Dynamical Quantities in Net Systems

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Abstract. The paper presents some known material of higher-level Petri nets in analogy to the dynamics of physical systems. Its aim is to widen the view on the role of the incidence matrix in the analysis of net systems. Rather than focusing on constant state quantities (S-invariants) and cyclic actions (T-invariants), i.e. on solutions of the homogeneous equation systems based on the incidence matrix, it studies dynamical quantities of net systems in general.

Keywords: Dynamics; Petri nets

1. Introduction

Petri nets are employed to a considerable extent in the analysis of systems with a combined continuous–discrete dynamics which in particular result from the computerised control of engineering processes. This inevitably raises the question how net systems with their *discrete* states and changes are related to the classical models in physics and engineering based on *infinitesimal* changes of states.

The sequel is an attempt to present some *known material* of higher-level Petri nets² from a particular perspective, namely in analogy to the dynamics of physical systems. It is also an attempt to complement, on a macroscopic level, the foundation that *Carl Adam Petri* has given to net theory in accordance with the laws of physics (see [Pet62, Pet82], for example).

Our purpose in looking at physics is to get orientation, not to reach it – like using the stars for navigation. Most likely there will be more irreconcilable differences than similarities between net systems and physical systems. However, both similarities and differences are equally important for understanding the role of Petri nets in engineering systems analysis.

In Mechanics, Kinematics describes the movement of a body as a sequence of spatial-geometrical positions, while Dynamics describes the same movement in terms of physical quantities that the body, in the course

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² Throughout this paper, the term *Petri nets* denotes the higher-level standard models of net theory like Predicate/Transition nets, Coloured Petri nets or Algebraic Petri nets which trivially include conventional transition nets. A standard model is one that has been derived by net theoretical abstractions and constructions from the basic model, the elementary net systems of conditions and events, rather than being defined only in terms of simulating annotated graphs of boxes, circles and connectors.

of moving, receives from and delivers to its environment (other bodies, fields). Hence the notion of an *exchangeable* quantity is essential to dynamics. The fundamental quantities that by their exchange control the movements are energy, linear momentum and angular momentum. The nature of the exchange of those quantities is particularly simple because it obeys universal conservation laws.

Dynamics does not depend on any geometrical view; it describes a process – a continuous sequence of states – as the receiving and delivering of physical (dynamical) quantities not necessarily proper to mechanics. This kind of description is of fundamental importance for all physics. The firm rules and principles that determine the dynamical description of physical systems and processes comprise a *General Dynamics* [Fal66].

Net theory also does not depend on any geometrical view – even though the graphical appearance of Petri nets is so prominent that it is often taken for the matter itself. As opposed to physics, however, which associates with every state quantity X both its values (that constitute the states) and its infinitesimal changes dX that constitute the processes, net theory describes the states of an object (*system*) as sets of coexisting state elements (*conditions*) and the processes as partially ordered sets of change elements (*events*) and treats state elements and change elements on an equal formal footing.

From elementary net systems of conditions and events, higher-level net models are derived through abstraction – like *Predicate Transition Nets* [GeL81] and *Coloured Petri Nets* [Jen92]. As computer-executable models, hierarchical coloured Petri nets [CPN96] are as powerful as higher-level programming languages, and as standard models of net theory they are subject to formal net theoretical analysis.

One particular aspect of the higher-level net analysis will be presented in this paper: the analysing of a system in terms of identifying *dynamic quantities* and studying their properties and relationships. This is tentatively called the *dynamics* point of view of Petri nets, in contrast to a *kinematics* point of view which deals, for example, with firing sequences and occurrence graphs.

The main part of this paper consists of a piecemeal confrontation of selected elements of net theory with sections of a presentation of the Dynamics of (classical rather than quantum mechanical) physical systems [Fal66].³ The running example is the Petri net model of a simple production scheme. An Appendix briefly introduces an experimental computer-algebra tool for Petri nets that supports the necessary symbolic calculations.

The reader is assumed to have a certain familiarity with Petri nets in general and higher-level nets (PrT nets, CP nets) in particular. This paper is not meant as an introduction to Petri nets, nor to modelling of engineering systems with Petri nets, nor to the dynamics of physical systems.

2. States and Quantities

2.1. Dynamics

A physical system is a totality of *states*. The processes in which the system takes part appear as transitions between states. Among the quantities of a system there are some which are called *state quantities* or *dynamical variables* in the sequel. They are related to the states in the following way:

Definition. A dynamical variable of a physical system is a measuring rule that assigns to each state exactly one value.

Examples are displacement, force, momentum, energy; *non-examples* are work, heat or duration since they belong to processes rather than states.

For identifying the states one selects a maximal number of *independent* variables; each may vary while all the others remain unchanged. Such a selection is called a *coordinate system*; the number of coordinates is the number of *degrees of freedom* of a system. If one knows, in addition to the coordinates fixing the states, the values of all other, dependent variables in every state – as a function of the coordinates – the system is completely known.

Let Σ^a and Σ^b be two systems that have, among others, the variables x^a and x^b . If Σ^a and Σ^b can be coupled in such a way that

$$x^a + x^b = \text{const.} \quad (1)$$

³ This presentation, although in German and a bit out of date, was chosen because with its intention and structure it was a perfect guideline for this paper.

Fig. 1. Two systems of the same type (*displacement*).

Fig. 2. Mass and spring coupled by exchange of coordinate x (dimension *length*).

while x^a , or x^b , can vary freely, the variables x^a and x^b are called *exchangeable*. Σ^a and Σ^b allow an exchange of 'the' variable x ; the amount of x that one system loses is gained by the other.

Example. See Figs 1 and 2.

Definition. Two variables x_1 and x_2 are called *comparable* if they are exchangeable themselves or there is a third variable that is exchangeable with both.

Hence with (1), two variables are comparable iff they can be *added*, if they have the same *dimension*. Since (1) denotes an interaction between Σ^a and Σ^b we finally get:

Every physical variable represents a particular interaction between a system and its environment.

Let Σ^a and Σ^b be two systems with the same number of degrees of freedom. If their variables are in pairs comparable, Σ^a and Σ^b are said to be *of the same type*. The following principle is based upon experience. It is of fundamental importance to the dynamical description of physical systems and their processes.

Principle. If Σ^a and Σ^b are systems of the same type, any coordinate system $\{x_1, \dots, x_n\}$ for Σ^a is a coordinate system for Σ^b as well. And conversely, if two systems have a coordinate system in common, they are of the same type.

In order to determine whether two given systems with n degrees of freedom are of the same type, it is sufficient to find in each system n independent variables such that they are pairwise comparable.

2.2. Net Theory

A net system is a totality of states *and* changes. As opposed to physics, which describes a process as a continuous sequence of states (with time as parameter), net theory describes the states as sets of coexisting state elements (*conditions*) and the processes as sets of change elements (*events*) equipped with a partial order *earlier*. And while physics associates with each variable x both its values (that constitute the states) and its infinitesimal changes dx in a process, net theory treats states and changes on an equal formal footing.

Every state determines which events *may* happen next – are *enabled* – and which *may* have happened last. The set of events that may happen at a state c is called the *en-ability* of c and is denoted by $\mathcal{E}(c)$. Every event e has a *pre-set* $\bullet e$ of conditions that cease to hold and a *post-set* $e^\bullet$ of conditions that begin to hold through an occurrence of e . This effect of an event also determines its en-ability. In its axiomatic foundation, general net theory [Pet77] states the following

Extensionality Principle. For every event e , both effect and possibility of occurring are uniquely determined by a pair of sets of conditions, $(\bullet e, e^\bullet)$. For any two states c_1, c_2 , the following holds:

$$c_1 \xrightarrow{e} c_2 \iff c_1 - c_2 = \bullet e \wedge c_2 - c_1 = e^\bullet \quad (2)$$

Physical quantities are real-valued; more precisely, their values are given as the product of a physical unit (*centimetre, gram, second, ...*) with a real number. In general, this is not the case for the quantities of net systems. Yet some minimal ‘numbership’ is required as well: it must be possible to add the values of a quantity to each other and to multiply them by *scalars* – usually integer numbers but rational or even real numbers are not excluded. A function that maps the states or the processes of a net system into some set of values is called a quantity only if it satisfies this requirement. *Non-examples* are all statements about states or processes; $3 \cdot \text{true}$, for example, is not a truth value.

A structure whose elements can be added and multiplied by the elements of a ring, A , is a module, an abelian group, M , together with an operation of A on M such that for all $a, b \in A$ and $x, y \in M$ we have

$$(a + b)x = ax + bx \quad a(x + y) = ax + ay \quad (3)$$

The markings of a place s of a higher-level Petri net, for example, belong to the module of all integer linear combinations of the elements of the data type (colour set) associated with s . Note that in the special case of conventional Petri nets, the module is the ring of integers itself.

Definition. A state quantity (process quantity) of a net system $\mathcal{N}$ is a function that maps the states (processes) of $\mathcal{N}$ into a module.

In the sequel, we shall encounter two kinds of modules as the value range of a quantity of a net system: (*linear*) *combinations* and *weighted sets*. Let A be the ring of integer, rational or real numbers, and let D be a set.

1. The module is the set of all *formal linear combinations* of elements of D with coefficients from A . In this case, the elements of D , the generators of the module, will be called *pieces*.
2. The module is the set of all *weighted sets* over D with weights in A , i.e. of all mappings of D into A . For any weighted sets $u, v \in [D \rightarrow A]$, scalar $c \in A$ and element $d \in D$, we have

$$(u + v)(d) = u(d) + v(d) \quad (4)$$

$$(cu)(d) = cu(d) \quad (5)$$

The ordering of A induces a partial ordering of $[D \rightarrow A]$. For $u, v \in [D \rightarrow A]$ we have

$$u \geq v \iff u(d) \geq v(d) \quad \text{for all } d \in D \quad (6)$$

Formally, combinations can be viewed as a special case of weighted sets, namely those which only for a finite number of elements have a non-zero weight. Conceptionally, however, and with respect to their usage we want to keep them separate.

If the values of two physical quantities x_1 and x_2 can be added, x_1 and x_2 are *comparable*; they have the same ‘dimension’. For example, energy (a state quantity) and work (a process quantity) have the same

dimension, $mass \cdot length^2 / time^2$. In an analogous way, if x is a quantity of a net system and D is the set such that the values of x are combinations or weighted sets over D , we call D the dimension of x .

Example: quantity: commercial value of a commodity
dimension: money
pieces: the coins and bills of convertible currency
unit: owner's currency | a key currency

The marking of a place s is an integer combination of pieces belonging to the colour set of s (in this case, the pieces are called tokens traditionally). Hence, the places of a net system are state quantities, the values being the markings at the respective states. Since the markings can be varied independently and determine the states, the places are the *coordinates* of a net system.⁴ Two net systems whose places are in pairs comparable – have the same colour sets – are of the same *type*.

The places being the coordinates, net systems in general have quite a large number of coordinates in comparison to physical systems. However, the numerical solution of the differential equations describing the physical system usually also leads to equation systems with a very large number of variables. Net theory just makes a different use of those many variables that are a consequence of the discrete approach.⁵

2.2.1. An Example

Figure 3 together with Table 1 depict the running example of this paper: the *Coloured Petri Net* (CPN, [Jen92]) model of a simple production scheme using batch processing [GeH95]. Several products or substances pass through various stages of production according to their ‘recipes’. The stages are identified with some components of the production site called *devices*. Hence a recipe is just a list of devices, for example $[D1, D2, D3, D1]$. Every incoming order designates a *batch* – the production of a certain integer quantity of the same substance. Depending on the availability of devices, several batches may be processed concurrently.

The declarations in Table 1 assign a formal meaning to the graphical presentation, Fig. 3, such that it can be executed by a tool like *Design/CPN* [CPN96]. Each place is associated with a data type, called its *colour set*; only tokens which belong to its colour set may appear on the place.⁶ In Fig. 3 some colour sets are shown; they are the basic ones like *Bch* (batch), *Dvc* (device), *Rcp* (recipe) or *Rdb* (recipe database), for example. Others that are considered auxiliary are hidden; they can be derived from the adjacent arc annotations.

The system appears divided into three subsystems. The innermost one represents the production proper, the processing of a batch as a double iteration over recipe and multiplicity. It is surrounded by an administration that accepts the orders and delivers the results. Production and administration share the fused pairs of places, (s_1, s'_1) , (s_3, s'_3) and (s_4, s'_4) . The shaded transitions, finally, are shared with an – invisible – environment that sends orders, accepts the deliveries, does the device maintenance and may develop new recipes. They are assumed to be shut off during simulation and analysis.

