Multicut is FPT*

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ABSTRACT

Let $G = (V, E)$ be a graph on n vertices and R be a set of pairs of vertices in V called *requests*. A *multicut* is a subset F of E such that every request xy of R is cut by F , *i.e.* every xy -path of G intersects F . We show that there exists an $O(f(k)n^c)$ algorithm which decides if there exists a multicut of size at most k . In other words, the MULTICUT problem parameterized by the solution size k is Fixed-Parameter Tractable.

General Terms

Algorithms

Keywords

Multicut, FPT, Parameterized Complexity

Categories and Subject Descriptors

G.2.2 [Mathematics of Computing]: DISCRETE MATHEMATICS—*Graph Theory, Graph algorithms*

1. INTRODUCTION

The study of cuts and flows is one of the most active field in combinatorial optimization. However, if the simplest case involving one source and one sink is algorithmically tractable, the problem becomes hard as soon as one deals with multiple terminals. For instance, given two requests (x_1, y_1) and (x_2, y_2) in a directed graph D , it is NP-complete to decide if there exist two disjoint directed paths, respectively from x_1 to y_1 and from x_2 to y_2 [10]. In fact, even deciding if two given vertices belong to a directed circuit is already hard. The picture changes when considering undirected graphs, in which case the celebrated result of Robertson and Seymour [22] asserts that given k requests $(x_1, y_1), (x_2, y_2), \dots, (x_k, y_k)$, one can decide in cubic time if

there exists k disjoint paths connecting all pairs x_i, y_i . The catch is of course that cubic time refers to the instance size, which we generally denote by n . From their work, the complexity of the k -path problem is $O(f(k)n^3)$, where f is by no mean polynomial, since the question is NP-complete when k is part of the input.

This result received considerable attention, both since this is the key tool for computing a given minor in a graph, but also because it has opened a breach in the classical NP-complete/P duality. Indeed, the difficulty of the k -path problem does not depend on the size of the instance, but rather on the number of paths we are looking for. In other words, the parameter containing the hardness of the problem is the number k of paths. In a more general way, a problem is *fixed parameter tractable* (FPT) with respect to the parameter k (e.g. solution size, treewidth, ...) if for any instance of size n it can be solved in time $O(f(k)n^c)$ for some fixed c . The reader is invited to refer to now classical books by Downey and Fellows [7], Flum and Grohe [8] and Niedermeier [21].

The dual problem of finding disjoint paths from a source s to a sink t is the cut problem where one asks for a set of vertices or edges which deletion separates s from t . Menger's theorem, or more generally LP-duality, asserts that the maximum number of disjoint st -paths is equal to the minimum size of a cut. This property no longer holds when considering multiple requests, where the maximum number of disjoint paths connecting requests is only an obvious lower bound for the size of a multicut, *i.e.* a set of vertices or edges which deletion separates x_i from y_i for every request $x_i y_i$. Formally, we have the following problem:

MULTICUT:

Input: A graph G , a set of requests R , an integer k .

Parameter: k .

Output: TRUE if there is a multicut of size at most k , otherwise FALSE.

The status of this problem is one of the long standing open problems in parameterized complexity. The main result of this paper is to provide an FPT algorithm for MULTICUT. The proofs being slightly less cumbersome in the edge case, we present our work in terms of edge-multicut rather than vertex-multicut. Our last section shows how to translate the algorithm to deal with vertices. Let us now give some formal definitions. Given a graph G and a set R of pairs of distinct vertices called *requests*, a *multicut* is a subset F of edges of G such that every xy -path, where xy is a request, contains an

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edge of F . Equivalently, the two endpoints (or *terminals*) of every request of R belong to different connected components of $G \setminus F$.

The MULTICUT problem is already hard on trees since VERTEX COVER is equivalent to MULTICUT in stars. Hence MULTICUT is NP-complete and Max-SNP hard. MULTICUT and its variants have raised an extensive literature. These problems play an important role in network issues, such as routing and telecommunication (see [5]).

MULTICUT IN TREES was already a challenging problem. Garg *et al.* [11] proved that it admits a factor 2 approximation algorithm. Guo and Niedermeier [15] proved that MULTICUT IN TREES is FPT with respect to the solution size. Bousquet *et al.* [1] provided a polynomial kernel.

On general instances, Garg *et al.* gave an approximation algorithm for MULTICUT within a logarithmic factor in [9]. However MULTICUT has no constant factor approximation algorithm if Khot's Unique Games Conjecture holds [2]. This fact is a further motivation to the study of the fixed parameterized tractability of MULTICUT. Guo *et al.* showed in [14] that MULTICUT is FPT when parameterized by both the treewidth of the graph and the number of requests. Gottlob and Lee in [12] proved a stronger result: MULTICUT is FPT when parameterized by the treewidth of the input structure, namely the input graph whose edge set is completed by the set of request pairs. Recently, Daligault *et al.* [6] proved that MULTICUT can be reduced to instances in which the graph G has treewidth bounded in terms of k .

Marx proved that MULTICUT is FPT for this latter parameterization [16]. A faster algorithm running in time $O^*(8 \cdot l)^k$ was given by Guillemot [13]. Marx *et al.* [17] obtained FPT results for more general types of constrained MULTICUT problems through treewidth reduction results. However their treewidth reduction techniques do not yield the tractability of MULTICUT when parameterized only by the solution size. Recently, Marx and Razgon obtained a factor 2 Fixed-Parameter-Approximation for MULTICUT in [18].

Marx and Razgon independently found a proof of the fact that MULTICUT is FPT, with a different approach, see [19].

The outline of our proof is quite pedestrian, even if some of the partial results are still a bit technical. Informally, our goal is to reduce our input graph to a subdivision of a graph with a bounded number of edges. The crucial tool for this reduction is to find a partition of the set of all "reasonable" solutions of the MULTICUT problem into a bounded number of subsets, in which the multicuts are "totally ordered" (this will be formalized in Subsection 5.6). This application of Dilworth's Theorem is maybe the most interesting aspect of our proof. However, it requires an important cleaning of our instance before being applied. It would be interesting to find an adequate partial order on multicuts at an earlier stage of our proof to shorten the argument and get more insight in the structure of multicuts.

