $H(\{V_1, V_2, V_3\}) = 35.6206$, this set of weighting coefficients: $(\alpha_1, \alpha_2, \alpha_3) = (0.398538, 0.505627, 0.0958351)$

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Remember now from eq. 31 that the variance of MIS is given by the following expression:

$$V(\alpha_1, \alpha_2, \alpha_3) = \int_0^\pi \frac{x^2(x^2 - \frac{x}{\pi})^2 \sin^2(x)}{\frac{2\alpha_1 x}{\pi^2} + \frac{\alpha_2}{8.764}(x^2 - \frac{x}{\pi}) + \frac{1}{2}\alpha_3 \sin(x)} dx - I^2$$

Substituting the values found above, we have that $V(0.398538, 0.505627, 0.0958351) = 25.803$. On the other hand, $V(1/3, 1/3, 1/3) = 33.4113$, and there is a gain of 29.47%. By iterating over all possible α_i values with step 0.01 we can find the minimum value for variance is 23.247, very near to our optimal. We can check the value for eq. 38:

$$\int_0^\pi \frac{x^2(x^2 - \frac{x}{\pi})^2 \sin^2(x)}{\frac{2\alpha_1 x}{\pi^2} + \frac{\alpha_2}{8.764}(x^2 - \frac{x}{\pi}) + \frac{1}{2}\alpha_3 \sin(x)} dx = 131.654$$

and values for eq. 36:

$$\int_0^\pi \frac{x^2(x^2 - \frac{x}{\pi})^2 \sin^2(x)}{(\frac{2\alpha_1 x}{\pi^2} + \frac{\alpha_2}{8.764}(x^2 - \frac{x}{\pi}) + \frac{1}{2}\alpha_3 \sin(x))^2} \frac{2x}{\pi^2} dx = 128.466$$

$$\int_0^\pi \frac{x^2(x^2 - \frac{x}{\pi})^2 \sin^2(x)}{(\frac{2\alpha_1 x}{\pi^2} + \frac{\alpha_2}{8.764}(x^2 - \frac{x}{\pi}) + \frac{1}{2}\alpha_3 \sin(x))^2} 8.764(x^2 - \frac{x}{\pi}) dx = 137.427$$

$$\int_0^\pi \frac{x^2(x^2 - \frac{x}{\pi})^2 \sin^2(x)}{(\frac{2\alpha_1 x}{\pi^2} + \frac{\alpha_2}{8.764}(x^2 - \frac{x}{\pi}) + \frac{1}{2}\alpha_3 \sin(x))^2} \frac{1}{2} \sin(x) dx = 114.445$$

Remember that for the optimal case all four values should be equal. We can see that the discrepancy is relatively small for the heuristically obtained α values. On the other hand, for α values $\{1/3, 1/3, 1/3\}$ the integrals evaluate to 139.262 and $\{143.513, 162.005, 112.267\}$ with a bigger discrepancy.

Suppose now relative costs are $c_1 = 1$, $c_2 = 6.24$, $c_3 = 3.28$. These costs are taken from experimentation. Using eq. 46, the approximative values close to optimal for weighting coefficients α_i are found to be $\{0.783361, 0.159209, 0.0574306\}$. Although the variance now increases a little $V(0.783361, 0.159209, 0.0574306) = 28.9247$, the relative efficiency increases from 1.2947 for equal sampling costs to 2.06141 when taking into account the different sampling costs. This presents improvement +106.1% in terms of sampling efficiency.

Appendix D - 2D analytical example with Lafortune-Phong BRDF model

Let us consider here the formula for the physically based variant of Phong model presented by Lafortune [1994]. We assume the whole lobe is above the surface. For simplicity, we will assume the case when the outgoing direction is along the surface normal.

$$f_r(\omega_o, \omega_i) = \rho_d/\pi + \rho_s \frac{n+2}{2\pi} \cos^n \theta \quad (57)$$

Let us consider also an environment map given by $R(\omega) = R \cdot \cos \theta$. As $d\omega = \sin \theta d\theta d\phi$ the outgoing radiance is given then by the integral:

$$R \int_0^{2\pi} \int_0^{\pi/2} (\rho_d/\pi + \rho_s \frac{n+2}{2\pi} \cos^n \theta) \cos \theta \cos \theta \sin \theta d\theta d\phi \quad (58)$$

and integrating over ϕ

$$2\pi R \int_0^{\pi/2} (\rho_d/\pi + \rho_s \frac{n+2}{2\pi} \cos^n \theta) \cos \theta \cos \theta \sin \theta d\theta \quad (59)$$

Changing variables $x = \cos \theta$,

$$2\pi R \int_0^1 \left((\rho_d/\pi + \rho_s \frac{n+2}{2\pi} x^n) x \right) x dx \quad (60)$$

We have thus the integral of the product $h_1(x)h_2(x)$, where $h_1(x)$ corresponds to the BRDF times cosine,

$$h_1(x) = (\rho_d/\pi + \rho_s \frac{n+2}{2\pi} x^n) x,$$

and $h_2(x) = x$ is the environment map. We can find then the optimal combination and the efficiency with respect to equal coefficients. For typical values, $n = 5$, $\rho_d = \rho_s = 0.5$, the relative efficiency for equal sampling cost is 2.38, with optimal coefficients $\alpha_1 = 0.89$, $\alpha_2 = 0.11$. If we suppose the environment map is very costly to sample, for instance $c_1 = 1$, $c_2 = 10$, then the relative efficiency is 15.62, with optimal coefficients $\alpha_1 = 0.99$, $\alpha_2 = 0.01$.

References

- KALOS, M., AND WHITLOCK, P. 1986. *Monte Carlo Methods: Basics*. Monte Carlo Methods. J. Wiley & Sons.
- LAFORTUNE, E. P., AND WILLEMS, Y. D. 1994. Using the Modified Phong BRDF for Physically Based Rendering. Technical Report CW197, Department of Computer Science, Katholieke Universiteit Leuven, Leuven, Belgium, November.
- NICODEMUS, F. E., RICHMOND, J. C., HSIA, J. J., GINSBERG, I. W., AND LIMPERIS, T. 1977. Geometric Considerations and Nomenclature for Reflectance. Monograph 161, National Bureau of Standards (US), October.
- RUBINSTEIN, R., AND KROESE, D. 2008. *Simulation and the Monte Carlo Method*. Wiley Series in Probability and Statistics. Wiley.