The annotations and global declarations of a Design/CPN model are derived from Standard ML [Pau91]. Brackets $[.]$ denote lists, the constructor $::$ combines head and tail of a list, and the operator $^{\wedge}$ concatenates two lists. The function *rev* reverses a list; *set* converts a list into a set. The CPN multi-set operators $\cdot$ and $\%$ denote the multiplication of a value by an integer scalar and by a Boolean coefficient, respectively. Transition guards and Boolean coefficients are lists of Boolean expressions.

The model is timed and uses parametric duration functions at the transitions. Tokens carry time stamps that indicate their earliest availability. The time stamps are set by the transitions relative to the simulation time according to their time regions and timed arc expressions (operator $@+$). In the sequel, time does not play an important role; see also Section 5.

To become familiar with the example, let us consider what happens with one of the tokens in the initial marking of place s_0 . The work item (batch) $(X, 3)$, ‘produce 3 units of substance X ’, appears on place s_0 as an incoming order. The processing of the batch begins with an occurrence of transition t_1 for the binding $\{c, r, x, n \leftarrow 3, [D1, D3], X, 0\}$. This takes the batch $(X, 3)$ from the incoming orders, s_0 , reads the recipe for X , $r = [D1, D3]$, from the recipe database, s_7 , and increases its counter n of current usage, allocates the set of

⁴ They have an ‘absolute zero’; the markings are non-negative.

⁵ Note that for this comparison the integer-valued places of the unfolded, equivalent conventional Petri net have to be considered.

⁶ The essence of such a colour set (in the mathematical sense) is that each element (‘colour’) has a finite representation, like the *integers* and unlike the *reals*, for example. In addition they are usually associated with some algebraic structure, and if they are compound the constituents of their elements can be determined by *pattern matching*.

Fig. 3. A simple production scheme.

devices listed in r , $set(r) = 1'D1 + 1'D3$, by removing it from the pool of available devices, s_8 . t_1 also puts an empty container for X 's for accumulating the finished units of a batch s_5 and initialises the status vector (*substance, units still to produce, steps still to go, steps already done*) on place s_1 .

All this happens, at the chosen level of abstraction, as a single indivisible event – an *occurrence* of t_1 . However, it may occur only if the binding for c and r satisfies the list of conditions stated in the *guard* of t_1 , $[c > 0, r \neq []]$, that is, if the count c and the recipe r are meaningful and non-trivial. And, of course, the corresponding tokens required must be present at the input places.

The identifiers x, c, r, n are typed, individual variables called *CPN variables* in Design/CPN. The CPN variables are declared in the global declarations (Table 1) that are part of the CPN model. A CPN variable occurs at a transition t if it occurs freely in its guard expression or in one of the adjacent arc expressions. Its scope is t or, more precisely, a single occurrence ('firing') of t specified by a particular binding. In a single

Table 1. Global declarations for the production scheme

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(* Global Declarations *)

(* Main Datatypes *)
color Cnt    = int; var c,c',n,n' : Cnt;
color Prd    = union X + Y + Z; var x,x' : Prd;
color Prd_t  = Prd timed;
color Dvc    = union D1 + D2 + D3 declare ms; var d,d' : Dvc;

color Bch    = product Prd * Cnt;
color Bch_t  = Bch timed;
color Stp    = Dvc;
color Rcp    = list Stp; var r,r' : Rcp;
color Rdb    = product Prd * Rcp * Cnt;

(* Auxiliary Datatypes *)
color P_C_C   = product Prd * Cnt * Cnt timed;
color P_R     = product Prd * Rcp timed;
color P_C_R   = product Prd * Cnt * Rcp timed;
color P_C_R_R = product Prd * Cnt * Rcp * Rcp timed;
color P_C_S_R_R = product Prd * Cnt * Stp * Rcp * Rcp timed;

(* Parameterized Durations *)
val dur = ref (fn (t:int,x:Prd,d:Dvc,r:Rcp) => 1);
fun d_1 (x:Prd,r:Rcp) = !dur(1,x,D1,r);
fun d_2 (x:Prd,d:Dvc) = !dur(2,x,d,[]);
fun d_3 (x:Prd,d:Dvc) = !dur(3,x,d,[]);
fun d_4 (x:Prd,r:Rcp) = !dur(4,x,D1,r);
fun d_5 (x:Prd)       = !dur(5,x,D1,[]);
fun d_6 (x:Prd,r:Rcp) = !dur(6,x,D1,r);

dur := (fn (1,_,_,r) => length r
      | (2,x,d,_) => (case x of X => 1 | Y => 2 | Z => 0)
                    * (case d of D1 => 3 | D2 => 2 | D3 => 1)
      | (6,x,_,r) => (case x of X => 1 | Y => 2 | Z => 0)
                    + (length r)
      | _ => 1 );

(* Parameterized Recipes *)
val rcp = ref (fn X => [D1,D3] | Y => [D1,D2,D3,D1] | Z => [] );
fun R (x:Prd) = !rcp (x)

(* Interface ON/OFF *)
val with_interface = ref false;
fun I () = !with_interface;

(* Initial Markings *)
val AvlDvcs_0 = 2 * Dvc;
val Recipes_0 = 1^(X,R X,0) + 1^(Y,R Y,0) + 1^(Z,R Z,0);

