Replies to the reviewers remarks about "Quick Extreme Hypervolume Contribution Algorithm"

A. Jaszkiewicz and P. Zielniewicz

Institute of Computing Science, Poznań University of Technology,
Piotrowo 2, 60-965 Poznań, Poland

Reviewers' comments and Authors' reactions

We would like to extend our deepest thanks to the Reviewers for the high evaluation of the paper and their valuable comments. We have duly reacted to all of these comments (see below) and, space permitting, implemented the predominant part of them.

Reviewer-1

The main problem is that this manuscript has been written to a very small target audience. The reader is assumed to be very familiar with the way of presenting the problem and the notation used. The authors do not help the reader in trying to make a connection to more widely used ways of denoting hypervolume calculation. It would be very desirable if the authors improved in this and, thus, made the manuscript easier to follow by a wider group of people.

Many readers are familiar with hypervolume defined as an integral (or area) using a reference point. The authors simply say that they use a "somehow non-standard" definition for their own convenience. However, they should make a connection to the well-known definition. And what here corresponds to the reference point?

We extended this remark to: "Note that the use of hypercuboid $[r_, r^*]$ in the presented definition is somehow non-standard, since usually only point r_* is used and called a reference point, however, our definition facilitates explanation of the proposed algorithm which breaks a problem into sub-problems corresponding to smaller hypercuboids. Our definition boils down to the standard one if $\forall_{p \in S} p \preceq r^*$."*

One more aspect making the manuscript inconvenient to read is the extensive usage of acronyms. Once again, it seems that the author assume that all readers are familiar with them so that so many acronyms are safe to be used - this is not true.

Because of the limited space and readability the use of long names of some algorithms would be undesired. We checked if all acronyms are defined before the first use, and added one missing reference to WFG algorithm.

The concept of sub-problem appears all of a sudden in the introduction but it is not defined.

Before using the term 'sub-problem' we state that the algorithms are based on divide and conquer scheme.

The topic as such is interesting since, as the authors say, hypervolume can be used in many ways. However, the authors do not clearly discuss the limitations of the approaches proposed - how many objectives are recommended? More discussion on this is needed.

We added in the Conclusions Section the remark that since efficient algorithms exists for problems with up to 4 objectives, the proposed method is recommended for 5 and more objectives.

Mentioning the order of objectives in the abstract is not understandable since the connection of this to the main topic has not been explained. It is not clear either, what is meant by iterations. The same question is valid in the introduction - the concept of iteration (iteration of what) must be defined before referring to it.

We added already in the abstract that the algorithm works in a divide and conquer manner. We reformulated the sentence to "We propose also a modification of the general IQHV algorithm scheme in which the order of objectives processing which influences construction of sub-problems is heuristically adapted and show that this modification improves efficiency of the algorithm."

Elaborate what is meant by "maximum joint hypervolume from a larger set".

We reformulated this sentence to "HECP may also be used for solving the hypervolume subset selection problem (HSSP), the goal of which is to select a predefined number of points from a larger set to maximize the hypervolume."

The concept of "empty reduced set" also needs clarification.

We changed to term "empty subset".

The acronym IWFG is never defined.

IWFG is not really an acronym. WFG is the name of a research group "Walking Fish Group". We do not think giving this extension is necessary.

The authors define H with two or three parameters. However, e.g. in formula (5) they use H with a single parameter but this notation has not been defined.

We use $A([r_, s])$ throughout the paper and explain that it is the hypervolume of hypercuboid $[r_*, s]$ to avoid any confusion.*

The authors should clarify what is meant by "hypervolume is calculated just once..."

We added an explanation that "the time of calculation of $A([r_, s])$ is negligible".*

Section 2 is basically a list of definitions, some clarifying text is needed between them.

Following this remark, we have added some introductory texts to this Section.

Notation $s > r$ should be defined.

We do not use this symbol any more and use just symbol $\preceq$.

Define contributor/contribution problem properly.

Following a remark of another reviewer, we now consider either $S = S'$ or $S \cap S' = \emptyset$.

Explain Figure 1, how should it be interpreted?

We added that the Figure shows ranges between lower and upper bound for several processes.

When referring to a "specific order of objectives", the authors should elaborate what the order implies.

We reformulated this sentence to "Conditions (8) and (9) assume that the sub-problems are built using a given order of objectives".

Explain tables better. E.g. what is the difference between Ref and Ref*?

We have reformatted the Tables. Methods Ref and Ref are now under a caption "Reference methods" to avoid confusion.*

Overall, Tables 1-2 are hard to read. Highlight winners or support getting a better overview in some other way.

The best results for contributor and contribution problems are highlighted now.