In Section 3 we develop some connectivity tools which are used to certify that some requests are irrelevant. In Section 4, we first compute a vertex-multicut Y of size $k + 1$ by iterative compression. We then reduce to the case where each component of $G \setminus Y$ has one or two attachment vertices in Y . We show that components with one attachment vertex have only a bounded (in k) number of terminals. For com-

ponents with two attachment vertices, we identify one path (called *backbone*) where only one edge is chosen in the multicut. In Section 5, we reduce to the case where the backbone is the only path connecting the two attachment vertices. Finally, we show that MULTICUT can be reduced to an instance which is a subdivision of a graph with a bounded number of edges and where the multicut selects a single edge in each of the paths corresponding to the subdivided edges. This case can easily be coded by a 2-SAT instance, which is solvable in polynomial time.

Acknowledgments. This proof was a long process in which several researchers gave us advice or participated in some of the partial results. One of our important tools, Lemma 7, is the crucial result of the reduction of MULTICUT to bounded treewidth, and was obtained in collaboration with Christophe Paul and Anthony Perez [6]. The fact that a vertex cutset Y could be obtained by iterative compression, a tool of [18], was brought to our attention by Sylvain Guillemot. Finally, we would like to thank Valentin Garnero who worked on this project as a part of his graduate research experience in June 2010.

2. PRELIMINARIES

A vertex which sends a request is called a *terminal*. The number of requests sent by a terminal is its *request degree*. We study MULTICUT variants with additional constraints on the deleted edges. In the original MULTICUT problem, we can delete any set of k edges, but in some more constrained versions we must delete a prescribed number of edges on some particular paths. The total number of deleted edges is called *deletion allowance* of the multicut problem. We will make extensive use of the term *bounded* which always implicitly means bounded in terms of the deletion allowance. Also, when speaking of FPT time, we always mean $O(f(d)n^c)$ where c is a fixed constant and d is the deletion allowance. Let us discuss further some of the operations we will often perform.

Reductions. These are computations where the output is a new instance which is equivalent to the original instance with respect to the existence of a solution. One of the most natural reductions concerns irrelevant requests, i.e. a request xy such that every multicut of $R \setminus xy$ actually cuts x from y , where R is the set of requests. If one can certify that a request xy is irrelevant, the reduction just consists in replacing R by $R \setminus xy$. The difficulty is obviously to certify that a request is irrelevant. Another easy reduction is obtained if we can certify that there exists a multicut which does not separate two given vertices u and v . In this case we simply contract u and v . Reductions are easy to control since we can perform them freely provided that some invariant polynomial in n decreases. For instance, request deletions can be performed at most n^2 times, and vertex contractions at most n times.

Branchings. In our algorithm, we often have to decide if the multicut we are looking for is of a particular type, where the number of types is bounded. We will then say that we branch over all the possible cases. This means that to compute the result of the current instance, we run our algorithm on each case, in which we force the solution to be of each given type. The output is TRUE if at least one of the outputs returns TRUE. To illustrate this, in the case of a graph G with two connected components G_1 and G_2 , both containing requests, we would branch over $k - 1$ instances,

depending of the number of edges (between 1 and $k-1$) that we delete from G_1 . This simple branching explains why we can focus on connected graphs.

Invariants. To prove that the total number of branches is bounded, we show that some invariant is modified at each branching step, and that the number of times that this invariant can be modified is bounded. We usually have several invariants ordered lexicographically. In other words, we have different invariants which we want to increase or decrease and can take a bounded number of values. These invariants are ordered, there is a primary invariant, a secondary, etc. Each branching must *improve* our invariant, *i.e.* the first invariant (with respect to priority order) which is changed by the branching must be modified according to the preference, increase or decrease, that we specified for it. For instance the primary invariant could be the number of deleted edges, which we want to decrease and the secondary invariant could be the connectivity of G , which we want to increase. If we can decrease the number of deleted edges we do so even if the connectivity of the graph decreases. Also, if a branching increases connectivity and leaves the number of deleted edges unchanged, we improve our invariant.

3. CONNECTIVITY IN FPT TIME

Dealing with minimum cuts can be done in polynomial time with usual flow techniques. However, dealing with k -edge cuts when k is some fixed value larger than the optimum is more difficult. We develop here some tools to deal in FPT time with bounded cuts.

3.1 Enumerating cuts in FPT time

The results in this subsection are of the same flavor as in [4], [6], [16] and [20]. Due to space restriction, we omit their proofs.

Let G be a connected graph on n vertices with a particular vertex x called *root*. We deal in this part with cuts, *i.e.* bipartitions of the vertex set of G . To fix one side of our cuts, we define a *cut* as a subset of vertices S containing x . The *border* of S is the set of edges of G with exactly one endpoint in S . We denote it by $\Delta(S)$. Its cardinality is denoted by $\delta(S)$. Recall that the function δ is submodular, *i.e.* $\delta(A) + \delta(B) \geq \delta(A \cap B) + \delta(A \cup B)$. Given a subset Y of vertices of $V(G)$, we denote its *complement* $V(G) \setminus Y$ by $\bar{Y}$. A cut S is a *left cut* if every cut $T \subsetneq S$ satisfies $\delta(T) > \delta(S)$. Note that every cut S contains a left cut S' with $\delta(S') \leq \delta(S)$.

LEMMA 1. *Left cuts are closed under union.*

LEMMA 2. *If S_1, S_2 are distinct left cuts, $\delta(S_1 \cup S_2) < \max(\delta(S_1), \delta(S_2))$.*

A cut S is *indivisible* if $G \setminus S$ is connected, otherwise it is *divisible*.

LEMMA 3. *If S is a divisible left cut and Y is a connected component of $G \setminus S$, the cut $\bar{Y}$ is an indivisible left cut with $\delta(\bar{Y}) < \delta(S)$.*

COROLLARY 4. *Every indivisible cut S contains an indivisible left cut S' with $\delta(S') \leq \delta(S)$.*

Given a vertex y , an xy -cut is a cut S such that $y \notin S$. We denote by C_k^y the set of indivisible left xy -cuts with border k . We also denote by $C_{<k}^y$ (resp. $C_{\leq k}^y$) the union of C_i^y for $i < k$ (resp. for $i \leq k$).

THEOREM 5. *The set $C_{\leq k}^y$ has size at most $k!$ and can be computed in FPT time.*

The value $k!$ in Theorem 5 can actually be improved to 4^k [3, 19].

3.2 Irrelevant Requests

We denote by C_k the union of C_k^y for all vertices y of G , by $C_{<k}$ the union of $C_{<k}^y$ for all vertices y of G , and by $C_{\leq k}$ the union of $C_{\leq k}^y$ for all vertices y of G .