(* Additional SML Code *)
val set = list_to_ms o remdupl; (* set from list *)

```

occurrence of t , all copies of the same individual variable at t are assigned to the same value. The same variables may be bound, independently, in other transition occurrences, including other occurrences of the same transition (e.g. for the other batch, $(Y, 2)$, waiting on place s_0).

The identifiers X, Y, Z and $D1, D2, D3$, in contrast to individual variables, are *constants* of the net system. They are declared globally in the declarations to denote the different substances and devices. Also the identifiers $Recipes^0$ and $AvlDvcs^0$ denoting the initial markings of the recipe database, s_7 , and the pool of available devices, s_8 , respectively, are syntactical constants, as well as all functions defined in ML like *set*, *rev*

and the duration functions $d_1, \dots, d_5$. Some of those ‘constants’, however, are more like *parameters*. Markings may be changed interactively between runs, and the definition of the durations contains a reference to a function *dur* that may be changed between runs as well.

Through transition t_1 one enters the kernel of the processing of a batch. It consists of two loops. The inner one is formed by transitions t_2 and t_3 and follows the recipe step by step until one unit of substance X is done. A single production step is identified, for the sake of simplicity, utilising a particular device (resource). When one unit is done, t_4 puts it on place s_3 for accumulation at s_5 . And, as long as more units have yet to be produced, t_4 also restores the recipe, decreases the count in the status vector, and enters another round; otherwise it keeps the recipe for control purposes on s_4 .

Transition t_5 accumulates the single units of a substance in the corresponding container that was put on place s_5 by transition t_1 . If the whole batch is done, t_6 puts the result ready for delivery on place s_6 , decreases the count of recipe usages on place s_7 and returns the set of devices that were reserved for the batch in the beginning to place s_8 .

The processing of the batch $(X, 3)$ alone takes 24 time units; at the end the token $(X, 3)$ on place s_6 carries the time stamp @24 if simulation starts at time zero. The processing of $(Y, 2)$ alone as well as the concurrent processing of both $(X, 3)$ and $(Y, 2)$ takes 55 time units; the available devices are sufficient to run both without interference.

2.2.2. Combinations of Pieces

The symbolic analysis of net systems deals with *expressions* for combinations of pieces, combinations of such expressions, and two kinds of transformations of expressions: *substitutions* and *distributions*. In essence it consists of calculating row sums and column sums of the transformed incidence matrix (see Section 2.2.3).

The arc expressions of a net system denote combinations of pieces. They map the set of bindings for the respective adjacent transition (transition ‘colours’) into the module of integer combinations of elements of the colour set of the respective adjacent place. Note that the so-called Boolean coefficients of arc expressions actually denote scalars. They map the truth value *false* onto *zero* (unit of addition) and *true* onto *one* (unit of multiplication).

$$[true] \cdot x = x \quad [false] \cdot x = 0 \quad [b_1, \dots, b_k] \cdot x = [b_1 \wedge \dots \wedge b_k] \cdot x \quad (7)$$

Let D be a data type and let u be an expression denoting a combination of pieces of D . We call D the *dimension* of u . The *scope* of u is assumed to be either a particular transition (e.g. if u is an arc expressions) which determines the set of typed individual variables that can occur in u or *global* if there are no free individual variables in u (e.g. if u denotes the initial marking of a place).

Expressions for combinations may be combined if they are *comparable*, i.e. if they have (1) the same dimension and (2) the same scope. The arc expressions around a single place – in one row of the incidence matrix – satisfy the first condition; their dimension is the colour set of that place. The arc expressions around a single transition – in one column of the incidence matrix – satisfy the second condition; their scope is that transition.

In order to form row sums and column sums the respective missing condition must be imposed by suitable transformations. There are two kinds of such transformations that will be used in the sequel:

1. *Substitutions*: that replace individual variables by individual terms within the given scope, i.e. observing the type constraints and not introducing new variables. The simultaneous substitution of variables x_i by terms t_i is written as $\{x_1 \leftarrow t_1, \dots, x_n \leftarrow t_n\}$, the symbol $\leftarrow$ indicating the binding and replacement.
2. *Distributions*: mappings that transform the pieces of the given data type D into combinations of pieces of another data type D' .⁷ With possibly several argument patterns p_i for pieces of D to distinguish different cases, a distribution is written as $\{p_1 \mapsto v_1 \mid \dots \mid p_n \mapsto v_n\}$, the symbol $\mapsto$ indicating the mapping.

Most often we write the application of a transformation to an expression as a product.⁸ Usually it is clear

⁷ ‘Distribution’ because it usually decomposes a given piece and recombines (distributes) the constituents into a linear combination.

⁸ It was with respect to this product that K. Lautenbach and the author were mistaken when – after having discovered the PrT nets – we tried to lift integer linear-algebraic techniques to the new level of abstraction. We considered the ring of polynomials over the set of arc expressions with integer coefficients as the proper generalisation of the ring of integers. We had some success – just enough to make us believe that we were on the right track [GeL81]. It was K. Jensen, however, who came up with the proper product – the product of functions [Jen81].

from the context which factor is the expression and which is the transformation. If needed we use ordinary juxtaposition if the transformation is on the left and the operator $\odot$ if it is on the right.

$$\tau u = \tau(u) = u \odot \tau \quad (8)$$

Example. Let c be of type *integer*, x, y of type $A = \{\underline{a}, \underline{b}, \underline{c}\}$, and r of type (A list).

$$\begin{aligned} \sigma &= \{c \leftarrow 3, r \leftarrow [\underline{a}, \underline{b}]\} && \text{substitution} \\ \delta &= \{(c, r) \mapsto c \cdot r\} && \text{distribution} \\ (c, r) \odot \sigma &= (3, [\underline{a}, \underline{b}]) \\ (c, r) \odot \delta &= c \cdot r \\ ((c, r) \odot \sigma) \odot \delta &= ((c, r) \odot \delta) \odot \sigma && \sigma \text{ and } \delta \text{ commute} \\ &= 3 \cdot [\underline{a}, \underline{b}] \end{aligned}$$

If distributions or substitutions are comparable, i.e. if the distributions have the same domain and range or the substitutions occur within the same scope, they can be combined:

$$u \odot (\tau_1 + \tau_2) = u \odot \tau_1 + u \odot \tau_2 \quad (9)$$

A combination of distributions δ_1 and δ_2 is again a distribution. Since δ_1 and δ_2 are comparable, their argument patterns can be unified and then mapped onto the combined image. For example,

$$\begin{aligned} \delta_1 &= \{(c, r) \mapsto c \cdot r\} \\ \delta_2 &= \{(c, [x, y]) \mapsto c \cdot x \cdot y\} \\ \delta_1 + \delta_2 &= \{(c, [x, y]) \mapsto (c \cdot [x, y] + c \cdot x \cdot y)\} \end{aligned}$$

In order to combine the arc expressions around a place – to form a row sum – one has to get rid of all the individual variables first; they are local to the transitions. Similarly, in order to combine the arc expressions around a transition – to form a column sum – one has to transform them first such that they all get the same dimension.

Let $X = (X_i)_{i \in I}$ be a vector of expressions of the same dimension and let $B = (B_i)_{i \in I}$ be a vector of substitutions (transition bindings) such that each B_i is within the scope of X_i and eliminates its individual variables. Then the *dot product* of X and B is

$$\langle X, B \rangle = \sum_{i \in I} X_i \odot B_i \quad (10)$$

Let $X = (X_i)_{i \in I}$ be a vector of expressions of the same scope and let $V = (V_i)_{i \in I}$ be a vector of distributions having the same range and such that for each i the domain of V_i is the dimension of X_i . Then the *dot product* of X and V is

$$\langle X, V \rangle = \sum_{i \in I} X_i \odot V_i \quad (11)$$

To end this section on pieces, let us do a little arithmetic with pieces that themselves are numbers, e.g. integers. They are underlined to distinguish them from the scalar coefficients. According to our definitions, the combination $\underline{2} + \underline{3}$ is different from $\underline{5}$, and $\underline{2} + \underline{2}$ equals $2 \cdot \underline{2}$ rather than $\underline{4}$. However, if we apply the distribution $\zeta = \{x \mapsto x \cdot (\)\}$, then $(\underline{2} + \underline{3}) \odot \zeta = 2 \cdot (\) + 3 \cdot (\) = 5 \cdot (\)$ and $(\underline{2} + \underline{2}) \odot \zeta = 2 \cdot (2 \cdot (\)) = 4 \cdot (\)$.

2.2.3. The Incidence Matrix

In essence, this paper is concerned with representing a net system by its *incidence matrix*, usually denoted as C . The incidence matrix of our running example, the production scheme in Fig. 3, is shown in Table 2.

Notation. For a matrix $A = (A_i^k)$, A_i denotes the i th row of A and A^k denotes the k th column of A . A^T denotes the *transpose* of A : $(A^T)_k^i = A_i^k$.

Dot product and matrix product are based upon the product between expressions and transformations (see (10) and (11)). Vectors are considered as one-column matrices.

$$\langle U, V \rangle = \sum_i U_i \odot V_i = U^T \cdot V \quad (12)$$

Table 2. Incidence matrix of the production scheme (border transitions and time regions omitted)

	t_1	t_2	t_3	t_4	t_5	t_6	M^0
$\mathcal{G}$	$[c > 0,$ $r \neq []]$		$[r \neq []]$	$[c > 0,$ $r = []]$	$[c \geq 0]$	$[c' = c]$	
s_0	$-(x, c)$						$1^*(X, 3)$ $+1^*(Y, 2)$
s_1	$(x, c, r, [])$	$-(x, c, d :: r, r')$	$(x, c, r, d :: r')$	$[c > 1]\%(x, c - 1,$ $rev(d :: r'), [])$			
s_2		(x, c, d, r, r')	$-(x, c, d, r, r')$	$-(x, c, d, r, r')$			
s_3				x	$-x$		
s_4				$[c = 1]\%rev(d :: r')$		$-r$	
s_5	$(x, 0, c)$				$-(x, c', c)$ $+(x, c' + 1, c)$	$-(x, c', c)$	
s_6						(x, c)	
s_7	$-(x, r, n)$ $+(x, r, n + 1)$					$-(x, r, n)$ $+(x, r, n - 1)$	$Recipes^0$
s_8	$-set(r)$					$set(r)$	$AvlDvc^0$

Each row C_s of the incidence matrix corresponds to a place s and each column C^t corresponds to a transition t . An entry C_s^t is an expression denoting a combination of pieces, namely the change at place s effected by an event at transition t . The minus sign distinguishes the input places of t , ${}^*t = pre(t)$, from its output places, $t^* = post(t)$. Accordingly C can be split into two matrices, *C and C^* , such that $C = -{}^*C + C^*$.

In addition there may be

- one row $\mathcal{G}$ with the *guards* of the transitions;
- one row $\mathcal{T}$ with the time increments of the transitions (if the model is timed);
- one column $\mathcal{D}$ with the colour sets of the places;
- one column M^0 with an initial marking of the places.

Table 2 shows the guards and the initial marking but does not show the colour sets, nor the time increments, nor the shaded border transitions $t_a, \dots, t_f$, nor the constituents of the fused places s_1, s_3 and s_4 (see Section 4.2).

An event at a transition t is determined by a binding, e , that assigns the individual variables of t to values such that the *guard* of t is satisfied. Its effect to a place s is given by applying e to the corresponding entry of the incidence matrix,

$$\Delta M_s = C_s^t \odot e \quad (13)$$

Keep in mind that the guard of t , $\mathcal{G}_t$, is a Boolean coefficient that is common to all arc expressions around t ; hence it is evaluated as part of $C_s^t \odot e$.

Event e is enabled – may occur – at state M if (see equations (4)–(6))

$$M - {}^*C^t \odot e \geq 0 \quad (14)$$

This is the so-called ‘weak enabling’ which is derived from (2) through a series of abstractions.

A process $M_1 \xrightarrow{P} M_2$ leads from state M^1 to state M^2 through a partially ordered set of event occurrences, P . If we ignore the order of occurring we get a T-vector, u , of combinations of events – a *T-marking* – that is called the *action* performed by the process. The state difference is

$$\Delta M = M^2 - M^1 = C \cdot u \quad (15)$$

A non-trivial action u is called a *cycle*⁹ iff $C \cdot u = 0$. Every process performing a cycle leads from a state to the same state again.

If δ is a distribution whose domain is the colour set of a place s , δ can be applied to all entries of row C^s of the incidence matrix; it is a distribution of the place s . Now let V be an S-vector such that for each place s the component V_s is a combination of distributions of s and all the V_s have the same range. Then the transposed incidence matrix C^T as well as every transposed marking M^T can be multiplied by the one column matrix V .

$$(C^T \cdot V)_t = \sum_{s \in S} C_s^t \odot V_s \quad (16)$$

$$M^T \cdot V = \sum_{s \in S} M_s \odot V_s \quad (17)$$

Obviously, any such S-vector V defines a state quantity of the net system represented by C . The dot product $M^T \cdot V$ is the value of V at state M , and soon (Section 6.2, formula (38)) it will turn out that $C^T \cdot V$ is the *gradient* of V , the T-vector of partial derivatives of V with respect to the transitions. If the gradient is the zero vector, V is constant.¹⁰

A state quantity V defined by a vector of (combinations of) distributions is linear in the markings. For any coefficient k and two markings $M^a = (M_1^a, \dots, M_m^a)$ and $M^b = (M_1^b, \dots, M_m^b)$,

$$\begin{aligned} V(M^a + M^b) &= V(M^a) + V(M^b) \\ V(k \cdot M^a) &= k \cdot V(M^a) \end{aligned} \quad (18)$$

Conversely, any linear state quantity V can be represented by an S-vector of distributions, $(V_1, \dots, V_m)$. Because of linearity, we have for any marking M

$$V(M) = V(M_1, \dots, M_m) = V \left(\sum_{s=1}^m (0, \dots, M_s, \dots, 0) \right) = \sum_{s=1}^m V(0, \dots, M_s, \dots, 0) \quad (19)$$

Hence we set

$$V_s = \{p_s \mapsto V(0, \dots, p_s, \dots, 0)\} \quad (20)$$

where p_s is a generic pattern for the colour set of place s .

3. Characteristic Functions

3.1. Dynamics

Two systems of the same type may be quite different dynamically. They are *equal* if all dependent variables are in pairs represented by the same functions of the coordinates. To tell them apart, however, it is sufficient to consider *one* particular function.

Fundamental Law of Dynamics. For any family of systems of the same type with n degrees of freedom there exists at least one way to select $n + 1$ variables $X_1, \dots, X_n, Y$ such that the X_i form a coordinate system and the function $Y = Y(X_1, \dots, X_n)$ identifies each individual system of the family. $Y = Y(X_1, \dots, X_n)$ is called the *characteristic function* of the individual system associated with the coordinate system $X_1, \dots, X_n$.

Note that characteristic function and coordinate system belong together. In general there are different characteristic functions associated with different coordinate systems.

For every system there is a coordinate system for which *energy* is the characteristic function. The two

⁹ Also known as a *T-invariant*.

¹⁰ Also known as an *S-invariant*.

systems of Fig. 1, for example, are characterised by the quite different energy functions

$$\begin{aligned}\Sigma^a: E &= Kx + E_0^a \quad (K = mg) \\ \Sigma^b: E &= \frac{\kappa}{2}x^2 + E_0^b \quad (x < 0)\end{aligned}\quad (21)$$

K (mass m times gravitational constant g) denotes the attractive gravitational force, $-\kappa x$ the restoring force exerted by the spring against the displacement x .

It is always possible to select coordinates $X_1, \dots, X_n$ in such a way that energy E is the characteristic function and is homogeneous of degree 1 in the coordinates.

$$E = \sum_{j=1}^n \xi_j X_j, \quad \xi_j = \frac{\partial E(X_1, \dots, X_n)}{\partial X_j} \quad (22)$$

The variables $E, X_1, \dots, X_n$ are called *extensive*, and the energy conjugates ξ_j are called *intensive*. The extensive variables are the natural exchange variables. Examples: energy (including mass as rest energy), linear momentum, angular momentum, electrical charge, entropy, particle numbers, volume. The dynamical variables – extensive and intensive ones – have fixed zeros; they are ‘absolute’ like ‘absolute’ temperature, pressure, charge, particle numbers.

3.2. Net Theory

Net theory answers the question whether two systems are equal by stating the *extensionality principle* (see equation (2)). This means that the *en-ability* $\mathcal{E}$ of a state c , the set of events enabled at c , is a *characteristic function* of net systems – the only one known.

$$e \in \mathcal{E}(c) \iff \bullet e - e^\bullet \subseteq c \wedge e^\bullet - \bullet e \subseteq \bar{c} \quad (23)$$

Hence the extensionality principle may also be read in the following way.

Fundamental Law of Net Theory. Two net systems Σ^a and Σ^b of the same type with n places are equal if their en-ability functions $\mathcal{E}^a = \mathcal{E}^a(s_1^a, \dots, s_n^a)$ and $\mathcal{E}^b = \mathcal{E}^b(s_1^b, \dots, s_n^b)$, are equal.

In higher-level net models classes of conditions are abstracted to places and classes of events are abstracted to transitions. A condition at place s is given by the presence of a piece – token – on s . An event at transition t is given by a binding of the individual variables of t satisfying the guard of t ; enabling and effect are described in (13) and (14). This defines the system variable en-ability, $\mathcal{E} = \mathcal{E}(M)$, for markings M . $\mathcal{E}(M)$ gives for every event the multiplicity of its enabling in M .

4. Composition of Systems

4.1. Dynamics

The composition of two systems is defined by the addition of their – *comparable* – characteristic functions. For example, the composition of the two separate systems in Fig. 1 yields

$$E = E^a + E^b = Kx^a + \kappa \frac{x^{b2}}{2} + E_0^a + E_0^b \quad (24)$$

Through the coupling $x^a + x^b = \text{const.} = 0$, $x = x^a = -x^b$ (see Fig. 2) we get

$$E = E^a + E^b = Kx + \kappa \frac{x^2}{2} + E_0 \quad (x < 0) \quad (25)$$

In the total system, processes of the form

$$\begin{aligned}X_1^a + X_1^b &= X_1 = \text{const.}, \dots, X_j^a + X_j^b = X_j = \text{const.}, \\ X_{j+1}^a &= \text{const.}, \dots, X_m^a = \text{const.}, \\ X_{j+1}^b &= \text{const.}, \dots, X_n^b = \text{const.}\end{aligned} \quad (26)$$

Fig. 4. Coupling of net systems by place fusion and by transition fusion.

are called exchange processes. The (extensive or intensive) variables X_k^σ form a complete coordinate system. With respect to the exchange process the variables $X_1, \dots, X_j$ are called *exchange variables* and their conjugates $\xi_1, \dots, \xi_j$ *contact variables*. The extensive variables are the preferred, ‘natural’ exchange variables yet not the only ones. In a stable equilibrium the contact variables have the same value.

For the exchange process $E^a + E^b = E = \text{const.}$ in Fig. 2 and equation (25), energy is the exchange variable and the conjugate force $\frac{\partial E}{\partial x}$ is the contact variable. In stable equilibrium, the two forces $\frac{\partial E_a}{\partial x} = K$ and $\frac{\partial E_b}{\partial x} = -Kx$ permanently have the same value: $x = x_0 = x(t=0) = -\frac{K}{\kappa}$; the system is at a standstill.

4.2. Net Theory

The composition of two net systems is defined by the addition (disjoint union) of their nets (graphs and matrices). Coupling can be achieved in two ways: through the fusion of places and the fusion of transitions. Figure 4 suggests that place fusion may be viewed as an abstraction from the ‘free exchange’ of tokens between comparable places (analogous to equation (1)), and transition fusion as an abstraction from the ‘synchronous communication’ between comparable transitions [ChH94].

The production scheme in Fig. 3 can be seen as the result of coupling the production with an administration; the places s_1 and s_3 are the result of fusing the respective border places, s_i and s'_i ($i = 1, 3, 4$). The system may be closed by fusing the shaded border transitions with corresponding complements in an environment.

5. Time

5.1. Dynamics

A process of a physical system appears as a continuous sequence of states. The variables are functions of the independent parameter *time*. The characteristic function of a system (see Section 3.1) becomes $Y = Y(X_1, \dots, X_n, t)$.

The coupled mass-spring system in Fig. 2 reacts to a displacement from its equilibrium with oscillating. We describe the states – the values of the single coordinate x – as a function of time, $x = x(t)$. The family of such functions, depending on the initial displacement, is determined by integrating the differential equation

$$\frac{d\mathbf{P}}{dt} = m \frac{d^2x}{dt^2} = K - \kappa x \quad (x < 0, K = -mg) \quad (27)$$

We get

$$x(t) = \hat{x} \cos \omega t + x_0 \quad \left(x < 0, x_0 = \frac{-mg}{\kappa}, \hat{x} = x(t=0) - x_0, \omega = \sqrt{\frac{\kappa}{m}} \right) \quad (28)$$

5.2. Net Theory

In net theory, a process $M_1 \xrightarrow{P} M_2$ is a set of event occurrences, *partially* ordered by a causal relation *earlier*, connecting two states. Every occurrence of an event determines a *point in time*; conditions hold for a *duration* greater than zero. The duration of a process is the maximum length of the maximal condition sequences. The total temporal order of event occurrence is stricter than the partial causal order *earlier*. A state can be determined only at times when no event occurs.

Let us now try to determine the course of a process in time explicitly. The occurrence of an event affects the enabling of other events through the starting and finishing of condition holdings. We now say that every event $e = \Delta t$ determines a *minimal duration* for all conditions that begin to hold with an occurrence of e . And the point in time of its occurrence is the earliest possible such that no condition finishes to hold before its minimal duration is over.

In timed models of *Design/CPN* [CPN96], the tokens carry *time stamps* that hold the earliest point in (simulation) time at which the token may be consumed by an event occurrence. The value of the time stamp of a token created is determined as a function of the point in time of the creation and an increment connected with the token in the corresponding arc expression; a time region of a transition may hold an increment that is common to all the tokens created. The point in time of a transition occurrence then is the maximal time stamp value of all tokens being consumed.

Time does not play a role for the rest of this paper. The time increments of the production scheme in Fig. 3 will be ignored.

6. Forms of State Quantities

6.1. Dynamics

Energy appears in different forms, and it can be converted from one form into another.

Examples.

- (a) *Energy of motion*: the change of energy dE appears together with a change of momentum; $d\mathbf{P} \cdot \mathbf{v} = \mathbf{v}(\mathbf{P})$ denotes the velocity of the body in motion.

$$dE = \mathbf{v} d\mathbf{P} = v_x dP_x + v_y dP_y + v_z dP_z$$

- (b) *Energy of stress*: the change of energy dE appears together with a displacement dx and a change dV of a volume V (component of force $K_x = \mathbf{K}(x)$, pressure $p = p(V)$).

$$dE = K_x dx \quad \text{respectively} \quad dE = p dV$$

- (c) *Thermal energy*: the change of energy dE appears together with a change of dS of entropy (absolute temperature $T = T(S)$).

$$dE = T dS$$

An energy form is defined in terms of an extensive coordinate X : $dE = \xi dX$, where the intensive variable ξ may depend on X (and other independent variables).

In general, if $X_1, \dots, X_n$ are coordinates of a system with n degrees of freedom, any infinitesimal change of a dependent variable $Y = Y(X_1, \dots, X_n)$ is a linear combination of n different *forms* in which Y appears in terms of the different coordinates.

$$dY = \frac{\partial Y}{\partial X_1} dX_1 + \dots + \frac{\partial Y}{\partial X_n} dX_n \quad (29)$$

6.2. Net Theory