In the observations, the authors say that there are differences between linear and convex data sets. This is most confusing since linear functions are convex. So, this should be discussed and elaborated better. Is there a conflict in the findings?

We added a remark that "Of course, formally speaking, linear data sets are also convex, but we use the original terminology of the authors of these sets."

The authors should be clearer about how to utilize the findings in practices. What kind of effect can the proposed approaches imply? How significant is the savings in time in practice? The authors should also elaborate what are the limitations of the approaches?

As it was stated before, we added in the Conclusions Section the remark that since efficient algorithms exists for problems with up to 4 objectives, the proposed method is recommended for 5 and more objectives.

There are inconsistencies in the list of references: in the way journal titles are sometimes written with only upper case initials and in article titles sometimes with capital initials in the whole title and sometimes not.

We corrected these inconsistencies.

In addition, ref 21 misses in which collection the article was published.
We added this information.

The series information is less relevant. And "cited By 0" should be removed.
We removed this text.

And sometimes journal titles are abbreviated and sometimes not. And Springer written in different ways. And Berlin, Heidelberg repeated in ref 32.
We applied this correction.

Overall, by partly rewriting the manuscript, being more careful, making it more reader-friendly, discussing limitations and not only benefits and connecting it to what most readers are familiar with will clearly improve the understandability and potential impact of this manuscript.

Minor comments (some examples of instances where more care is needed, this is not an exhaustive list)

- multiobjective methods → multiobjective optimization methods
We applied this correction.
- which goal → goal of which
We applied this correction.
- in greedy manner → in a greedy manner
We applied this correction.
- objective space → set S in the objective space R^m
We applied this correction.
- definition?
We applied this correction.
- nothe?
We applied this correction.
- the bounds the QHV-BQ?
We applied this correction.
- this methods
We applied this correction.
- comparison of the two times?
We applied this correction.

Reviewer-2

In "Quick Extreme Hypervolume Contribution Algorithm", the authors propose a version of the quick hypervolume algorithm to compute efficiently the smallest (or largest) hypervolume contributors (and their hypervolume contributions). The original algorithm is thereby only slightly modified to retrieve the smallest (or largest) hypervolume contributors, exploiting the fact that some subproblems do not need to be looked at anymore if the computed upper and lower bounds tell that no smallest (or largest) contributor can be found in certain subproblems. The main contribution (from my viewpoint) of the paper is to change the order in which the subproblems are produced. Instead of a fixed or random order in which the objectives were touched in the previous version of the algorithm, the authors propose to sort the objectives—splitting the problem into smaller subproblems such that more domination happens earlier in the computation with the result of smaller solutions sets to handle and, consequently, a reduced computation time. Numerical comparisons of the CPU time to compute the minimum and maximum contributors and contributions on three types of Pareto front shapes in 6 dimensions (from 5 to 10 objectives) show that the proposed approach outperforms the best algorithms from the literature and that the sorting objectives idea can reduce the computation time not only for the quick hypervolume algorithm. The comparison with a theoretical algorithm, with the knowledge about the minimum/maximum contributor, shows that the proposed algorithm is only slightly slower.

Overall, the paper is easy to follow, the contribution becomes clear and the paper delivers what it promises. The idea of sorting the objectives to quickly reduce the number of considered points in the subproblem is interesting and the numerical results show that this works well in practice (and for several algorithms). The footnote on page 4 suggests that the authors will provide the source code once the paper is accepted such that the contribution will certainly provide a step forward for the EMO community. I therefore suggest to definitely accept the manuscript as paper.

Comments and Remarks:

- Equation 5 seems faulty. $H([r_*, s])$ is not defined. Do you mean $\Lambda(s, [r_* r^*])$? This happens a few times later as well.
As suggested we use $\Lambda([r_ r^*])$ throughout the paper and explain, at the first use, that it is hypervolume of hypercuboid $[r_*, s]$.*
- "Efficient algorithms exist only for the number of objectives $m = 3$ [3] and $m = 4$ [16]." does not look correct, because also the two-objective case is obviously efficiently solvable.
We added this remark.
- The abbreviations QEHC and WFG are not introduced at first usage.
We added the abbreviation of QEHC in the Introduction and added a missing

reference to WFG.

- "definition"
We applied this correction.
- page 3, item (3), the $\prec$ symbol is not defined.
We do not use this symbol any more.
- double "the" in "of achieving the the condition"
We applied this correction.
- "outperforms" $\rightarrow$ "outperforms"
We applied this correction.

