APPENDIX

Crossover Can Simulate Bounded Tree Search on a Fixed-Parameter Tractable Optimization Problem

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PROOF OF LEMMA 4.4

We define the *orthogonal subcode* of $\mathcal{C}(\ell)$ with respect to any $\mathbf{z} \in \mathbb{R}^k$ as

$$\mathcal{C}_{\mathbf{z}}^{\perp}(\ell) := \left\{ c_i \in \mathcal{C}(\ell) : \sum_{j=1}^k c_i[j]z[j] = 0 \right\},$$

i.e., the set of all code words in $\mathcal{C}(\ell)$ that are orthogonal to $\mathbf{z}$ in $\mathbb{R}^k$.

Let $\mathbf{x} \in \mathbb{R}^k$ be an arbitrary real-valued nonzero vector. We proceed by induction on $|\text{supp}(\mathbf{x})|$. We decompose $\mathbf{x}$ into a linear combination $\mathbf{x} = \mathbf{y} + \mathbf{z}$ where $|\text{supp}(\mathbf{y})| = |\text{supp}(\mathbf{x})| - 1$ and $|\text{supp}(\mathbf{z})| = 1$. Set $a \in [k]$ to the unique nonzero index in $\mathbf{z}$. Choose $b \in \text{supp}(B_{\ell}(a))$ arbitrarily (note that $a \neq 0$ as $0 \notin [k]$ so such a choice is always possible).

We define a map

$$\phi_b : [k] \rightarrow [(k+1)/2 - 1] \cup \{0\} := i \mapsto B_{\ell-1}^{-1}(B_{\ell}(i) \setminus \{b\})).$$

Intuitively, $\phi_b(i)$ is the index element in $[(k+1)/2 - 1]$ obtained by removing the bit in the b -th position of the length- ℓ bitstring representation of i .

We consider the orthogonal subcode $\mathcal{C}_{\mathbf{z}}^{\perp}(\ell)$ with respect to $\mathbf{z}$ and argue that the simplex code $\mathcal{C}(\ell-1)$ is isomorphic to $\mathcal{C}_{\mathbf{z}}^{\perp}(\ell)$ in the following sense. For each $i, j \in [k]$, if $c_i \in \mathcal{C}_{\mathbf{z}}^{\perp}(\ell)$ and $B_{\ell}(j)[b] = 0$, then

$$c_i[j] = c_{\phi_b(i)}[\phi_b(j)].$$

In other words, the code words of $\mathcal{C}(\ell-1)$ can be recovered by taking each length- k code word in $\mathcal{C}_{\mathbf{z}}^{\perp}(\ell)$ and deleting the elements in positions j such that $B_{\ell}(j)[b] = 1$. To prove this, note that under these conditions on i and j ,

$$\begin{aligned} c_i[j] &= \sum_{q=1}^{\ell} B_{\ell}(i)[q]B_{\ell}(j)[q] \pmod{2} \\ &= \left(\sum_{q \in [\ell] \setminus \{b\}} B_{\ell}(i)[q]B_{\ell}(j)[q] + B_{\ell}(i)[b]B_{\ell}(j)[b] \right) \pmod{2} \\ &= \sum_{q \in [\ell] \setminus \{b\}} B_{\ell}(i)[q]B_{\ell}(j)[q] \pmod{2}, \quad \text{since } B_{\ell}(j)[b] = 0 \\ &= c_{\phi_b(i)}[\phi_b(j)]. \end{aligned}$$

It remains to show there is a bijection between the elements of $[(k+1)/2 - 1]$ and the corresponding elements of $[k]$. For each $i \in [(k+1)/2 - 1]$, there are exactly two distinct strings $i', i'' \in [k]$ in the preimage of $\phi_b(i)$ whose length- ℓ bitstring representations differ only by the bit in the b -th position. Without loss of generality, suppose $B_{\ell}(i')[b] = 1$. We claim that either $c_{i'}$ or $c_{i''}$ is in the orthogonal subcode. If $c_{i'} \in \mathcal{C}_{\mathbf{z}}^{\perp}(\ell)$, then the claim is shown,

otherwise, suppose $c_{i'}$ is not in the orthogonal subcode. Then

$$\begin{aligned} 0 &\neq \sum_{r=1}^k c_{i'}z[r] = c_{i'}[a] \\ &= \sum_{q=1}^{\ell} B_{\ell}(i')[q]B_{\ell}(a)[q] \\ &= \sum_{q \in [\ell] \setminus \{b\}} B_{\ell}(i')[q]B_{\ell}(a)[q] + B_{\ell}(a)[b] \end{aligned}$$

since $B_{\ell}(i')[b] = 1$. Since $B_{\ell}(i'')[b] = 0$, and the strings only differ in the b -th position, changing the i -th bit to zero changes the parity of the sum, and we conclude that $c_{i''} \in \mathcal{C}_{\mathbf{z}}^{\perp}$. Finally, it is easy to see that each $j \in [k]$ with $B_{\ell}(j)[b] = 0$ corresponds to exactly one element of $[(k+1)/2 - 1]$.

Let $\hat{\mathbf{y}} \in \mathbb{R}^{(k+1)/2-1}$ be the vector constructed from $\mathbf{y}$ as $\hat{\mathbf{y}}[\phi_b(j)] = \mathbf{y}[j]$ for each $j \in [k]$ such that $B_{\ell}(j)[b] = 0$. Thus $\hat{\mathbf{y}}$ is obtained from $\mathbf{y}$ by removing the indexes of $\mathbf{y}$ in which the bitstring representation of the index contains a 1 in the b -th position. It is easy to see that if $\mathbf{y}$ is orthogonal to $c_i \in \mathcal{C}_{\mathbf{z}}^{\perp}(\ell)$, then $\hat{\mathbf{y}}$ is necessarily orthogonal to $c_{\phi_b(i)} \in \mathcal{C}(\ell-1)$, since orthogonality is preserved by index deletion. Moreover, $|\text{supp}(\hat{\mathbf{y}})| \leq |\text{supp}(\mathbf{y})|$ since no nonzero elements are created under the transformation.

Then $\mathbf{x} = \mathbf{y} + \mathbf{z}$ is orthogonal to the code words in the orthogonal subcode $\mathcal{C}_{\mathbf{z}}^{\perp}(\ell)$ that are also orthogonal to $\mathbf{y}$. By induction, $\hat{\mathbf{y}} \in \mathbb{R}^{(k+1)/2-1}$ is orthogonal to at least $2^{(\ell-1)-|\text{supp}(\hat{\mathbf{y}})|-1}$ code words in $\mathcal{C}(\ell-1)$. By the congruence between $\mathcal{C}_{\mathbf{z}}^{\perp}(\ell)$ and $\mathcal{C}(\ell-1)$, and since $|\text{supp}(\hat{\mathbf{y}})| \leq |\text{supp}(\mathbf{y})|$, $\mathbf{y}$ is orthogonal to at least

$$2^{(\ell-1)-|\text{supp}(\mathbf{y})|-1} = 2^{\ell-|\text{supp}(\mathbf{x})|-1}$$

code words in $\mathcal{C}_{\mathbf{z}}^{\perp}(\ell) \subseteq \mathcal{C}(\ell)$. Finally, all $2^{\ell} - 1$ code words in $\mathcal{C}(\ell)$ are orthogonal to the zero vector, which fulfills the base case, completing the proof. $\square$