Since the places are the coordinates of a net system, we now want to study in which form a state quantity of a net system may appear at the different places.

There are no infinitesimal changes in a net system. The ‘smallest’ changes are single events. Let Δt be an event at transition t given by a binding e for t . The effect of an occurrence of Δt on a single place s is $C_s^t \odot e$; see (13). We now set out to interpret, and write, C_s^t as the *partial derivative* of coordinate s with respect to transition t and rewrite (13) as

$$\Delta s = \Delta M_s = C_s^t \odot e \cong \frac{\partial s}{\partial t} \Delta t \quad (30)$$

The symbol $\cong$ is to denote an analogy whose power we have yet to explore. There is no claim that the use of some notation of calculus may lead to more than certain similarities. In fact the similarities are limited to *linear* quantities of net systems (see (18) and (20)). In this restricted area, however, they seem quite striking and worth exploring. And they are not purely accidental. At the end of our exploration, when we have seen more of the purpose and power of this analogy, the proper foundation in calculus will be shown (Section 7.3).

To begin with, let us have a look at the effect of an event Δt on a linear state quantity given by an S-vector of distributions.

$$\Delta V = V(\Delta M) = \sum_s V_s(\Delta s) = \sum_s V_s \left(\frac{\partial s}{\partial t} \Delta t \right) \quad (31)$$

We interpret the distributions V_s as partial derivatives of the variable V with respect to the coordinates s ,

$$\frac{\partial V}{\partial s} \cong V_s \quad (32)$$

and get

$$\Delta V = \frac{\partial V}{\partial s_1} \Delta s_1 + \dots + \frac{\partial V}{\partial s_n} \Delta s_n \quad (33)$$

Matching this to (29) we call the distribution V_s the *form* in which the state quantity V appears at the place s . V_s shows in which way the pieces on place s contribute to the value of quantity V .

If the two derivatives make sense or, at least, are consistent so far there should also be the partial derivative of V with respect to transition t such that

$$\Delta V = \frac{\partial V}{\partial t} \Delta t \quad (34)$$

$$\frac{\partial V}{\partial t} = \sum_s \frac{\partial V}{\partial s} \frac{\partial s}{\partial t} \quad (35)$$

The following two parallel derivations for the effect ΔV of an event $e = \Delta t$ on V demonstrate that this is

Table 3. Three linear variables of the production scheme given by their forms

	load		recipe		count
s_0	$(x, c) \mapsto c \cdot x$	$-$	$\mapsto$	$\mathbf{0}$	$(-, c) \mapsto [c > 0] \cdot c'()$
s_1	$(x, c, -, -) \mapsto c \cdot x$	$(-, -, r, r')$	$\mapsto$	$1'(rev(r') \wedge r)$	$(-, c, -, -) \mapsto [c > 0] \cdot c'()$
s_2	$(x, c, -, -, -) \mapsto c \cdot x$	$(-, -, d, r, r')$	$\mapsto$	$1'(rev(d :: r') \wedge r)$	$(-, c, -, -, -) \mapsto [c > 0] \cdot c'()$
s_3	$x \mapsto 1 \cdot x$	$-$	$\mapsto$	$\mathbf{0}$	$- \mapsto 1'()$
s_4	$x \mapsto \mathbf{0}$	r	$\mapsto$	$1 \cdot r$	$- \mapsto \mathbf{0}$
s_5	$(x, c', -) \mapsto c' \cdot x$	$-$	$\mapsto$	$\mathbf{0}$	$(-, c', -) \mapsto [c \geq 0] \cdot c'()$
s_6	$(x, c) \mapsto c \cdot x$	$-$	$\mapsto$	$\mathbf{0}$	$(-, c) \mapsto [c > 0] \cdot c'()$
s_7	$- \mapsto \mathbf{0}$	$(-, r, n)$	$\mapsto$	$-n \cdot r$	$- \mapsto \mathbf{0}$
s_8	$- \mapsto \mathbf{0}$	$-$	$\mapsto$	$\mathbf{0}$	$- \mapsto \mathbf{0}$

indeed the case.

$$\Delta V = \sum_s V_s (C_s^t \odot e) \stackrel{!}{=} \sum_s (C_s^t \odot V_s) \odot e = \left(\sum_s C_s^t \odot V_s \right) \odot e = (C^T \cdot V) \odot e \quad (36)$$

$$\Delta V = \sum_s \frac{\partial V}{\partial s} \left(\frac{\partial s}{\partial t} \Delta t \right) = \sum_s \left(\frac{\partial V}{\partial s} \frac{\partial s}{\partial t} \right) \Delta t = \left(\sum_s \frac{\partial V}{\partial s} \frac{\partial s}{\partial t} \right) \Delta t = \frac{\partial V}{\partial t} \Delta t \quad (37)$$

The essential step in (36) is due to the fact that distributions and substitutions commute. Equation (36) also shows how to compute $\frac{\partial V}{\partial t}$ as the dot product of C^t and V . Consequently, the matrix product of C^T with the one column matrix V yields the vector of partial derivatives with respect to the transitions, the *gradient* of V , ∇V .

$$\nabla V = \left(\dots, \frac{\partial V}{\partial t}, \dots \right) = (C^T \cdot V) \quad (38)$$

If the gradient is the zero vector, V is constant, $Y = \text{const.} = Y(M^0)$; it is what in the literature is known as an *S-invariant* or a *place semi-flow*.

If V can be split into two non-trivial, non-negative variables U and W such that $V = U - W$, i.e. if V has mixed signs, and V is constant, U and W are, up to a constant, copies of each other: $U = W + V(M^0)$. The notion *copy* does not appear in the dynamics of continuous systems but is of fundamental importance in digital information processing. *Digital* data are distinguished from *analogue* data by the fact that exact copies can be made of digital data arbitrarily often: $\text{copy}(\text{copy}(\dots(\text{copy}(d))\dots)) = d$.

It seems high time for some examples. Table 3 shows three state quantities of the production scheme (Fig. 3) that are given by the forms in which they appear at the different places. The quantity *load* is the actual workload of the system, the sum of all batches currently being present somewhere, in one of the five following forms:

$load(M) =$	$M(s_0) \odot \{(x, c) \mapsto c \cdot x\}$	order waiting
$+$	$M(s_1) \odot \{(x, c, \rightarrow, -) \mapsto c \cdot x\}$	batch (remainder) being started
$+$	$M(s_2) \odot \{(x, c, \rightarrow, \rightarrow, -) \mapsto c \cdot x\}$	batch at a station
$+$	$M(s_3) \odot \{x \mapsto 1 \cdot x\}$	single unit done
$+$	$M(s_5) \odot \{(x, c, -) \mapsto c \cdot x\}$	units on shelf
$+$	$M(s_6) \odot \{(x, c) \mapsto c \cdot x\}$	product ready for delivery

Similarly the quantity *recipe* gives the recipes for those batches currently being processed. *Count* is an auxiliary quantity that looks at the meaningful, i.e. non-negative multiplicities of the batches. It can be used to prove that the condition $c > 0$ in the guard of t_4 that comes in handy for the reduction of partial derivatives at t_4 is redundant.

Table 4 shows the result of applying each form of *load* to all entries of the corresponding row of the incidence matrix. In addition, the bottom row shows the resulting column sums: the gradient of quantity *load*. Now note that at t_6 , the guard says $c' = c$. Hence the derivative is zero. At t_4 , the guard says $c > 0$. Formally,

Table 4. Gradient of the workload, $C^\top \cdot load$

	t_1	t_2	t_3	t_4	t_5	t_6
Γ	$[c > 0, r \neq []]$		$[r \neq []]$	$[c > 0, r = []]$	$[c' = c]$	
s_0	$-c \cdot x$					
s_1	$c \cdot x$	$-c \cdot x$	$c \cdot x$	$[c > 1] \cdot (c - 1) \cdot x$		
s_2		$c \cdot x$	$-c \cdot x$	$-c \cdot x$		
s_3				x	$-x$	
s_4						
s_5	$0 \cdot x$				$-c' \cdot x + (c' + 1) \cdot x$	$-c' \cdot x$
s_6						
s_7						
s_8						$c \cdot x$
$\nabla load$	0	0	0	$-[c > 1] \cdot (x - c \cdot x)$	0	$-c' \cdot x + c \cdot x$

Table 5. Gradient of the variable *recipe*

	t_1	t_2	t_3	t_4	t_5	t_6
Γ	$[c > 0, r \neq []]$		$[r \neq []]$	$[c > 0, r = []]$	$[c \geq 0]$	
$\nabla recipe$	$(rev[])^{\wedge} r$ $-r$	$-rev(r')^{\wedge} (d :: r)$ $+rev(d :: r')^{\wedge} r$	0	$[c > 1] \cdot (rev[])^{\wedge} rev(d :: r')$ $-rev(d :: r')^{\wedge} r$ $+ [c = 1] \cdot rev(d :: r')$	0	0

a guard is a Boolean coefficient that is common to all arc expressions. We can multiply the derivative at t_4 with $[c > 0]$. Since $[c > 0] = [c = 1] + [c > 1]$ and $[c > 0] \cdot [c > 1] = [c > 1]$, we get

$$-[c > 0] \cdot [c > 1] \cdot (x - c \cdot x) = -[c > 1] \cdot (x - c \cdot x) = [c = 1] \cdot (x - c \cdot x) = \mathbf{0}$$

The gradient of quantity *load* is zero; the value *load* is constant. The value of *load* for some marking M is the dot product of M and *load*.

$$\begin{aligned} load(M) &= load(M^0) = ((\underline{X}, 3) + (\underline{Y}, 2)) \odot \{(x, c) \mapsto c \cdot x\} + \dots 0 \dots \\ &= 3 \cdot \underline{X} + 2 \cdot \underline{Y} \end{aligned}$$

In the same way the gradient of the quantity *recipe* is calculated. The result is shown in Table 5. Since the calculation was done with the help of the experimental software package rather than with paper and pencil, the details are given in Section A.4 of the Appendix.

The partial derivatives with respect to transitions t_1 and t_2 reduce to zero due to the properties of the list operator *rev*. And at t_4 , this time both $c > 0$ and $r = []$ of the guard are used to reduce the partial derivative to zero. Hence we have

$$recipe(M) = const. = recipe(M^0) = M_0^\top \cdot recipe = \mathbf{0}$$

In contrast to *load* the forms of *recipe* have mixed signs.

$$recipe(M) = recipe^{(\geq 0)}(s_1, s_2, s_4) - recipe^{(\leq 0)}(s_7) = \mathbf{0}$$

$$recipe^{(\geq 0)}(M) = recipe^{(\leq 0)}(M)$$

A copy of the recipe of every batch being processed, distributed over the places s_1 , s_2 and s_4 , is kept on place s_7 .

7. Process Quantities

7.1. Dynamics

Quantities like *work* or *heat* belong to processes rather than states. If, for example, the trajectory of a point mass m is $P = \vec{r}(t) = (x(t), y(t), z(t)) (t_a \leq t \leq t_e)$ and the force acting on m is the vector field $\vec{F}(x, y, z)$, the work done by $\vec{F}$ in the course of P is the line integral

$$W = \int_P \vec{F} d\vec{r} = \int_{t_a}^{t_b} \left(\vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt} \right) dt \quad (39)$$

If the work depends on the initial and terminal states only rather than on the course of the process itself, the force is *conservative* (without friction, in particular). In this case there exists a state quantity $U = U(x, y, z)$ such that $\vec{F}$ is the negative gradient field of the variable U , $\vec{F} = -(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}) = -\nabla U$. U is unique up to a constant U_0 and is called the *potential function* of $\vec{F}$. The $-$ sign is to ensure that if $\vec{F}$ is the gravitational force then U is the potential energy. $W = U(s_2) - U(s_1)$ is the work done *against* $\vec{F}$ by going from state (position) $\vec{r}_1$ to $\vec{r}_2$.

7.2. Net Theory

In net systems a process $M_1 \xrightarrow{P} M_2$ is a partially ordered finite set of event occurrences that lead from the initial to the terminal state. Let $\{\Delta t_i\}_{0 \leq i \leq n}$ be a ‘firing sequence’, i.e. an enumeration of the event occurrences of P that is compatible with the partial order $<$ (*earlier*),

$$\Delta t_i < \Delta t_k \Rightarrow i < k \quad (40)$$

Index i corresponds to the parameter time; the transition occurrences are the time points.

Let $\mathcal{F} = (\dots, \mathcal{F}_t, \dots)$ be a T-vector of expressions denoting combinations of pieces of the same type such that for each t the individual variables of $\mathcal{F}_t$ belong to t . The T-vector $\mathcal{F}$ uniquely determines a linear process quantity W by virtue of

$$W(P) = \sum_i \mathcal{F}_t \odot \Delta t_i = \mathcal{F}^T \cdot P \quad (41)$$

which we now interpret, in analogy to (39), as the *line integral* of the *field* $\mathcal{F}$ along the process P .

$$W(P) = \sum_i \mathcal{F}_t \odot \Delta t_i \cong \int_P \mathcal{F} dT \quad (42)$$

In analogy to the forms in which a state quantity appears at the places, we call the components $\mathcal{F}_t$ the *forms* of the process quantity W at the transitions t .

Example. The gradient of a linear state quantity V (see equation (38)) gives the forms in which the work on V is done at the transitions during a process.

Non-Example. In general, in the presence of concurrency, the duration of a process cannot be determined by a field.

The following facts on the relationship between derivatives and integrals will provide a kind of final test for the consistency of the notions and notation transferred from physics to net theory.

Theorem 7.1. If $\mathcal{F} = \nabla V$ then $\int_P \mathcal{F} dT$ is the change $\Delta V = V(M_2) - V(M_1)$ effected by some process $M_1 \xrightarrow{P} M_2$.

Proof. Immediate consequence of (36). $\square$

Theorem 7.2. The derivative of a distribution $\{(x_1, \dots, x_n) \mapsto u\}$ is its image expression u , and the indefinite integral of an expression u is the distribution $\{(x_1, \dots, x_n) \mapsto u\}$ where the x_i are the free individual variables of u .

Proof. Consider the following trivial net:

Let u be an arbitrary expression denoting a combination of pieces in terms of the individual variables $x_1, \dots, x_n$. Let V be the linear quantity defined by the corresponding distribution $\delta = \{(x_1, \dots, x_n) \mapsto u\}$. Let p be an arbitrary combination of pieces on s and, with pattern matching $\equiv$, $\hat{p} = (x_1, \dots, x_n) \equiv p$ the corresponding combination of substitutions. Then according to our definitions

$$\frac{dV}{dt} = (x_1, \dots, x_n) \odot \delta = u \quad (43)$$

$$\int_0^p u dt = u \odot \hat{p} = p \odot \delta = V(p) \quad (44)$$

$\square$

Theorem 7.3. For a field $\mathcal{F}$ there exists a potential function, a state quantity U such that $\mathcal{F} = -\nabla U$, iff for any two processes P and P' connecting the same pair of states the value of the line integral of $\mathcal{F}$ is the same:

$$M_1 \xrightarrow[P']{P} M_2 \Rightarrow \int_P \mathcal{F} dT = \int_{P'} \mathcal{F} dT \quad (45)$$

or, equivalently, iff for any cyclic process P the integral equals 0 :

$$M \xrightarrow{P} M \Rightarrow \oint_P \mathcal{F} dT = 0 \quad (46)$$

Proof.

- (a) Assume there is a state quantity U with $\mathcal{F} = -\nabla U$ and let P, P' be such that $M_1 \xrightarrow{P} M_2$ and $M_1 \xrightarrow{P'} M_2$. Then $\int_P \mathcal{F} dT = -U(M_2 - M_1) = \int_{P'} \mathcal{F} dT$.
- (b) Assume that $\int_P \mathcal{F} dT = \int_{P'} \mathcal{F} dT$ for any P, P' with $M_1 \xrightarrow{P} M_2$ and $M_1 \xrightarrow{P'} M_2$. Let $U_0 = U(M_0) = 0$ for the initial marking M_0 . For any reachable marking M there is a process P such that $M_0 \xrightarrow{P} M$. Set $U(M) = -\int_P \mathcal{F} dT$; U is a proper mapping since its value does not depend on the path leading from M_0 to M .

U is linear and can be presented as an S-vector of distributions $(\dots, U_s = \frac{\partial U}{\partial s}, \dots)$ (see (20)). For any event $e = \Delta t$,

$$-\mathcal{F}_t \odot e = U(\Delta M) = \sum_s U_s (C_s^t \odot e) = \left(\sum_s \frac{\partial U}{\partial s} \frac{\partial s}{\partial t} \right) \odot e = \frac{\partial U}{\partial t} \odot e \quad (47)$$

Hence $\mathcal{F}_t = -\frac{\partial U}{\partial t}$, $\mathcal{F} = -\nabla U$. $\square$

An Example

For the production scheme in Fig. 3, let some *production costs* be given in units $\underline{u}$ per substance and per transition by the T-vector

$$cost = \begin{array}{c|c} & \hline t_1 & 0\underline{u} \\ t_2 & K_2(x, d)\underline{u} \\ t_3 & 0\underline{u} \\ t_4 & 0\underline{u} \\ t_5 & K_5(x)\underline{u} \\ & \hline \end{array}$$

where

$$K_2(x, d) = \begin{array}{c|ccc} & D_1 & D_2 & D_3 \\ \hline X & 4.5 & 2.7 & 3.1 \\ Y & 3.0 & 2.0 & 5.0 \\ Z & 0.0 & 0.0 & 0.0 \end{array} \quad \text{and} \quad K_5(x) = \begin{array}{c|c} & \hline X & 1.4 \\ Y & 1.0 \\ Z & 0.0 \end{array}$$

The vector *cost* is the cost field of production. If, for example, P is the process of producing three copies of item X , the costs made by P are

$$\int_P cost \, dT = cost^T \cdot P = \begin{array}{l} 3 \cdot K_2(X, D1) \underline{u} \\ + \quad 3 \cdot K_2(X, D3) \underline{u} \\ + \quad 3 \cdot K_5(X) \underline{u} \end{array} = 27.0 \underline{u}$$

In this simple example, at any stage of production the remaining costs for processing the current work load can be predicted. There is a state quantity $COST$ such that for any process $M_1 \xrightarrow{P} M_2$, the costs of P , $\int_P cost \, dT$, are $COST(M_1) - COST(M_2)$. $COST$ can be defined by its forms in the following way:

$$COST = \begin{array}{c|ccc} & & & \hline s_0 & (x, c) & \mapsto & c \cdot (K_5(x) + H(x, r)) \underline{u} \\ s_1 & (x, c, r, r') & \mapsto & (H(x, r) + (c - 1) \cdot H(x, rev(r')^{\wedge} r)) \underline{u} \\ s_2 & (x, c, d, r, r') & \mapsto & (H(x, r) + (c - 1) \cdot H(x, rev(d :: r')^{\wedge} r)) \underline{u} \\ s_3 & - & \mapsto & 0 \underline{u} \\ s_4 & - & \mapsto & 0 \underline{u} \\ s_5 & (x, c', c) & \mapsto & (c - c') \cdot K_5(x) \underline{u} \\ s_6 & - & \mapsto & 0 \underline{u} \\ s_7 & - & \mapsto & 0 \underline{u} \\ s_8 & - & \mapsto & 0 \underline{u} \\ & \hline \end{array}$$

where $H(x, r)$ is an auxiliary function that for substance x and remainder r of its recipe determines the costs at transition t_2 :

$$H = \{(x, []) \mapsto 0 \mid (x, [d]) \mapsto K_2(x, d) \mid (x, r^{\wedge} r') \mapsto H(x, r) + H(x, r')\}$$

The quantity $COST$ is a potential function for the cost field *cost*. The negative of the product of the incidence matrix C with $COST$ reduces to *cost* (by means of the guards of t_1 and t_4); *cost* is the negative gradient field of $COST$, $-\nabla COST = cost$. The calculation shown in Table 6 was done by means of the software tool; details are given in the Appendix (see A.5).

If we extend the system by the shaded border transitions t_a : ‘accept new order’ and t_b : ‘deliver product’, there are cyclic processes with non-zero costs and hence no potential function can exist for the cost field (Theorem 7.3).

Table 6. Calculating of a negative gradient field.

$-\nabla COST =$		t_1	—	$[c > 0, r \neq []] \cdot H(x, r) \cdot \underline{u}$
			+	$[c > 0, r \neq []] \cdot H(x, rev([])) \cdot \underline{u}$
			+	$[c > 0, r \neq []] \cdot c \cdot H(x, r) \cdot \underline{u}$
			—	$[c > 0, r \neq []] \cdot c \cdot H(x, rev([])) \cdot \underline{u}$
		t_2	—	$H(x, r) \cdot \underline{u}$
			+	$H(x, d :: r) \cdot \underline{u}$
			—	$H(x, rev(r') \wedge (d :: r)) \cdot \underline{u}$
			+	$H(x, rev(d :: r') \wedge r) \cdot \underline{u}$
			+	$c \cdot H(x, rev(r') \wedge (d :: r)) \cdot \underline{u}$
			—	$c \cdot H(x, rev(d :: r') \wedge r) \cdot \underline{u}$
		t_3		$0 \cdot \underline{u}$
		t_4		$[c > 0, r = []] \cdot H(x, r) \cdot \underline{u}$
			—	$[c > 0, r = []] \cdot H(x, rev(d :: r') \wedge r) \cdot \underline{u}$
			+	$[c > 0, r = []] \cdot c \cdot H(x, rev(d :: r') \wedge r) \cdot \underline{u}$
			—	$[c > 0, r = []] \cdot [c > 1] \cdot H(x, rev(d :: r')) \cdot \underline{u}$
			—	$[c > 0, r = []] \cdot [c > 1] \cdot c \cdot H(x, rev([]) \wedge rev(d :: r')) \cdot \underline{u}$
			+	$2 \cdot [c > 0, r = []] \cdot [c > 1] \cdot H(x, rev([]) \wedge rev(d :: r')) \cdot \underline{u}$
		t_5		$K_5(x) \cdot \underline{u}$
		t_6		$[c' = c] \cdot c \cdot K_5(x) \cdot \underline{u}$
			—	$[c' = c] \cdot c' \cdot K_5(x) \cdot \underline{u}$
		t_1	$0 \cdot \underline{u}$	t_1
		t_2	$H(x, [d]) \cdot \underline{u}$	t_2
		t_3	$0 \cdot \underline{u}$	t_3
		t_4	$0 \cdot \underline{u}$	t_4
		t_5	$K_5(x) \cdot \underline{u}$	t_5
		t_6	$0 \cdot \underline{u}$	t_6
		$= cost$		

7.3. The Incidence Matrix and Partial Derivatives

Now that we have seen the extent to which the analogy between dynamics and net theory works, let us return to the question whether the similarities are purely accidental.

Assume that C is the incidence matrix of a conventional Petri net without loops. Then the entries of C are just integer numbers; the markings are place vectors of non-negative integers, and for every marking (state) $M = (s_1, \dots, s_n)$ in the full reachability class of the initial marking M^0 there is a transition vector $u = (t_1, \dots, t_m)$ of integers such that (see equation (15))

$$M = M^0 + C \cdot u = M^0 + \sum_{1 \leq k \leq m} t_k \cdot C^k \quad (48)$$

Note that we identify the transition t_k and the variable t_k telling the number of times t_k occurs. If t_k is negative, M is in the past of the occurrences of t_k .

Let us now consider two generalisations which are purely mathematical rather than net theoretical. We ignore the restriction that the markings of places are non-negative, i.e. we ignore the enabling rule for transitions. And we allow arbitrary changes, in particular arbitrarily small changes, i.e. we allow the s_i and t_k to be *real* numbers. Then

$$M = (s_1, \dots, s_n) = M(t_1, \dots, t_m) = M^0 + C \cdot u \quad (49)$$

is a continuous vector function in m variables. With equation (48) we get trivially:

$$\frac{\partial M}{\partial t_k} = \left(\frac{\partial s_1}{\partial t_k}, \dots, \frac{\partial s_n}{\partial t_k} \right) = C^k \quad \frac{\partial s_i}{\partial t_k} = C_i^k \quad (50)$$

In other words, our notation making use of partial derivatives is firmly based on calculus. We have just lifted equation (50) and its implications to higher-level Petri nets. The similarities we observed between dynamics and net theory are a matter of course rather than accidental. They give additional evidence to the fact that Petri nets are adequate models of systems with combined continuous and discrete dynamics, that Petri nets are truly inter-disciplinary. And this, in turn, is a consequence of *Petri's* careful founding of net theory in accordance with the laws of physics (see [Pet62, Pet82], for example).

8. Conservation Laws

Laws of conservation and invariance are of fundamental importance for the understanding of systems and their embedding into environments. They can tell us that something is impossible – like a *perpetuum mobile*, for example, delivering more energy than it receives. They can help to adjust the border between a system and its environment, i.e. not to ignore certain phenomena if there is significant loss or gain of a quantity that should not change. The fact that the value of a particular variable, like *load* of the production scheme, remains constant within given system borders may help to prove that a design is correct with respect to a specific requirement.

8.1. Dynamics

In Section 2.1 the exchange of variables between systems defined their comparability. This does not mean, however, that all variables can change by exchange only, i.e. through an interaction such that equation (1) is satisfied. Yet there are variables for which this is the case. They are of immense importance since they satisfy a universal *conservation law*. This leads to the following classification of dynamical variables:

1. Variables that change their value by exchange only, that satisfy a universal conservation law. Processes changing such a variable need the participation of other systems.
2. Variables that may change in other ways but pure exchange. For those variables the processes can be divided into
 - (a) those during which the respective variable changes by exchange only, and
 - (b) all others.

In contrast to conservation one may talk of generation or production and of destruction or consumption. Processes that only change variables of the second kind do not necessarily depend on other systems: they may run ‘automatically’.

Energy is *the* example of a variable of the first kind.

First Law. Energy can be exchanged only between dynamical systems; it cannot be generated or destroyed (conservation law).

Entropy is an example of a variable of the second kind. Its value can decrease by exchange only but may increase by generation as well.

Second Law. If during a process entropy decreases in one system – $\Delta S < 0$ – then by the same process entropy must increase in another system by at least $|\Delta S|$.

One may say that entropy satisfies half of a conservation law. Processes which generate entropy cannot be reversed since the second law above excludes the destruction of entropy; they are *irreversible*. Processes conserving entropy, however, are *reversible*.

8.2. Net Theory

In this section four state quantities will be studied that have been identified for net systems in general, independent of any interpretation or application. The crucial question is whether they satisfy some conservation laws. And although the answer will be negative in all four cases, it seems worthwhile to understand the reasons.

The four quantities are

- *en-ability*, $\mathcal{E}$, the characteristic function of net systems;
- *decision content*, $\mathcal{D}$, the result of *choosing* between possible modes of behaviour;
- *information*, $\mathcal{I}$, the quantitative measure for $\mathcal{D}$;
- *enlogy*, $\mathcal{L}$, closely intertwined with $\mathcal{I}$.