THEOREM 6. *Every set K with at least $\alpha(k, k') := k'^{\binom{k+2}{2}-1}$ vertices of G contains a subset K' of size k' such that every left cut S with $\delta(S) \leq k$ satisfies either $S \cap K' = \emptyset$ or $|K' \setminus S| \leq k$. The set K' can be computed in FPT single exponential time.*

PROOF. Observe that the result trivially holds when $k' \leq k$. So we can assume that $k' > k$. Consider a graph G rooted at x . We prove the result by induction on k .

For $k = 1$, $\alpha(1, k') = k'^2$. The complements of a left cut with a border of size 1 form a collection of disjoint sets of vertices, which induces a partition of K . There is either a class K' of this partition containing at least $\sqrt{|K|} \geq k'$ elements, or a set K' of size at least $\sqrt{|K|} \geq k'$ which elements are chosen in different classes. In both cases K' satisfies the induction hypothesis.

Assume now that $k > 1$. We distinguish two cases.

Assume that there exists an indivisible left cut S with $\delta(S) \leq k$ and $|K \setminus S| \geq k'^{\binom{k+1}{2}-1}$. By induction, we extract from $K \setminus S$ a subset K' with size k' such that every left cut T with $\delta(T) \leq k-1$ satisfies either $T \cap K' = \emptyset$ or $|K' \setminus T| \leq k-1$. To conclude, let us consider a left cut S' with $\delta(S') = k$. If $S' = S$, then $S \cap K' = \emptyset$ by definition of K' , hence we assume that S' is distinct from S . Observe that $K' \setminus S'$ is equal to $K' \setminus (S \cup S')$. By Lemma 2, $S \cup S'$ is a left cut with border at most $k-1$, so $K' \setminus S'$ is either K' or has size at most $k-1$. Hence the conclusion of Theorem 6 holds.

Conversely, assume that all indivisible left cuts S with $\delta(S) \leq k$ satisfy $|K \setminus S| < k'^{\binom{k+1}{2}-1}$. Let us form a graph H with vertex set K and where vv' is an edge when there exists an indivisible left cut S with $\delta(S) \leq k$ such that $\{v, v'\} \cap S = \emptyset$. The degree of a vertex v of H is less than $d := k'^{\binom{k+1}{2}-1} \cdot k!$. Indeed, the number of indivisible left xv -cuts with border at most k is at most $k!$, and we have assumed that all indivisible left cuts S with $\delta(S) \leq k$ satisfy $|K \setminus S| < k'^{\binom{k+1}{2}-1}$. Note that d is less than $k'^{\binom{k+1}{2}-1} \cdot k'^k$ since $k' \geq k$. There is a stable set K' in H of size at least $\frac{|K|}{d} \geq k'$ since $\binom{k+2}{2} - \binom{k+1}{2} - k = 1$. Every indivisible left cut S with $\delta(S) \leq k$ isolates at most one vertex of K' as K' is stable in H . Hence every left cut S with $\delta(S) \leq k$ isolates at most k vertices of K' .

The induction hypothesis holds in both cases, which concludes the proof of Theorem 6. $\square$

The next result is our key tool in reducing MULTICUT to graphs of bounded treewidth in [6], but the following easy argument is more straightforward. The proof of Theorem 4 of [6] implies that the following result holds with $h(\ell) = \ell^{O(\ell)}$, and can be computed in time $\ell^{O(\ell)} n^c$.

THEOREM 7. *Every set K with at least $h(\ell) := \ell \cdot 2^{\ell} + 1$ vertices of G contains a vertex y such that every cut S with $\delta(S) + |S \cap K| \leq \ell$ is such that $y \notin S$. Moreover, y is computed in FPT time.*

In other words, vertex y verifies the following: whenever the deletion of a set of a edges isolates x from all but b elements of K , with $a + b \leq \ell$, then the vertex y is also isolated from x .

PROOF. We create a new vertex z joined to all the vertices of K to form a new graph G' . In this proof we change our point of view and consider that z is our root vertex for G' , i.e. z plays for G' the same role as x plays for G . In G' , the set C of indivisible left zx -cuts with border at most ℓ has size at most $\ell!$ by Theorem 5. Hence, since the size of K is at least $\ell \cdot 2^{\ell} + 1$, there exists a subset T of K of size at least $\ell + 1$ such that for every cut S in C , we have either $T \subseteq S$ or $T \cap S = \emptyset$. We compute such a set T in FPT time. We then pick y in T . Let us prove that y satisfies the conclusion of Theorem 7.

In the graph G , consider a set A of a edges which isolates x from all the elements of K save a subset B of size b with $a + b \leq \ell$. Let F be the set of edges $A \cup \{zb : b \in B\}$ of G' . Note that F is a zx -edge cut. We denote by X the component of x in $G' \setminus F$. Since $V(G') \setminus X$ is an indivisible zx -cut with border at most ℓ , it contains by Corollary 4 a indivisible left zx -cut S with border at most ℓ , in other words S belongs to C .

Let us first observe that the set T cannot be disjoint from S . Indeed T has size $\ell + 1$ and each vertex of T is adjacent to z , thus the border of S would exceed ℓ . Hence T is included in S , thus the set of edges A isolates T from x , and in particular y from x . This concludes our proof. $\square$

COROLLARY 8. *In a multicut problem instance with deletion allowance k , the maximum request degree can be reduced to at most $h(k + 1)$ in FPT time.*

PROOF. Let x be a vertex which sends requests to all vertices of K , where $|K| \geq h(k + 1)$. By Theorem 7, there is a vertex y of K such that every subset S containing x such that $\delta(S) + |S \cap K| \leq k + 1$ is such that $y \notin S$. We simply delete the request xy from the set of requests. Indeed, let F be a multicut of size at most k of this reduced instance. Let S be the component of x in $G \setminus F$. Since F is a multicut, no element of $K \setminus y$ belongs to S . Moreover $\delta(S) \leq k$ since at most k edges are deleted. Thus $\delta(S) + |S \cap K|$ is at most $k + 1$, which implies that $y \notin S$. In other words, even if we do not require to cut x from y , any multicut of the reduced instance will cut the request xy . Therefore the request xy can be deleted from R . $\square$

3.3 Cherry Reduction

An *x-cherry*, or simply *cherry* is a connected induced subgraph C of G with a particular vertex x called *attachment vertex* of C such that there is no edge from $C \setminus x$ to $G \setminus C$ and no request has its two terminals in $C \setminus x$. In other words the only requests inside an x -cherry have origin x . Note that we can always assume that the restriction of a multicut to an x -cherry C is the border of a left cut of C . If $u \in C \setminus x$, a request $uv \in R$ is *irrelevant* if every multicut F with at most k edges of the reduced instance on $R \setminus uv$ and such that $F \cap C$ is the border of a left cut in C actually separates u from v .

