Supplementary information for “Evolvability and Complexity Properties of the Digital Circuit Genotype-Phenotype Map” by Alden H. Wright and Cheyenne L. Laue, GECCO 2021

The computation of Tononi complexity

Tononi, Edelman, Sporns (1998) [3] give an elegant information-theoretic explanation of their measure of neural complexity defined in Tononi, Sporns, Edelman (1994) [4]. Their objective is to measure both the existence of functional clusters of neurons and the integration between these clusters. Here, we are looking for functional clusters of gates, and we want to show that these clusters are integrated. A functional cluster of gates is defined as a set of gates that interact much more strongly with each other than with the rest of the circuit. Thus, a functional cluster can be defined as a subset of gates for which the statistical dependence within the subset (integration) is much larger than that between the subset and rest of the circuit (mutual information). As defined below, mutual information from information theory Cover and Joy [1] can be used to measure the statistical dependence of two subsets of gates.

Following Box 1 of Tononi, Edelman, Sporns (1998) [3], we define entropy, mutual information, and complexity of the r gates of a circuit. Our approach to applying information theory to circuits is based on that of Macia and Solé (2009) [2]. When the circuit is executed, or mapped to its phenotype, each gate has an output bit string of length 2^m where m is the number of inputs to the circuit. We form a binary matrix, which we will call X , whose rows are the output strings of the gates. Each column of this matrix is a bit string, and so the columns of X can be interpreted as a string of symbols over the alphabet of bit strings of length r . The **entropy** of X is the entropy of this string. More precisely,

$$H(X) = - \sum_{i=1}^{2^r} p(x_i) \log p(x_i),$$

where the x_i are bit strings of length r and $p(x_i)$ is the relative frequency of x_i in the string of columns of X . Entropy is high when the components of this string are all different (high uncertainty) and low when many components are repeated (low uncertainty).

As an example, we apply this to the gates of the circuit given in Figure 1 of the paper. In this example (and in the paper), we use logarithms of base 2 to compute Tononi complexity. The following table shows the states of the input nodes and gates of this circuit:

Input 1	1	1	1	1	0	0	0	0
Input 2	1	1	0	0	1	1	0	0
Input 3	1	0	1	0	1	0	1	0
Gate 4 OR	1	1	1	1	1	1	0	0
Gate 5 AND	1	0	0	0	1	0	0	0
Gate 6 XOR	0	1	1	1	0	1	0	0

As described in the paper, we use the same input node states for all evaluation of circuits with a given number of inputs. The 3 by 8 matrix X is given by the last 3 rows of this table:

The entropy $H(X)$ is the entropy of the string whose symbols are the columns of X , namely 110, 101, 101, 101, 110, 101, 000, 000. These symbols have relative frequencies (probabilities) $p(110) = \frac{1}{4}$, $p(101) = \frac{1}{2}$, $p(000) = \frac{1}{4}$. Thus, $H(X) = -\frac{1}{4}\log\frac{1}{4} - \frac{1}{2}\log\frac{1}{2} - \frac{1}{4}\log\frac{1}{4} = \frac{3}{2}$.

We also need to be able to compute the entropy of a subset of the gates of the circuit. Since gates correspond to rows of X , a subset of the gates corresponds to a subset of the rows of X .

To compute the entropy of a subset of gates, we compute the entropy of the matrix whose rows are the rows of X that correspond to the subset. For the subset consisting of gates 4 and 6, this matrix has rows corresponding to rows 1 and 3 of X :

1	1	1	1	1	1	0	0
0	1	1	1	0	1	0	0

If X is the matrix of a circuit and Y is the matrix of a subset of the gates of the circuit, then the **mutual information** between Y and the rest of the system $X - Y$ is defined by:

$$M(Y; X - Y) = H(Y) + H(X - Y) - H(X).$$

Mutual information is a measure of statistical dependence that expresses how much information is provided about the state of Y by ‘knowing’ the state of X (and vice versa). Mutual information is nonnegative and symmetric, and is zero in the case of statistical independence. Mutual information has the advantage of that it captures both linear and non-linear dependencies.

Let X_j^k , for $j = 1, 2, \dots, \binom{r}{k}$, denote the matrices of the gate subsets of cardinality k [3]. The **complexity** of a circuit is defined by:

$$C(X) = \frac{1}{2} \sum_{k=1}^r \frac{1}{\binom{r}{k}} \sum_{j=1}^{\binom{r}{k}} M(X_j^k; X - X_j^k).$$

As explained and illustrated by [3], complexity is high when there is high mutual information between the intermediate-sized subsets of X . They give several alternate mathematically equivalent formulations of complexity which we implemented and checked for consistency.