En-ability (23) has been discussed in Section 3.2; the other three quantities will be introduced a little further down in Section 8.2.2: enlogy (51), decision content (52) and information (53). None of these quantities are linear; they cannot be represented by forms. This must not surprise; if *en-ability*, for example, the characteristic function of net systems, were linear, net theory would just be part of the theory of linear systems.

8.2.1. En-ability

Obviously en-ability does not observe a conservation law in a strict sense. By an event occurrence any number of events may gain or lose en-ability. Yet the *extensionality principle* of net theory (see Section 3.2) that makes the en-ability *the* characteristic function of a net system seems to play a similar role that the conservation law of energy plays in the dynamics of physical systems. Hence the main consequences of the extensionality principle will be formulated as follows.

First Law. The enabling of an event is determined by its effect. An event *gains* its en-ability through the occurrence of some other event, and it *loses* its en-ability by either occurring itself or by the occurrence of some other, *conflicting* event.

8.2.2. Enlogy and Information

Roughly speaking, information is connected with choosing between modes of behaviour and enlogy with the resources needed for representing information. Using these terms we may talk informally about the example in Fig. 5 as follows:

- In state $\{s_0\}$ the system contains neither information nor enlogy.
- In the phase following the *free choice* $\{t_1 | t_2\}$ two alternative states are possible: $\{s_1, s_2\}$ and $\{s_3, s_4\}$. Hence 1 *bit* of information has entered the system through the decision. (If there were a third outgoing transition leading into a third alternative state, $\log_2 3$ *bit* – one ‘*trit*’ – would enter the system; see also discussion of Fig. 7, below.

This information appears in our example in two copies that are to control concurrent subprocesses. However, the system does not contain two bits of – *independent* – information; the *knowledge* of one bit entails the knowledge of the other. We say that the system also contains 1 *bit* of *enlogy* and observe that the copying of information requires the same amount of enlogy.

- The two bits leave the system through the two concurrent merges (co-decisions) $\{t_3 | t_4\}$ and $\{t_5 | t_6\}$. State $\{s_5, s_6\}$ contains neither information nor enlogy.

The co-decisions are concurrent but not independent. In the reversed system they must be supplied with copies of the same bit of information; otherwise the reverse processes may deadlock.

- The four intermediate states $\{s_1, s_6\}$, $\{s_2, s_5\}$, $\{s_3, s_5\}$ and $\{s_4, s_6\}$ still hold one bit of information but no enlogy anymore.

Hence it is impossible to say by which of the two co-decisions the enlogy and the information leaves.

It seems very difficult to define enlogy $\mathcal{L}$ for net systems in general. Hence the following considerations are restricted to a special class which will be called *synchronisation schemes*. They combine synchronisation graphs (marked graphs) [CHE71, GeL73] and PrT nets [GeL81]. Their purpose is that choices will depend only on the values of tokens rather than on the distribution of tokens as well.

Synchronisation schemes are defined by the following restrictions.

1. Every place

- either has exactly one incoming and one outgoing arc
- or its input and output transitions are the same – it forms a set of loops.

Fig. 5. Net system with a *free choice* between modes. Fig. 6. Net system with a *race* between two modes.

2. The subnet without shared places is strongly connected.
3. All colour sets are finite.
4. All arc expressions denote a single token; there no are multiple arcs.
5. The behaviour is cyclic: the reachability graph is strongly connected.

The ‘grey’ markings of a synchronisation scheme where the tokens are not distinguished by their values but counted only are called its *phases*. The grey markings of the shared places do not change. The subnet without shared places is called its *flow graph*; under the phases it forms a live and bounded synchronisation graph.

Two known subclasses of synchronisation schemes are *information flow graphs* [Pet67] and *bipolar synchronisation schemes* [GeT84]. Both classes have live and safe (1-bounded) underlying synchronisation graphs; in both classes all places have the same binary colour set. Another subclass consists of the *marked-graph-like PrT nets* [LaP85].

The two net systems on the left of Figs 5 and 6 can be transformed into equivalent synchronisation systems (on the right) through a sequence of equivalence transformations of PrT nets [Gen90]. In both cases the colour sets consist of at most two elements. The *free-choice* example is 1-bounded; the *race* example is 2-bounded. The production scheme (Fig. 3) is a non-example in several respects. Two places are shared; two arc expressions contain Boolean coefficients; and without the border transitions t_0 and t_6 , it is not cyclic.

Next to the phases the concept of a *mode* of behaviour [Fur76] is needed. Its definition does not necessarily depend on synchronisation schemes but modes are particularly simple there.

Let Z be an elementary cyclic process of a synchronisation scheme Σ and z the corresponding T-vector of combinations of bindings; then $C \cdot z = 0$ for the incidence matrix C . The T-vector z is called a *mode* of Σ ; the set of all modes is denoted as $\mathcal{M}(\Sigma)$. Every elementary solution of $C \cdot z = 0$ with non-negative integer coefficients that can be executed is a mode. In an execution of a mode of a synchronisation scheme, all transitions occur the same number of times.

For a particular marking M , the set of modes that can be executed starting at M reflects the degree of *non-determinism* of M ; it is denoted by $\mathcal{M}(M)$. The examples in Figs 5 and 6 both have two modes which are folded on top of each other to get the synchronisation scheme. Figure 7 has three modes that have to be unified first before the synchronisation scheme can be formed.

Fig. 7. Folding of an unnested double *if-then-else*. Fig. 8. Transformation of a simple *reader-writer* scheme.

The three examples above can be transformed into mere flow graphs, without shared places. This is not true for the generalisation of the – symmetric – race in Fig. 6 to the asymmetric *reader-writer* example (Fig. 8). Here the *write* mode has an arc multiplicity greater than one. The transformation into a synchronization scheme requires a proper resource allocation scheme [Sha79].

c	$\{s_0, \dots\}$	$\{s_1, s_2, \dots\}$	$\{s_1, s_4, \dots\}$	$\{s_2, s_3, \dots\}$	$\{s_3, s_4, \dots\}$
$\mathcal{L}$	0	0	1	1	0
$\mathcal{J}$	0	0	1	1	0

Elementary net system (left)

M	$\{s_0, \dots\}$	$\{s_{1,2} = a+b, \dots\}$	$\{s_{1,2} = a, s_{3,4} = b, \dots\}$	$\{s_{1,2} = b, s_{3,4} = a, \dots\}$	$\{s_{3,4} = a+b, \dots\}$
$\mathcal{L}$	0	1	1	1	1
$\mathcal{J}$	0	0	1	1	0

Synchronisation system (right)

Fig. 9. Representation-dependent enlogy (variant of the race example).

Enlogy

Let (Σ, M^0) be a synchronisation scheme with a given initial marking. Let τ be one of its phases (grey markings), $\tau = \{g_1, \dots, g_n\}$. For each of the grey tokens g_i the data type D_i of the corresponding place determines the range of values that g_i may assume. Hence the set of colorations of τ that are possible in principle is the Cartesian product $\prod_{i=1}^n D_i$. Let $\mu(\tau)$ denote the colorations of τ that actually belong to the marking class $[M^0]$. Usually it is a much smaller subset.

The ratio between the number of markings feasible at phase τ and the number of those markings that are actually possible (reachable) at τ is a measure of the order imposed by the system. Its logarithm to base 2 is called the *enlogy* of (all states at) phase τ and denoted by $\mathcal{L}$; the unit is *bit*.¹¹

$$\mathcal{L}(M) = \log_2 \frac{\prod_{i=1}^n |D_i|}{|\mu(\tau)|} \quad (M \in \mu(\tau)) \quad (51)$$

In the *free-choice* example in Fig. 5, the phase $\{s_{1,3}, s_{2,4}\}$ following the decision allows four different colorations but only two are system states. The enlogy is $\log_2 \frac{4}{2} = 1 \text{ bit}$ as told in our narration above.

Assume that at a phase τ , the data type D of a grey token has more elements than actually appear in any of the reachable markings of τ . (The reason could be that the net is to represent a whole family of systems by virtue of different initial markings.) Then the additional – redundant – enlogy reflects the extra cost of implementing D . Enlogy is a *representation dependent* quantity.

This is not a matter of economy only. Look at a variant of the *race* example that is shown in Fig. 9. In the elementary net system, the event t_0 does not add enlogy while in the equivalent synchronisation scheme, it adds 1 *bit*.

En-ability and information, however, are preserved by the equivalence transformations of PrT nets which are based on the isomorphism of reachability graphs [Gen90].

¹¹ Enlogy and its relative, enlogic structure, were coined by C. A. Petri [Pet67, Pet76].

Decision Content and Combinatorial Information

The set of all modes of a system, $\mathcal{M}(\Sigma)$, indicates the amount of behavioural freedom in Σ . The more modes are possible at a particular state M , i.e. the larger the set $\mathcal{M}(M)$ is, the more information is needed in order to predict what may happen next. Conversely, the more modes are excluded at M , i.e. the larger the set $\mathcal{M}(\Sigma) - \mathcal{M}(M)$ is, the larger is the decision content of M . The quantity

$$\mathcal{D}(M) = \mathcal{M}(\Sigma) - \mathcal{M}(M) \quad (52)$$

is called the – combinatorial – *decision content* of state M . The quantity

$$\mathcal{I}(M) = \log_2 \frac{|\mathcal{M}(\Sigma)|}{|\mathcal{M}(M)|} \quad (53)$$

is called the – combinatorial – *information* of state M ; its unit is the *bit*.¹²

The examples in Figs 5 and 6 both have two modes: $|\mathcal{M}(\Sigma)| = 2$. At the initial states $\{s_0\}$ and $\{s_0, s_{1,4}\}$ none of the modes is excluded; the information is $\mathcal{I}(M^0) = \log_2 \frac{2}{2} = 0 \text{ bit}$. The decisions $t_{1,2}$ and $t_{1,4}$ respectively exclude one mode: $|\mathcal{M}(M^1)| = 1$. One bit of information enters the system, $\mathcal{I}(M^1) = \log_2 \frac{2}{1} = 1 \text{ bit}$.

The example in Fig. 7 has three modes. At the initial state $\{s_0\}$ no mode is excluded; the information is 0 bit. Depending on the outcome of decision $t_{1,2}$ either one or two modes are excluded. The information added is $\log_2 \frac{3}{1} \text{ bit}$ for $x = a$ and $\log_2 \frac{3}{2} \text{ bit}$ for $x = d$. Decision $t_{3,4}$ adds either no information ($x = y = a$) or exactly one bit ($x = d \wedge (y = b \vee y = c)$). In both cases, the information becomes $(\log_2 3 = \log_2 \frac{3}{2} + \log_2 \frac{2}{1}) \text{ bit}$. In a mirrored way the information leaves the system.

The production scheme in Fig. 3 has a mode for every feasible batch.¹³ The actual information flow depends on a number of parameters. At t_1 the batches may race for the available resources. For the two batches in the given initial marking the available devices just suffice. There is no conflict; they can be produced concurrently. If more orders are accepted races may occur and information is needed to resolve them.

As a last example assume a synchronisation scheme where at all transitions the binding of the individual variables at the output arcs is a function of the binding of the variables at the input arcs. Such a system has only one mode. The information is constant zero; the system is closed with respect to information flow. A special case are the live and bounded marked graphs where the data type of all places is *unit* = $\{()\}$. If we had tried – because of its role in ‘information processing’ – to use information as a characteristic function for net systems, any two net systems closed under information flow with the same set of typed places would be considered the same.

Whenever a conflict is resolved in a net system Σ , the information contained in Σ increases. And if Σ is cyclic, the information will decrease eventually. Only conflict-free systems are closed under information flow. If by a closer look into the environment of Σ the outcome of a decision (conflict resolution) can be predicted, then the environment contains the information. It enters Σ either by exchange (*flux*) or by copying (*influence*) [Pet77]. If, on the other hand, the source of the information is inaccessible, if even the most precise reconstruction of the situation before the decision cannot guarantee the same outcome, then there is a *random* decision and information is *generated*. This is in contradiction to the *principle of local determinacy* [Pet77] stating that if a conflict is resolved in a net system, the information comes from an environment.

Examples. The numbers of the national lottery; the decision of a close race, like an asynchronous input or a shared resource allocation, by an *arbiter*; analogue-to-digital conversion (‘fuzzification’) close to the border between two intervals.

In general, conflict resolution in the presence of *confusion* [Smi96] requires the generation of information. And since the holding of information needs resources and resources are limited, information that is generated must be destroyed eventually. Information does not even satisfy half of a conservation law like entropy as we might have hoped.

Second Law. The decision of a *race*, i.e. of a conflict in a situation of *confusion*, requires the generating of information. The limitation of resources requires the destroying of information (recycling).

¹² Combinatorial decision content and information were first coined by Holt and Commoner in [HoC70].

¹³ In practice, an upper bound for the size of batches keeps the number of modes finite.

Fig. 10. The dynamics of net systems – survey.

Information Work

The information work done during a process $M_1 \xrightarrow{P} M_2$ is

$$W_{\mathcal{J}} = \Delta \mathcal{J} = \mathcal{J}(M_2) - \mathcal{J}(M_1) \quad (54)$$

The *free choice* example in Fig. 5 demonstrates that by the same event, depending on the current state, either information or enlogy may leave the system. This is due to the fact that several copies of the same bit of information may exist concurrently.

The consequence is that we cannot, in general, tell which information work is done by which event. Information work has no forms, in general; the work done by a process, for example the *cost of computation* [War68], cannot be calculated as the line integral over a field (of ‘information forces’). When looking at earlier work on modes, decision content and information in net systems [Fur76, Bes82], it seems that the intricate interplay of information and enlogy could not be understood without higher-level nets and their equivalence transformations. And the thermodynamical approach taken in [Joh88] may only work if combinatorial information is replaced by statistical information as in information theory.

Table 7. General dynamics and net theory: similarities and differences.

	Physical systems	Net systems
Modeling	physical, continuous	informatical, discrete
Coordinates	independent state variables	places
Values	<i>real number · unit</i>	$\sum (\text{number} \cdot \text{piece})$
State	vector of coordinate values	set of coexisting <i>conditions</i> (case, marking)
Change	infinitesimal change dx (of a coordinate x)	elementary change (<i>event</i>) Δt (at a transition t)
Process	continuous sequence of states (parameter time)	set of event occurrences with partial order <i>earlier</i>
Time	parameter of process	events: time points conditions: non-zero durations
Coordinate transf'ns	equivalence of different coordinate systems	equivalence transformations of PrT nets
Characteristic function	$Y = Y(x_1, \dots, x_n)$ (energy preferred)	$\mathcal{E}(c) = \{e \mid \bullet e - e^\bullet \subseteq c \wedge e^\bullet - \bullet e \subseteq \bar{c}\}$ en-ability (imperative)
Forms	$\frac{\partial Y(x_1, \dots, x_n)}{\partial x_i} dx_i$ (variable Y)	$\frac{\partial V(s_1, \dots, s_n)}{\partial s_i} \Delta s_i$ (linear variable V) $\frac{\partial W(t_1, \dots, t_m)}{\partial t_j} \Delta t_j$ (linear process quantity W)
Composition	addition of comparable characteristic functions	addition of disjoint nets
Coupling	exchange of coordinates, $x^a + x^b = \text{const.}$, x^a free	fusion of places fusion of transitions
Flow	exchange	flux (exchange) and influence (copying)
Laws of dynamics	1 st law: energy 2 nd law: entropy	extensionality principle (en-ability) generation of information in situations of confusion

9. Conclusion

This paper presented some known material of higher-level Petri nets from a particular perspective, namely in analogy to the dynamics of physical systems. The aim was to widen the view on the role of the incidence matrix in the analysis of net systems. Rather than focusing on constant state quantities (S-invariants) and cyclic actions (T-invariants), i.e. on solutions of the homogeneous equation systems based on the incidence matrix, we propose the study of dynamical quantities of net systems in general.

Figure 10 shows the main ingredients of the dynamical analysis of net systems. It exhibits the typical S – T symmetry resulting from the fact that net theory treats the notions *state* and *change* on an equal footing. This symmetry is reflected not only by the linear-algebraic duality between the incidence matrix C and its transpose, but it also appears closely related to the two complementary forms in which data appear as structured sets of values. As pieces, data are stored and transmitted – ‘displaced’ over distances in time and space; as bindings, data are processed and combined with other data. As pieces they belong to the states, as bindings they belong to the processes.

Table 7 summarises our comparison of the general dynamics of physical systems to a potential net dynamics that was outlined in this paper. It shows some promising similarities as well as severe differences.

The decomposition into forms, in analogy to the forms of energy, is limited to linear quantities. The main dynamical quantities of net systems, however, *en-ability*, *information* and *enlogy*, are non-linear.

There is no equivalent to the first law of dynamics (conservation of energy) nor to the second law (semi-conservation law for entropy). However, the situations in which information is created can be localised. They are situations of *confusion* [Smi96] where the decision of *races* requires special elements with meta-stable states (arbiters).

Information work in general cannot be expressed in terms of forms associated with the transitions, not even for the special class of synchronisation systems, as was shown by the analysis of the intricate interplay of information and enlogly in Section 8.2. Calculation of the information content of states and of the information work done in the course of processes requires to determine the set of all possible modes of behaviour through the analysis of the reachability graph – quite an unpleasant fact.

In spite of these difficulties the analysis of net systems in terms of dynamical quantities, supported by an adequate computer-algebra tool, seems a worthwhile endeavor.

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Fig. 11. Structure of the analysis package.

A. Appendix: Symbolic Calculations

To complement the theoretical parts of this paper, this Appendix briefly describes the experimental software package that supported symbolic calculations based on the incidence matrix like those in Sections 6.2 and 7.2. The package is still quite experimental. Its original purpose was to conduct some experiments concerning the purely symbolic or computer-algebraic analysis of higher-level net systems – the kind K. Lautenbach and the author once envisioned after their discovery of *predicate transition nets* but failed to realise by taking an unsuitable approach (solving equations in a ring of polynomials, cf. [GeL81]).

When the package became useful enough to do some non-trivial computations, it had a great impact on the dynamical view of net systems presented in this paper and thence, in return, developed a broader purpose and had to undergo many revisions.

A.1. An Experimental Analysis Package

The tool is an addition to *Design/CPN* [CPN96] and has been written in *Standard ML* [Pau91]. Its overall structure is depicted in Fig. 11.

1. By means of the *Design/CPN*'s *ML* interface to the graphical model and its database, the executable *CPN* model is translated into an equivalent incidence matrix representation. The net inscriptions (character strings) are transformed into an internal representation (kind of syntax tree) facilitating symbolic manipulation.
2. Dot product and matrix product are implemented based on the product between expression and transformation – the application of (combinations of) transformations on (combinations of) expressions.
3. A large number of *reduction* rules for combinations of expressions (equivalences, simplifications) are built in; they can be switched on and off interactively.

4. The (intermediate) results of a calculation are kept in a workspace *IT*. There are several formats for them to visualise.
5. Form sheets derived from the model facilitate the definition of markings and processes, of variables and fields, of functions and parameters.
6. The built in general reduction rules can be augmented by user-defined rules and declarations written in *ML*.

A.2. The Incidence Matrix

The incidence matrix is the non-hierarchical equivalent of the executable version of the CPN model. It is created while *Design/CPN* is in its simulator state. Our example is the production scheme, Fig. 3 with border transitions omitted and time being ignored.

<pre>(IgnoreTime (); SetFusion (); getIncMatrix (); getInitMarks (); SubNet.T [t_1]; show_Mat_S (!IncMatrix); show_FusionSets (); print_Marking ());</pre>	<pre>Time stamps IGNORED. ----- Fusion ON. ----- t_1: c,n,r,x Guard: [(c>0),(r<>[])] Pre: s_8: (set r):ms s_0: (x,c) s_7: (x,r,n) Post: s_1: (x,c,r,[]) s_5: (x,0,c) s_7: (x,r,(1+n)) ----- t_2: c,d,r,r',x Pre: s_1: (x,c,([d]^~r),r') Post: s_2: (x,c,d,r,r') ----- t_3: c,d,r,r',x Guard: [(r<>[])] Pre: s_2: (x,c,d,r,r') Post: s_1: (x,c,r,([d]^~r')) ----- t_4: c,d,r,r',x Guard: [(c>0),(r=[])] Pre: s_2: (x,c,d,r,r') Post: s_1: [(c>1)]%(x,(~1+c),(rev([d]^~r')),[]) s_3: x s_4: [(c=1)]%(rev([d]^~r')) ----- t_5: c,c',x Pre: s_3: x s_5: (x,c',c) Post: s_5: (x,(c'+1),c) ----- t_6: c,c',n,r,x Guard: [(c'=c)] Pre: s_5: (x,c',c) s_4: r s_7: (x,r,n) Post: s_8: (set r):ms s_7: (x,r,(~1+n)) s_6: (x,c) ===== FG_S1: s_1,s_1' FG_S3: s_3,s_3' FG_S3: s_4,s_4' ===== s_0: 1'(X,3)+1'(Y,2) + 0'(x,c) s_1: 0'(x,c,r,r') s_2: 0'(x,c,d,r,r') s_3: 0'x s_4: 0'r s_5: 0'(x,c,c') s_5: 0'(x,c) s_7: Recipes_0 + 0'(x,r,n) s_8: AvlDvc_0 + 0'd =====</pre>
---	---

The representation is transition based (column-wise). In addition to the arc expressions in the *pre* and *post*, the *guard* and the free individual variables (the binding type) of each transition are listed. The naming conventions for multiple instances of places and transitions in a hierarchical model are those of the occurrence graph analyser of *Design/CPN*. The elements of place fusion sets are listed separately.

When this *read-only* master matrix has been created, the simulator state can be left. The actual analysis operates on two temporary working copies – primal and transposed form – derived from the master matrix. Submatrices based on subsets of places and transitions can be formed, and place fusion can be switched on and off.

A.3. Processes

The effect of a process $M_1 \xrightarrow{P} M_2$ is the marking difference $\Delta M = M_2 - M_1$. It is calculated as the matrix product between the incidence matrix C and the action (one-column matrix) performed by P (see equation (15)).

Let P be the process of executing the order ‘*three times product X*’ represented by the token $(X, 3)$ in the initial marking of place s_0 . Combining the events (bindings) at each transition yields the action *action_x_3*. The product of C with *action_x_3* is first shown in non-reduced form. Applying the built-in reduction rules for Boolean coefficients, numbers and lists yields the net effect of *action_x_3* as expected.