THEOREM 9. *Let C be an x -cherry of a instance with deletion allowance k . We can find in FPT time a set $K(C)$ of at most $b(k) := h(k + 1) \cdot \alpha(k, h(2k + 1)) = k^{O(k^3)}$ terminals in $C \setminus x$, such that if F is a set of at most k edges which cuts all requests with one endpoint in $K(C)$ and such that $F \cap C$ is the border of a left cut, then F actually cuts all requests with an endpoint in $C \setminus x$.*

The proof's ideas are similar to the three previous proofs, and we omit it due to space restrictions.

Let C be a cherry of a graph G with deletion allowance k . A subset $\mathcal{L}$ of the edges of C is *active* when, if a multicut F of size at most k exists, then there exists a multicut F' of size at most $|\mathcal{L}|$ such that $F' \setminus C = F \setminus C$ and $F' \cap C \subseteq \mathcal{L}$. When the set $\mathcal{L}$ is clear from the context, we say by extension that edges of $\mathcal{L}$ are *active*.

LEMMA 10. *Let C be an x -cherry of a graph G with deletion allowance k , and let K be the set of all terminals of $C \setminus x$. Let $\mathcal{L}(C)$ be the union of all borders of cuts of $C_{\leq k}^y$, where $y \in K$. Then $\mathcal{L}(C)$ is active, and has size at most $k|K|k!$.*

PROOF. Assume that F is a multicut with size at most k . Let S be the component of x in $C \setminus F$. Let T be a left cut with $T \subseteq S$ and $\delta(T) \leq \delta(S)$. If a component U of $\bar{T}$ does not intersect K , then $F \setminus \Delta(U)$ is still a multicut. Hence we can assume that all components U of $\bar{T}$ intersect K , in which case $\Delta(U) \in C_{\leq k}^y$ for some y in K , hence $\Delta(T)$ is included in $\mathcal{L}(C)$. The set $F' = (F \setminus C) \cup \Delta(T)$ is a multicut, and the size bound for $\mathcal{L}(C)$ follows from Theorem 5. $\square$

THEOREM 11. *Let $H_1, H_2, \dots, H_p$ be x -cherries of a graph G with deletion allowance k such that $H_1 \setminus x, H_2 \setminus x, \dots, H_p \setminus x$ are pairwise disjoint. Assume that for every i , $U_i := H_1 \cup \dots \cup H_i$ is a cherry. Then every U_i has a bounded active set $\mathcal{L}_i$ such that $\mathcal{L}_j \cap U_i \subseteq \mathcal{L}_i$ whenever $i \leq j$.*

PROOF. By Theorem 9, we can reduce the set of terminals in U_1 to a bounded set K_1 . The set $\mathcal{L}_1 = \mathcal{L}(U_1)$ is bounded and active by Lemma 10. The requests of $C_1 \setminus K_1$ are irrelevant in U_2 since they are irrelevant in U_1 , hence we can assume that Theorem 9 applied to U_2 yields a set of terminals $K_2 \subseteq K_1 \cup C_2$. Let $\mathcal{L}_2$ be the active edges associated to K_2 . Note that if an edge $e \in \mathcal{L}_2$ is in U_1 , it must belong to some $C_{\leq k}^y$ for some $y \in K_2 \cap U_1$. Since $K_2 \subseteq K_1 \cup C_2$, we have $y \in K_1$, hence $e \in \mathcal{L}_1$, which is the property we are looking for. We extract K_3 from $K_2 \cup C_3$, and iterate our process to form the sequence $\mathcal{L}_i$. $\square$

4. REDUCING TO COMPONENT MULTICUT

Let $G = (V, E)$ be a connected graph, and R be a set of requests. A *vertex-multicut* Y is a subset of V such that every xy -path of G where $xy \in R$ contains an element of Y . Let A be a connected component of $G \setminus Y$. We call Y -*component*, or *component*, the union of A and its set of neighbors in Y . Let C be a component, the vertices of $C \cap Y$ are the *attachment vertices* of C .

4.1 Component Multicut

Here is our first intermediate problem.

COMPONENT MULTICUT:

Input: A connected graph $G = (V, E)$, a vertex-multicut Y , a set of requests R , a set I of q integers such that:

- (i) There are q Y -components $G_1, \dots, G_q$ with two attachment vertices x_i, y_i . The other components have only one attachment vertex.
- (ii) Every G_i has a $x_i y_i$ -path P_i called *backbone* of G_i . Moreover, the deletion of every edge of P_i decreases the edge connectivity in G_i between x_i and y_i .
- (iii) The set I consists of q non-negative integers $f_1, \dots, f_q$ such that $f_1 + \dots + f_q \leq k - q$.

Parameter: k .

Output: TRUE if there exists a multicut F such that:

- (i) every path P_i contains exactly one edge of F ,
- (ii) every G_i contains exactly $1 + f_i$ edges of F ,
- (iii) the solution F splits Y , i.e. the connected components of $G \setminus F$ contain at most one vertex of Y .

Otherwise, the output is FALSE.

We call Y a *cutset*, i.e. a vertex-multicut which must be split by the solution F . The edges of G which do not belong to the backbones are called *free edges*. The backbone P_i , in which only one edge is deleted, is the crucial structure of G_i . Indeed, the whole proof consists of modifying G_i step by step to finally completely reduce it to the backbone P_i . Here f_i is the number of free edges one can delete in G_i . Observe that $k - q - f_1 - \dots - f_q$ free edges can be deleted in components with one attachment vertex. Our first reduction is the following:

THEOREM 12. *MULTICUT can be reduced to COMPONENT MULTICUT in FPT time.*

The remaining of Section 4 is devoted to the proof of Theorem 12. We first construct a cutset Y . Then we prove that we can reduce to Y -components with one or two attachment vertices. Finally, we show that we can assume that every component with two attachment vertices has a path in which exactly one edge is chosen in the solution. This is our backbone.

4.2 The cutset Y .

We use iterative compression to prove the existence of a vertex-multicut of size $k + 1$, as was done in [18].