Reviewer-3

The paper proposes an algorithm to compute a point in a set S' with minimum/maximum hypervolume contribution to a reference set S , considering m objectives. The algorithm proposed is interesting and relevant, and the results presented show a good performance when computing the minimum/maximum contribution from a set of 1000 points with $S'=S$ and $m \geq 5$. However, there are some important issues that must be fixed/addressed particularly those concerning problem definitions and the experiments. Moreover, the writing quality could (and should) be improved, some parts could be better explained.

Major comments:

- In the sections previous to Section 2, the text does not explain clearly enough the extreme contribution (contributor) problem. It is not clear whether the intention is to compute the minimum/maximum contribution of the points in a set S' to the set S' itself or to an arbitrary set S . Since considering an arbitrary set S is considerable different from restricting the problem to cases where $S=S'$, this must be clarified, otherwise the text is ambiguous and may be misleading.
We added that "The extreme contributor/contribution may be searched within the same set of points to which the contribution is calculated or within an external set of points."
- When describing the state-of-the-art algorithms, clearly indicate whether the algorithm solves the extreme contribution problem for the case of the contributions of points in S' to an arbitrary set S , or whether they are specific to a particular case such as $S=S'$. For example, it is said that "IWFG algorithm is at present the best algorithm for HMinCP in the general case", however, I believe that IWFG only considers the case where $S=S'$, and this is not clearly stated in the text. Moreover, the text should explicitly mention

which are the state-of-the-art algorithms for $m < 5$.

We added remark that "Note that this implementation (of IWFG) assumes $S = S'$."

- After introducing Definitions 2.4, 2.5 and 2.6, it is said "Note that in the last two definitions we allow all cases $S = S'$, $S \cap S' = \emptyset$, and $S \cap S' \neq \emptyset$." In my opinion, there is an important issue arising from considering that some points of S' are in S and others are not. Consider the following example: $r_* = (0,0)$, $S = \{(9,10)\}$ and $S' = \{(9,10), (10,10)\}$. Note that, given expression (4), although the point $(9,10)$ is dominated by the point $(10,10)$, the former has greater contribution to S than the latter, which are 90 and 10, respectively. This is counter-intuitive and may be misleading. In my opinion, it would be better to restrict the definitions to the cases where $S = S'$ or where their intersection is empty. Otherwise, the case where some points in S' are in S and others are not should be discussed in detail and it should be explained why is it relevant to consider such a case.
We agree and we clearly state now that we allow either $S = S'$ or $S \cap S' = \emptyset$.

- The definitions and expressions in Sections 2 and 3 must be carefully checked and fixed as some of them are incorrect. For example, the expression (5) is correct only if r^* weakly dominates s . Another example is the first expression in Section 3.1, it is correct only if r^* weakly dominates s^p .
We agree. We added this assumption.

- I think that the content of Section 3.1 should have been introduced before Section 3 because it is background information and not something new.
We changed sub-sections 3.1, 3.2, and 3.3 to Sections.

- Regarding Algorithm 1, I think it has too much text, it is a mixture of mathematical notation and text description. For example, it would be much simpler to present the following 'if' condition "if $S^{\wedge}s$ contains one or two points" as "if $|S^{\wedge}s| \leq 2$ ". I think it would have been much better if Algorithm 1 was presented just using mathematical notation, with numbered lines and accompanied by a much more detailed description in Sections 3.2 and 3.3 referring to specific lines of the algorithm.
We added a clear statement that the algorithm is presented in a pseudocode. We believe that this form of presentation using some natural text is more clear.

- The authors should report the time and space complexities of the proposed algorithm.
We added the following analysis to the paper "In the worst case, the lower and upper bounds will not allow to stop any process before finding the exact value of $HC(s_, S, [r_*, r^*])$, which is equivalent to running QHV-BQ algorithm for each process until the end. Such situation may happen if the hypervolume contribution of each point in S' is exactly the same. According to [20] the*

worst case time and space complexities of QHV-BQ are both $\mathcal{O}(d^{n-1})$. Thus, the worst case time and space complexities of QEHC are both $\mathcal{O}(|S'|d^{n-1})$. In practice, the use of equation (5) for calculation of the hypervolume contribution and the use conditions (15) and (16) to stop unnecessary processes usually allow for a significant speed-up.”