```
val action_X_3 = MakeAction [
  (t_1, [(1, "(x,c,n,r)" <-+ "(X,3,0,[D1,D3]))]),
  (t_2, [(1, "(x,c,d,r,r)" <-+ "(X,3,D1,[D3],[ ])",
    (1, "(x,c,d,r,r)" <-+ "(X,3,D3,[ ],[D1])"),
    (1, "(x,c,d,r,r)" <-+ "(X,2,D1,[D3],[ ])",
    (1, "(x,c,d,r,r)" <-+ "(X,2,D3,[ ],[D1])"),
    (1, "(x,c,d,r,r)" <-+ "(X,1,D1,[D3],[ ])",
    (1, "(x,c,d,r,r)" <-+ "(X,1,D3,[ ],[D1])"))]),
  (t_3, [(1, "(x,c,d,r,r)" <-+ "(X,3,D1,[D3],[ ])",
    (1, "(x,c,d,r,r)" <-+ "(X,2,D1,[D3],[ ])",
    (1, "(x,c,d,r,r)" <-+ "(X,1,D1,[D3],[ ]))]),
  (t_4, [(1, "(x,c,d,r,r)" <-+ "(X,3,D3,[ ],[D1])",
    (1, "(x,c,d,r,r)" <-+ "(X,2,D3,[ ],[D1])",
    (1, "(x,c,d,r,r)" <-+ "(X,1,D3,[ ],[D1])"))]),
  (t_5, [(1, "(x,c',c)" <-+ "(X,0,3)",
    (1, "(x,c',c)" <-+ "(X,1,3)",
    (1, "(x,c',c)" <-+ "(X,2,3)"))]),
  (t_6, [(1, "(x,c',c,n,r)" <-+ "(X,3,3,1,[D1,D3]))]),
];
EFFECT action_X_3;
REDUCE_IT ();
```