LEMMA 13. *MULTICUT can be solved in time $O(f(k)n^c)$ if MULTICUT given a vertex multicut of size at most $k + 1$ can be solved in time $O(f(k)n^{c-1})$.*

PROOF. By induction on n , we solve MULTICUT in time $f(k)(n - 1)^c$ on $G - v$, where $v \in V(G)$. If the output is FALSE, we return FALSE, otherwise the output is a multicut F of size at most k . Let X be a vertex cover of F of size at most k . Thus $X \cup \{v\}$ is a vertex-multicut of our instance, so we solve MULTICUT in time $f(k)n^{c-1} + f(k)(n - 1)^c$ which is at most $f(k)n^c$. $\square$

So we can assume that we have a vertex-multicut Y of size at most $k + 1$.

LEMMA 14. *We can assume that the solution F splits Y .*

PROOF. To any solution F is associated the partition of $G \setminus F$ into connected components. In particular this induces a partition of Y . We branch over all possible partitions of Y . In a given branch, we simply contract the elements of Y belonging to the same part of the partition corresponding to the branch. $\square$

During the following reduction proof, the size of the set Y will never decrease. Since one needs $k + 1$ edges to separate $k + 2$ vertices, the size of Y cannot exceed $k + 1$, otherwise we return FALSE. Hence our primary invariant is the size of Y , and we immediately conclude if we can make it increase.

4.3 Reducing Attachment Vertices

Our second invariant, which we intend to maximize, is the number of Y -components with at least two attachment vertices. This number cannot exceed k , since any solution splits Y . Our third invariant is the sum of the edge connectivity between all pairs of vertices of Y , which we want to increase. This invariant is bounded by $k \binom{|Y|}{2}$ since the connectivity between two elements of Y is at most k . Note that this invariant never decreases when we contract vertices.

LEMMA 15. *If C is a Y -component with at least three attachment vertices, we improve our invariant.*

PROOF. Let x, y, z be attachment vertices of C . Let λ be the edge-connectivity between x and y in C . Let $P_1, \dots, P_\lambda$ be a set of edge-disjoint xy -paths. A *critical edge* is an edge which belongs to some xy -edge cut of size λ . Note that every critical edge belongs to some path P_i . A *slice* of C is a connected component of C minus the critical edges. Given a vertex v of C , the *slice of v* , denoted by $SL(v)$, is the slice of C containing v . Let $B(z)$ be the *border* of $SL(z)$, i.e. the set of vertices of $SL(z)$ which are incident to a critical edge. Note that $B(z)$ intersects every path P_i on at most two vertices, namely the leftmost vertex of P_i belonging to $SL(z)$ and the rightmost vertex of P_i belonging to $SL(z)$. In particular, $B(z)$ has b vertices, where $b \leq 2\lambda$.

We branch over $b + 1$ choices to decide whether one of the b vertices of $B(z)$ belongs to a component of $G \setminus F$ (where F is our solution) which does not contain a vertex of Y . When this is the case, the vertex is added to Y , hence we increase our primary invariant. In the last branch, all the vertices of $B(z)$ are connected to a vertex of Y in $G \setminus F$. We branch again over all mappings f from $B(z)$ into Y . In each branch, the vertex $v \in B(z)$ is connected to $f(v) \in Y$ in $G \setminus F$. Hence we can contract every vertex $v \in B(z)$ to the vertex $f(v)$ of Y . This gives a new graph G' . We denote by S' the subgraph $SL(z)$ after contraction of the vertices of $B(z)$. Observe that S' is a Y -component of G' .

If x and y belong to S' , the edge connectivity between x and y has increased. Indeed, there is now a path P joining x to y inside S' , in particular P has no critical edge. Thus the connectivity between x and y has increased, so our invariant has improved. We assume without loss of generality that x does not belong to S' .

If S' contains an element of Y distinct from z , then S' is a Y -component with at least two attachment vertices. Moreover, there exists a path P in $C \setminus S'$ from x to $B(z)$. Hence we have created an extra Y -component with at least two attachment vertices in G' , which improves our second invariant.

In our last case, S' only intersects Y on z . Therefore $B(z)$ is entirely contracted to z . In particular z is now incident to a critical edge e . So there exists an xy -cut A with $\delta(A) = \lambda$ and $e \in \Delta(A)$. Without loss of generality, we assume that $z \notin A$ (otherwise we consider the yx -cut $\bar{A}$). We denote by B the vertices of $\bar{A}$ with a neighbor in A . In particular, B contains z , has size at most λ , and every xy -path in C contains a vertex of B . Let us denote by L the set $A \cup B$ and by R the set $\bar{A}$. Note that $L \cap R = B$. We now branch to decide in which components of $G \setminus F$ the elements of B are partitioned. If an element of B is not connected to Y in $G \setminus F$, we improve our invariant. If each element of B is contracted to a vertex of Y , both L and R in the contracted graph are Y -components with at least two attachment vertices (respectively $\{x, z\}$ and $\{y, z\}$). We again improve our invariant. $\square$

4.4 Backbones

We now assume that every component has at most two attachment vertices. Let $G_1, \dots, G_q$ be the components of G with two attachment vertices. We denote by λ_i the edge connectivity of G_i between its two attachment vertices x_i, y_i . Recall that our third invariant is just the sum of the λ_i .

LEMMA 16. *We can assume that x_i and y_i have degree λ_i in G_i .*

PROOF. Let A be the unique left $x_i y_i$ -cut with $\delta(A) = \lambda_i$ in the graph G_i rooted in x_i . Let B be the set of vertices of A with a neighbor in $\bar{A}$. We now branch to decide how the components of $G \setminus F$ partition B . If an element of B is not connected to Y in $G \setminus F$, we improve our invariant. If an element of B is contracted to y_i , we increase λ_i . Hence all elements of B are contracted to x_i . Therefore A becomes an x_i -cherry, hence $A \setminus x_i$ is removed from G_i . The degree of x_i inside G_i is now exactly λ_i . We apply the same argument to reduce the degree of y_i to λ_i . $\square$

We now branch over all partitions of k into $k_0 + k_1 + \dots + k_q = k$, where k_i is the number of edges of our solution chosen in G_i when $i > 0$, and k_0 is the number of edges chosen in the y -components for $y \in Y$.