- Regarding the experiments, I would have liked to see some results considering the case where the intersection of S and S' is empty, as well as some plots showing how the runtime increases with the number of input points. *We added an experiment showing running times of the proposed QEHC-r and QEHC-s methods for data sets with 10 objectives composed of randomly selected 100, 200, . . . , 1000 points. The running times of the proposed QEHC-r and QEHC-s methods grow relatively slowly with the number of points. This is in line with previous experiments showing that the running times of exact methods for HMinCP scale well with the number of points [6], while growing exponentially with the number of objectives.*
- In Section 4, the authors iterate over WFG to compute hypervolume contributions and find the maximum contributor/contribution and use this as a baseline to compare their algorithms. This is not the most appropriate adaptation of WFG to compute the contribution of each point. In reference [16], the authors used a different adaption of WFG (which they called WFG-c) which achieved a similar performance to IWFG in the experiments presented in [16]. They exploit the working principle of WFG, since it already calls a function to compute a contribution, it is more advantageous to iterate over such function. The authors should compare the proposed algorithms against WFG-c, which should be easy to do. *This would indeed be interesting. We are not convinced, however, that the modification of WFG to WFG-c would be so simple. The description of this modification in [16] is fairly general: "Version 1.11 of WFG was adapted to iterate over the function used to compute the contribution of a single point, which is faster than iterating over WFG as such." and the code of this modification is not available. Furthermore, at the moment of correcting the paper, the web page dedicated to WFG (<http://www.wfg.csse.uwa.edu.au/hypervolume/>) is not available for several months (in our experiment we used executable code of WFG). Taking into account the limited time for correction of the paper, we were not able to implement this algorithm and to perform this additional experiment. We added this as a direction for future research in the Conclusions Section.*
- In Section 4, why the algorithm proposed in [29] was not used for comparison, particularly for the maximum contributor/contribution problem? According to reference [16], that is one of the recommended algorithms to compute all hypervolume contributions. *Indeed, it would be an interesting algorithm for comparison. Due to the limited time for correction of the paper as well because of the limited space,*

we were not able to perform this additional experiment. We added this as a direction for future research in the Conclusions Section.

- From my understanding of the proposed algorithm, the memory requirements may not be linear with respect to the input size. The authors should add information about the memory used by the algorithm in the experiments and should discuss this aspect.

As it was stated above, we added the analysis of the worst case time complexity to the paper.

- In many sentences, the word "the" is missing. A few examples are:

- "... and show that this modification improves efficiency of" should be "... and show that this modification improves the efficiency of"

We applied this correction.

- "Because of wide use of hypervolume ..." should be "Because of the wide use of the hypervolume indicator"

We applied this correction.

- "... (until exact value of ..." should be "... (until the exact value of ..."

We applied this correction.

- "Initialize the priority queue Qs with subproblem defined by ..." should be "Initialize the priority queue Qs with the subproblem defined by ..."

We applied this correction.

- and so on.

We corrected the paper in the above places and in several other places.

Minor comments:

- In page 1, replace "... is the space of m maximized objectives." by something like "... is the space of m objectives to be maximized."

We applied this correction

- In page 1/2, where is "either HECP with minimum or maximum contribution is the best choice for solving HSSP in a greedy way.", explain this.

We reformulated this statement to "In the former case, HECP with the maximum contribution needs to be solved, while in the latter case, HECP with the minimum contribution is used." In this way we avoid the term "best choice" which could be confusing.

- In page 2, where is "... outperforms straightforward adaptations of IQHV and WFG.", this is the first time WFG is introduced. WFG must be introduced before and a reference to this must be provided.

We added missing reference to a paper about WFG and added a remark that it is among state-of-the-art algorithms for exact hypervolume calculation.

- In page 2, replace "Section Three" by "Section 3".
We applied this correction.
- In page 2, expression 3: I think that using " $\text{proj}[r_*, r^*]$ " is odd. I suggestion to use intersections, for example, something like: $[r_*, r^*] \cap [-\inf, p]$.
We applied this version $H(S, [r_, r^*]) = \Lambda\left(\bigcup_{p \in S, r_* \leq p} [r_*, r^*] \cap [-\infty, p]\right)$*
- In page 3, point (4), it mentions that s^p is selected and mentions an iteration. However, at this point, it is not clear from where or why s^p is selected nor which loop the text refers to.
We added before that IQVH works in a divide and conquer manner.
- In page 4, I suggest not to refer to the heuristic as "objectives sorting" as this is different from the objectives ordering mentioned in references [12] and [19]. I believe that "Subproblem sorting" or "Hypercube sorting" would be more adequate.
The use of "Subproblem sorting" or "Hypercube sorting" would suggest that the sub-problems and corresponding hypercubes are constructed first and then sorted, which is not true. The order of objectives influences the construction of sub-problems and corresponding hypercubes. Thus, we are in the opinion that the term "objectives sorting" is the most appropriate.
- In page 8, in Section 5, clearly state that the conclusions about the performance of the proposed algorithm refer to the case where $m \geq 5$.
We added this remark to Conclusions Section.
- In page 8, references [16] and [17] are the same.
We corrected this mistake.