We continue the example by computing the complexity of the circuit of figure 1 of the paper. We will use the notation XS to denote the matrix corresponding to a subset S of the gates of the circuit. Thus, $X\{4,5,6\}$ denotes the matrix X for all gates and $X\{4,6\}$ denotes the matrix for gates 4 and 6. (The matrix $X\{4,6\}$ is shown above.) There are 3 cardinality 1 sets of gates, namely $X_1^1 = X\{4\}$, $X_2^1 = X\{5\}$, $X_3^1 = X\{6\}$, and 3 cardinality 2 subsets, namely $X_1^2 = X\{4,5\}$, $X_2^2 = X\{4,6\}$, $X_3^2 = X\{5,6\}$.

$$H(X\{4,5\}) = H(X_1^2) = \left(\frac{1}{4}\right)\log\left(\frac{1}{4}\right) + \frac{1}{2}\log\left(\frac{1}{2}\right) + \frac{1}{4}\log\left(\frac{1}{4}\right) = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}$$

$$H(X\{6\}) = H(X_3^1) = \frac{1}{2}\log\left(\frac{1}{2}\right) + \frac{1}{2}\log\left(\frac{1}{2}\right) = 1$$

$$M(X_1^2; X - X_1^2) = M(X_1^2; X_3^1) = H(X_1^2) + H(X_3^1) - H(X) = \frac{3}{2} + 1 - \frac{3}{2} = 1.$$

Similarly,

$$H(X\{4,6\}) = H(X_2^2) = \left(\frac{1}{4}\right) \log\left(\frac{1}{4}\right) + \frac{1}{2} \log\left(\frac{1}{2}\right) + \frac{1}{4} \log\left(\frac{1}{4}\right) = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}$$

$$H(X\{5\}) = H(X_2^1) = \frac{1}{4} \log\left(\frac{1}{4}\right) + \frac{3}{4} \log\left(\frac{3}{4}\right) = 2 - \frac{3}{4} \log(3)$$

$$M(X_2^2; X_2^1) = H(X_2^2) + H(X_2^1) - H(X) = \frac{3}{2} + \left(2 - \frac{3}{4} \log(3)\right) - \frac{3}{2} = 2 - \frac{3}{4} \log(3)$$

$$H(X\{5,6\}) = H(X_3^2) = \frac{3}{2}$$

$$H(X\{4\}) = H(X_1^1) = 2 - \frac{3}{4} \log(3)$$

$$M(X_3^2; X_1^1) = \frac{3}{2} + 2 - \frac{3}{4} \log(3) - \frac{3}{2} = 2 - \frac{3}{4} \log(3)$$

$$\begin{aligned} C(X) &= \frac{1}{2} \left(\frac{1}{3} \left(M(X_1^1; X_3^2) + M(X_2^1; X_2^2) + M(X_3^1; X_1^2) \right) + \frac{1}{3} \left(M(X_1^2; X_3^1) + M(X_2^2; X_2^1) + M(X_3^2; X_1^1) \right) \right) \\ &= \frac{1}{3} \left(\left(2 - \frac{3}{4} \log(3) \right) + \left(2 - \frac{3}{4} \log(3) \right) + 1 \right) = \frac{5}{3} - \frac{1}{2} \log(3) = 0.8742 \end{aligned}$$

Thus, we have computed the Tononi complexity $C(X)$ of the circuit of Figure 1 of the paper. Clearly this computation is exponential in the number of gates in the circuit. We do not attempt to compute complexity for circuits with more than 20 gates within simulations.

Evolvability results for NAND gate circuits

One of the reviewers of this paper expressed the concern that the results of the paper might depend on the choice of logic gates used in a circuit. To address this concern, we show evolvability results when circuits use only the NAND gate. Such circuits are universal in that they can compute any Boolean function.

First, we show evolution and sampling evolvability versus robustness.

As in Figure 6 of the paper, there is a negative relationship for evolution evolvability and a positive relationship for sampling evolvability.

We also show evolution and sampling evolvability versus Tononi complexity.

As in Figure 8 of the paper, there is a positive relationship for evolution evolvability and a negative relationship for sampling evolvability.

Bibliography

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