LEMMA 17. *Every component G_i can be deleted or has a backbone.*

PROOF. If $k_i \geq 2\lambda_i$, we simply delete in G_i the edges incident to x_i and y_i , reduce our parameter, and improve our invariant. So we can assume that $k_i \leq 2\lambda_i - 1$. Let $P_1, P_2, \dots, P_{\lambda_i}$ be some edge-disjoint $x_i y_i$ -paths. Our algorithm now branches $2\lambda_i$ times, where the branches are called B_j and B'_j for $j = 1, \dots, \lambda_i$. In the branch B_j , we assume that there is only one edge of our solution selected in P_j , and this edge belongs to an $x_i y_i$ -cut of size λ_i . In the branch B'_j , we assume that all the edges of our solution selected in P_j are not critical. Let us show that any solution F belongs to one of these branches. If F does not belong to any branch B'_j , this means that F uses at least one critical edge in each P_j . But since $k_i \leq 2\lambda_i - 1$ some P_j only intersects F on one edge, which is therefore critical. Hence F is a solution in the branch B_j . Thus this branching process is valid. In the branch B_j , we contract all non critical edges of P_j , therefore P_j is the backbone we are looking for. In the branch B'_j , we contract all critical edges of P_j , hence the connectivity λ_i increases. We thus improve our invariant. $\square$

This concludes the proof of Theorem 12.

5. BACKBONE MULTICUT IS FPT

5.1 Backbone Multicut

We introduce here the problem BACKBONE MULTICUT, which is a generalization of COMPONENT MULTICUT. Our goal is to show that BACKBONE MULTICUT is solvable in FPT time, which implies that COMPONENT MULTICUT is FPT, which in turns implies that MULTICUT is FPT thanks to Theorem 12.

BACKBONE MULTICUT:

Input: A connected graph $G = (V, E)$, a set of half-requests R , a set Y of vertices, a set B of q variables, a set C of clauses, a set I of q integers such that:

- (i) G has q Y -components called G_i with two attachment vertices $x_i, y_i \in Y$, with $i = 1, \dots, q$. Moreover, G_i has a backbone P_i and the x_i, y_i -connectivity in G_i is λ_i . Recall that the edges of G which are not in backbones are called *free edges*.
- (ii) The set R contains *half-requests*, i.e. sets of triples (u, y, v) , informally meaning that vertex u sends a request to vertex v via y , where $y \in Y$. Also, Y is a u, v -cut for every half-request $(u, y, v) \in R$.
- (iii) The set B contains q integer-valued variables $c_1, \dots, c_q$. Each variable c_i corresponds to the deletion of one edge in the backbone P_i . Formally, if the edges of P_i are $e_1, \dots, e_{\ell_i}$, ordered from x_i to y_i , the variable c_i can take all possible values from 1 to ℓ_i , and $c_i = r$ means that we delete the edge e_r in P_i .
- (iv) The clauses in C have four possible types: $(c_i \leq a \Rightarrow c_j \leq b)$, or $(c_i \leq a \Rightarrow c_j \geq b)$, or $(c_i \geq a \Rightarrow c_j \geq b)$, or $(c_i \geq a \Rightarrow c_j \leq b)$.
- (v) The set I consists of q non-negative integers $f_1, \dots, f_q$ summing to a value at most k . Each integer f_i corresponds to the number of free edges of the solution which are chosen in G_i .

Parameter: k .

Output: TRUE if:

- (i) There exists an assignment of the variables of B which satisfies C ,
- (ii) There exists a subset F of at most k free edges of G , which contains f_i free edges in G_i for $i = 1, \dots, q$.
- (iii) The union F' of F and the backbone edges corresponding to the variables of B splits Y and intersects every half-request of R , i.e. for every half-request $(u, y, v) \in R$ every path between u and v containing y intersects F' .

Otherwise, the output is FALSE.

Note that the deletion allowance of BACKBONE MULTICUT is $k + q$. COMPONENT MULTICUT directly translates into BACKBONE MULTICUT with an empty set of clauses, and where each request is simulated by one or two half-requests. This section is devoted to the proof of the following result.

THEOREM 18. *BACKBONE MULTICUT can be solved in FPT time.*

Figure 1: A component in a BACKBONE MULTICUT instance. The path at the bottom is the backbone. Note that each edge of the backbone does indeed belong to a λ -cut.

5.2 Invariants

Our primary invariant is the sum of the f_i , which starts with value at most k and is non-negative. Any branch in which we can decrease it will be considered solved. Our secondary invariant is the sum of the $\lambda_i - 1$, called the *free connectivity*, which we try to increase. Observe that this invariant is bounded above by k . For our last invariant, recall that the *slice* $SL(v)$ of some vertex v in a component G_i is the connected component containing v of G_i minus its *critical* edges, *i.e.* edges of λ_i -cuts. Observe that since the edges of P_i are critical, the slices of distinct vertices in P_i do not intersect. The *slice connectivity* of a vertex v in P_i is the $x_i y_i$ -edge-connectivity of $G_i \setminus SL(v)$. We denote it by $sc(v)$. For instance, if the set of neighbors of v intersect every $x_i y_i$ -path in $G_i \setminus P_i$ then we have $sc(v) = 0$. Conversely, if $v \in P_i$ has only neighbors in P_i , we have $sc(v) = \lambda_i - 1$. The *slice connectivity* sc_i of G_i is the maximum of $sc(v)$, where $v \in P_i$. Our third invariant is the sum sc of the sc_i , for $i = 1, \dots, q$, and we try to minimize this invariant. Observe that at any step, sc is at most k .

Our goal is to show that we can always improve our invariant, or conclude that $\lambda_i = 1$ for all i . We consider a component G_i with $\lambda_i > 1$, say G_1 . To avoid cumbersome indices, we assume that the attachment vertices of G_1 are x and y , and that their edge-connectivity is denoted by λ instead of λ_1 . Moreover, we still denote by P_1 the backbone of G_1 , but we assume (against our previous notations) that $P_1, P_2, \dots, P_\lambda$ is a set of edge-disjoint xy -paths in G_1 . We visualize x to the left and y to the right (see Figure 1). Hence when we say that a vertex $u \in P_i$ is to the left of some $v \in P_i$, we mean that u is between x and v on P_i .

5.3 Contracting Edges

In our proof, we contract edges of the backbone and also free edges which are not critical. At each step, we always preserve the fact that the edges of the backbone are critical.

When contracting an edge of the backbone P_1 , we need to modify several parameters. Assume that the edges of P_1 are $e_1, \dots, e_\ell$. The variable c_1 represents the edge of P_1 which is deleted in our multicut. Now assume that the edge $e_i = v_i v_{i+1}$ is contracted. All the indices of the edges which are at least $i + 1$ are decreased by one. All the constraints associated to the other backbones are not affected by the transformation. However, each time a clause contains a literal $c_1 \geq j$, where $j > i$, this literal must be replaced by $c_1 \geq j - 1$. Similarly each occurrence of $c_1 \leq j'$ for $j' \geq i$ must be replaced by $c_1 \leq j' - 1$. If a set of edges is contracted, we perform the contractions one by one.