```
s_0: ~ (X,3)
s_1: [(1>1)]%(X,0,(rev([D3]^^[D1])),[ ])
+ [(2>1)]%(X,1,(rev([D3]^^[D1])),[ ])
+ [(3>1)]%(X,2,(rev([D3]^^[D1])),[ ])
- (X,1,([D1]^^[D3]),[ ])
+ (X,1,[D3],([D1]^^[ ]))
- (X,1,([D3]^^[ ]),[D1])
- (X,2,([D1]^^[D3]),[ ])
+ (X,2,[D3],([D1]^^[ ]))
- (X,2,([D3]^^[ ]),[D1])
- (X,3,([D1]^^[D3]),[ ])
+ (X,3,[D3],([D1]^^[ ]))
- (X,3,([D3]^^[ ]),[D1])
+ (X,3,[D1,D3],[ ])
s_4: [(1=1)]%(rev([D3]^^[D1]))
+ [(2=1)]%(rev([D3]^^[D1]))
+ [(3=1)]%(rev([D3]^^[D1]))
- [D1,D3]
s_6: (X,3)
=====
s_0: - (X,3)
s_6: (X,3)
=====
```

A.4. State Quantities and their Gradients

Next we demonstrate how to fill the gaps that were left in Section 6.2 when calculating the gradient of the quantity *recipe*.