The collection of paths $P_2, \dots, P_\lambda$ can be affected during our contractions since it can happen that a path P_i with $i \geq 2$ contains both endpoints of a contracted edge uv . In such a case, we remove from P_i the loop formed by the contraction, *i.e.* the subpath of P_i between u and v . We thus preserve our path collection.

5.4 Choosing a Stable Edge

Let v be a vertex of P_1 . The *tag* of v is the subset $t(v) := \{i : P_i \cap SL(v) \neq \emptyset\}$, *i.e.* the set of indices of the paths intersecting the slice of v . Note that $t(v)$ contains 1. Observe also that the slice connectivity of G_1 is the maximum of $\lambda - |t(v)|$, where v belongs to P_1 . By extension, the *tag* of an edge $v_i v_{i+1}$ of the backbone P_1 is the ordered pair $(t(v_i), t(v_{i+1}))$. When speaking of an XY -edge, we implicitly mean that its tag is (X, Y) . In particular, the edge of P_1 which is selected in our solution has a given tag. We branch over the possible choices for the tag XY of the deleted edge of P_1 . Let us assume that the chosen edge has tag XY .

LEMMA 19. *If $X \neq Y$, we improve our invariant.*

PROOF. Since only one edge is cut in the backbone, we can contract all the edges of P_1 with tags different from XY . Observe that when contracting some UV -edge of P_1 , the tag of the resulting vertex contains $U \cup V$ since the slice of the resulting vertex contains the union of both slices (it can actually be larger). After our contraction, all the edges of P_1 between two consecutive occurrences of XY -edges are contracted, hence the tag of every vertex of P_1 now contains $X \cup Y$. In particular, the slice connectivity of G_1 decreases while the free connectivity is unchanged. Thus our invariant improved. $\square$

Therefore we may assume that we choose an XX -edge in our solution. Let us contract all the edges of P_1 which are not XX -edges. By doing so, we now have that the tag of every vertex of P_1 contains X . After this contraction, our instance is modified, hence we have to branch again over the choice of the tag of the edge chosen in our solution. Any choice different from XX increases the slice connectivity. Hence we can still assume that the tag of the chosen edge is XX .

The slice connectivity of G_1 is $\lambda - |X|$. An XX -edge uv of the backbone is *unstable* if, when contracting uv , the tag of the vertex $u = v$ increases (*i.e.* strictly contains X). Otherwise uv is *stable*. We branch on the fact that the chosen XX -edge is stable or unstable.

LEMMA 20. *If the chosen XX -edge is unstable, we improve the invariant.*

The proof of Lemma 20 and of the following Lemma 21 are omitted due to space restrictions, but are easily obtained via simple branching.

5.5 Contracting Slices

In this part, we assume that the chosen edge of P_1 is a stable XX -edge. A vertex v of P_1 is *full* if v belongs to every P_i , where $i \in X$ (see Figure 2). Our goal in this subsection is to show that we can reduce to the case where $X = \{1, \dots, \lambda\}$. By the previous section, any branching increasing the tag of the chosen edge would improve the invariant. So we assume that in all our branchings, the chosen edge is still a stable XX -edge.

Figure 2: All backbone vertices of this component are full, where $X = \{1, 2, 3, 4\}$, with the backbone P_1 at the bottom.

LEMMA 21. *We can assume that all backbone vertices are full.*

Note that after performing the reduction of Lemma 21, if $v_i v_{i+1}$ is a stable XX -edge, then for every vertex $w \in P_j$, with $j \in X$, which lies between v_i and v_{i+1} in P_j , every wY -path contains v_i or v_{i+1} . In particular, if $X = \{1, \dots, \lambda\}$ then v_i and v_{i+1} are xy -cut vertices in G_1 .

LEMMA 22. *We can assume that $X = \{1, \dots, \lambda\}$.*

In other words, we can reduce to the case where every vertex of P_1 is a cutvertex of G_1 .

PROOF. Assume that X is not equal to $\{1, \dots, \lambda\}$. We show that we can partition our component G_1 into two components G_1^1 and G_1^2 . This partition leaves the free-connectivity unchanged, but decreases the slice connectivity. A vertex v_i of the backbone P_1 is *left clean* if the edge $v_{i-1}v_i$ of P_1 is a stable XX -edge, but the edge $v_i v_{i+1}$ of P_1 is not. It is *right clean* if the edge $v_i v_{i+1}$ is a stable XX -edge, but the edge $v_{i-1}v_i$ is not. Finally, v_i is *clean* if both $v_{i-1}v_i$ and $v_i v_{i+1}$ are stable XX -edges. When enumerating all left clean and right clean vertices from left to right, we obtain the sequence of distinct vertices $r_1, l_1, r_2, l_2, \dots, r_p, l_p$ where the r_i are right clean and the l_i are left clean vertices. Observe that x and y do not appear in the sequence since their tags are $\{1, \dots, \lambda\}$. Let us consider a pair r_i, l_i . We say that a vertex v of G_1 is *between* r_i and l_i if every path from v to x or y intersects $\{r_i, l_i\}$. Let B_i be the set of vertices which are between r_i and l_i . Let B be the union of B_i for $i = 1, \dots, p$. Let G_1^1 be a copy of the subgraph induced by B on G_1 . Observe that G_1^1 has p connected components, since $l_i \neq r_{i+1}$. We contract in G_1^1 the vertices l_i and r_{i+1} , for all $i = 1, \dots, p-1$, hence making G_1^1 connected. We finally identify in G_1^1 the vertex x with r_1 and y with l_p . The backbone P_1^1 of G_1^1 simply consists of the edges of the original backbone. To construct G_1^2 , we remove from G_1 all the vertices of B which are not left clean or right clean vertices. Hence no stable XX -edge is left in G_1^2 . We contract all the backbone edges of G_1^2 . Formally, all the vertices between x and r_1 are contracted to a vertex $w_1 := x$, more generally all the vertices between l_i and r_{i+1} are contracted to a new vertex called w_{i+1} , and finally all the vertices between l_p and y are contracted to $w_{p+1} := y$. We now add the path $w_1, w_2, \dots, w_{p+1}$ which is the backbone P_1^2 of G_1^2 . We correlate the edges of the backbone of G_1^1 and G_1^2 by adding clauses implying that the chosen edge of P_1^2 is $w_i w_{i+1}$ if and only if the chosen edge of P_1^1 is between r_i and l_i . We finally branch to split the number of free edges f_1 chosen