```
val recipe = MakeQuantity [
  (s_1, [(1, "(_,_,r,r)" +> 1'(rev(r')^r) ")]),
  (s_2, [(1, "(x,_,d,r,r)" +> 1'(rev(d::r')^r) ")]),
  (s_4, [(1, "r" +> 1'r ")]),
  (s_7, [(1, "(_,r,n)" +> ~n'r ")]),
];
```

The gradient of *recipe*, the vector of its partial derivatives with respect to the transitions, is calculated as the matrix product of the transposed incidence matrix, C^T , and the vector *recipe* viewed as a one-column matrix. The raw, non-reduced result is simplified considerably by means of the reduction rules for list expressions.

```

grad recipe; t_1:      ((rev[ ])^^r)
              -
              r
t_2:      ((rev([d]^^r'))^^r)
              -
              ((rev r')^^[d]^^r)
t_4:      [(c>1)]%((rev[ ])^^((rev([d]^^r'))))
              -
              ((rev([d]^^r'))^^r)
              + [(c=1)]%(rev([d]^^r'))
=====
REDUCE_IT (); t_4: ~ ((rev r')^^[d]^^r)
              + [(c=1)]%(rev r')^^[d]
              + [(c>1)]%(rev r')^^[d]
=====

```

To get rid of the remaining defect at t_4 we first multiply it by the guard of t_4 – a Boolean coefficient common to all arc expressions around t_4 . Then we exploit the equation $r = []$ in the guard and use the fact that $[c > 0] = [c = 1] + [c > 1]$.

```

( Guards_to_IT [t_4];
  print_IT ();
  ApplyEqus.T [t_4];
  REDUCE_IT ();
  Define.T ([t_4],[ ])
    "[c>0]" "[c=1]+[c>1]";
  EvalDefs ();
  REDUCE_IT () );
t_4: ~ [(c>0),(r=[ ])]%((rev r')^^[d]^^r)
      + [(c>0),(r=[ ])]^^[(c=1)]%((rev r')^^[d])
      + [(c>0),(r=[ ])]^^[(c>1)]%((rev r')^^[d])
=====
t_4: [(c=1)]%((rev r')^^[d])
      - [(c>0)]%((rev r')^^[d])
      + [(c>1)]%((rev r')^^[d])
=====
=====

```

A.5. Fields and Potentials

Our example of a field in a net system was the field of costs for the production scheme (see Section 7.2). In the analysis tool the definition of the cost field looks as follows (the *unit* value () replacing the cost unit $\underline{u}$).

```

val cost = MakeField [
  (t_1, "" ),
  (t_2, "K^2(x,d)'( )"),
  (t_3, "" ),
  (t_4, "" ),
  (t_5, "K^5(x)'( )"),
  (t_6, "" ),
];

```

The line integral of field along a path (process) is calculated as the dot product of the field with the action performed by a process – for example, the *action_X_3* known from above.

```

( value_T cost action_X_3;
  print_IT () );
      3*(K^2 (X,D1))'( )
      + 3*(K^2 (X,D3))'( )
      + 3*(K^5 X)'( )
=====

```

The actual value depends on the actual definition of the transition cost functions K^2 and K^5 that are regarded as parameters.

```

( ResetDefs ();
  Define ([ ],[ ]) "K^2" "( (X,d) | (Y,d) | (Z,d) | _ ) +->
    (K_2.X(d) | K_2.Y(d) | K_2.Z(d) | 0 )";
  Define ([ ],[ ]) "K_2.X" "(D1|D2|D3|_) +-> (4.5|2.7|3.1| 0 )";
  Define ([ ],[ ]) "K_2.Y" "(D1|D2|D3|_) +-> ( 3 | 2 | 5 | 0 )";
  Define ([ ],[ ]) "K_2.Z" "(D1|D2|D3|_) +-> ( 0 | 0 | 0 | 0 )";

  Define ([ ],[ ]) "K^5" "(X|Y|Z|_) +-> (1.4 | 1 | 0 | 0)";

  REDUCE_IT () );
=====
27.0'( )

```

To end this short demonstration of symbolic calculations based on the incidence matrix, let us prove that the state quantity *COST* is indeed a potential of the field *cost* as claimed in Section 7.2. The calculation starts with defining $negCOST = -COST$ since by convention we look for the negative gradient field of a potential.

```

val negCOST = MakeQuantity [
  (s_0, [(1, "(x,c) +-> ~ c*(H(x,r) + K^5())'() )]),
  (s_1, [(1, "(x,c,r,r') +-> ~ (H(x,r)+(c-1)*H(x,(rev r')^~r))'() )]),
  (s_2, [(1, "(x,c,d,r,r') +-> ~ (H(x,r)+(c-1)*H(x,(rev (d::r'))^~r))'() )]),
  (s_5, [(1, "(x,c',c) +-> ~ (c-c')*K^5(x)'() )])
];

```

The next step consists of three substeps:

1. The gradient of $negCOST = -COST$ is calculated. The lengthy, non-reduced intermediate result we know already from section 7.2; it shrinks considerably when the built-in reduction rules are applied.
2. The partial derivative at t_4 is reduced by exploiting its guard. First the identity $[c > 1] = [true] - [c = 1]$ that follows from $c > 0$ is applied once again; then the equation $r = []$ is applied at t_4 as well as $c' = c$ at t_6 .
3. Only now the definition of the – recursive – auxiliary function H is provided and evaluated.
4. The final result is exactly the field $cost$ as defined above.

```

( grad negCOST;
  print.IT () );
t_1:      ((H (x,((rev[ ])^~r))) * ~1))'()
+
+      ((H (x,((rev[ ])^~r))) * c))'()
-
-      ((H (x,r)) * ~1 * c)'()
+
+      ((H (x,r)))'()
t_2:      ((H (x,((rev([d]^~r'))^~r))) * ~1))'()
-
-      ((H (x,((rev r')^~[d]^~r))) * ~1))'()
+
+      ((H (x,((rev([d]^~r'))^~r))) * c))'()
-
-      ((H (x,((rev r')^~[d]^~r))) * c))'()
-
-      ((H (x,([d]^~r)))')'()
+
+      ((H (x,r)))'()
t_4:      ((H (x,((rev[ ])^~(rev([d]^~r'))))) * ~2))'[(c>1)]%()
+
+      ((H (x,((rev[ ])^~(rev([d]^~r'))))) * c))'[(c>1)]%()
+
+      ((H (x,(rev([d]^~r')))))'[(c>1)]%()
-
-      ((H (x,((rev([d]^~r'))^~r))) * ~1))'()
-
-      ((H (x,((rev([d]^~r'))^~r))) * c))'()
-
-      ((H (x,r)))'()
t_5:      ((K^5 x) * ~1 * ~1)'()
t_6:      ((K^5 x) * (c'))'()
-
-      ((K^5 x) * c * ~1)'()
=====

```

```

REDUCE.IT ();
t_2:      (H (x,([d]^~r)))'()
-
-      (H (x,r))'()
t_4:      ((H (x,((rev r')^~[d]))) * c)'[(c>1)]%()
+
+      (H (x,((rev r')^~[d])))'[(c>1)]%()
+
+      ((H (x,((rev r')^~[d]^~r))) * c)'()
-
-      (H (x,((rev r')^~[d]^~r)))'()
+
+      (H (x,r))'()
t_5:      ((K^5 x))'()
t_6:      ((K^5 x) * c)'()
+
+      ((K^5 x) * c)'()
=====

```

```

( Define.T ([t_4],[ ])
  "[c>1]" "[true] - [c=1]";
  EvalDefs ();
  ApplyEqs [t_4,t_6];
  Reduce.IT () );
t_2:      (H (x,([d]^~r)))'()
-
-      (H (x,r))'()
t_4:      (H (x,[ ]))'()
t_5:      ((K^5 x))'()
=====

```

```

( Define ([ ],[ ]) "H"
  "( (.,[ ]) | (s,[d]2) ) +-> (0 | K^2(s,d) + H(s,r))";
  REDUCE.IT () );
t_2:      K^2(x,d)'()
t_5:      K^5()'()
=====

```

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