Figure 3: At the top, a component G_1 before transformation in the proof of Lemma 22. At the bottom, G_1^1 consists of B_1 and B_2 , and G_1^2 is depicted in the center. The dashed edges in G_1^1 and G_1^2 are correlated, and similarly the dotted edges in G_1^1 and G_1^2 are correlated.

in G_1 into $f_1^1 + f_1^2 = f_1$, the respective free edges deleted in G_1^1 and G_1^2 . Let us call G' the graph G in which G_1 is replaced by G_1^1 and G_1^2 . Note that the free edges of G_1 are partitioned into the free edges of G_1^1 and of G_1^2 . Observe that the free-connectivity of G and G' are equal. However, the slice connectivity has decreased in G' , since its value is 0 in G_1^1 and strictly less than $\lambda - |X|$ in G_1^2 . Indeed, for $i = 1, \dots, p-1$, the edge $l_i l_{i+1}$ is either unstable or the tag of one of its endpoints strictly contains X . Hence contracting all vertices between l_i and r_{i+1} strictly increases the tag of the resulting vertex in G_1^2 . Hence we improve our invariant. Figure 3 gives an example of this transformation.

We now have to prove that there exists a multicut in G' if and only if there exists one in G which uses a stable XX -edge. This comes from the following observation: Let $e = v_j v_{j+1}$ be a stable XX -edge of P_1 between r_i and l_i . Let G_e be obtained from G by deleting e , contracting x with all the vertices of P_1 to the left of v_j , and contracting y with all the vertices of P_1 to the right of v_{j+1} . Let G'_e be obtained from G' by deleting e in P_1^1 , deleting the edge $w_i w_{i+1}$ correlated to e in P_1^2 , contracting x with all the vertices of P_1^1 to the left of v_j and all the vertices of P_1^2 to the left of w_i , and contracting y with all the vertices of P_1^1 to the right of v_{j+1} and all the vertices of P_1^2 to the right of w_{i+1} . The key fact is that G_e is equal to G'_e . Hence the multicuts in G and G' selecting the edge e are in one to one correspondence. $\square$

The proof of Lemma 22 produces a new component, hence a new edge to be chosen in a backbone. This increases the deletion allowance by 1, but the number of free edges has not increased.

5.6 Reducing the Lemons

We now assume that each vertex of the backbone P_1 of G_1 intersects all other paths P_i . Let $v_i v_{i+1}$ be an edge of the backbone P_1 . The v_i -cherry C_i is the set of all vertices u of G_1 such that every uY -path contains v_i .

The *lemon* M_i of $v_i v_{i+1}$ is the set consisting of v_i , v_{i+1} and of all vertices u of G_1 which do not belong to a cherry and

Figure 4: The left subgraph L_i of v_i . The set C_i is the v_i -cherry, and M_i is the lemon of the backbone edge $v_i v_{i+1}$.

such that every ux -path in G_1 contains v_i and every uy -path in G_1 contains v_{i+1} . Observe that when contracting $v_i v_{i+1}$, the lemon M_i becomes part of the v_i -cherry, where v_i denotes the resulting vertex. We denote by L_i the union of all C_j with $j \leq i$ and all M_j with $j < i$. We call L_i the *left subgraph* of v_i . Similarly, the *right subgraph* R_i of v_i is the union of all C_j with $j \geq i$ and all M_j with $j > i$. See Figure 4.

If a multicut F selects the edge $v_i v_{i+1}$ in the backbone, then the vertices $x, v_1, \dots, v_i$ all lie in the same connected component of $G \setminus F$. When these vertices $x, v_1, \dots, v_i$ are contracted to x , the set L_i becomes an x -cherry. Half-requests through y with an endpoint in L_i are automatically cut since F splits Y . Consider the terminals T_i of half-requests of L_i which are routed via x . By Theorem 9 we can reduce T_i to a bounded set of terminals K_i (note that these half-requests become equivalent to usual requests, since L_i is now an x -cherry). This motivates the following key definition.

By Lemma 10, we define $\mathcal{L}_i$ to be a bounded active set of edges in the x -cherry obtained from L_i by contracting vertices $x, v_1, \dots, v_i$. By Theorem 11, we can compute such sets $\mathcal{L}_i$ so that $\mathcal{L}_j \cap L_i \subseteq \mathcal{L}_i$ when $i \leq j$.

Let us say that a multicut F selecting $v_i v_{i+1}$ in P_1 is *proper* if $F \cap L_i$ is included in $\mathcal{L}_i$.

LEMMA 23. *If there exists a multicut F of size at most k containing the backbone edge $v_i v_{i+1}$, then there is a proper multicut F' of size at most k containing $v_i v_{i+1}$.*

PROOF. Consider a multicut F containing $v_i v_{i+1}$. As the set $\mathcal{L}_i$ is active in the cherry obtained by contracting the path $x, v_1, \dots, v_i$ in L_i , there exists a multicut F' of size k such that $F' \setminus L_i = F \setminus L_i$ and $F' \cap L_i \subseteq \mathcal{L}_i$. Hence F' is proper and contains $v_i v_{i+1}$. $\square$

We denote by $\mathcal{L}$ the set of all subsets F of size at most k contained in some $\mathcal{L}_i$. We denote by c the maximum size of a set $\mathcal{L}_i$. Note that c is bounded in terms of k .

For two sets $F_i \subseteq \mathcal{L}_i$ and $F_j \subseteq \mathcal{L}_j$ with $j \geq i$, let us write $F_i \preceq F_j$ when $F_j \cap L_{i+1} \subseteq F_i$. Observe that $\preceq$ is a partial order. A subset $\mathcal{F}$ of $\mathcal{L}$ is *correlated* if:

- elements of $\mathcal{F}$ have the same size, and
- $\mathcal{F}$ is a chain for $\preceq$, i.e. if for every F_i and F_j in $\mathcal{F}$, with $F_i \subseteq \mathcal{L}_i$, $F_j \subseteq \mathcal{L}_j$ and $j \geq i$, we have $F_j \cap L_{i+1} \subseteq F_i$.

LEMMA 24. *There is a partition $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_{k(2c)^k}$ of $\mathcal{L}$ into $k(2c)^k$ correlated sets.*

PROOF. Let us prove by induction on $\ell = 0, \dots, k$ that there exists no antichain for $\preceq$ in $\mathcal{L}$ consisting of $(2c)^\ell + 1$ sets of size at most ℓ . This clearly holds for $\ell = 0$. Assume that this holds for $\ell - 1$. By contradiction, let $A = \{F_1, F_2, \dots, F_{(2c)^\ell + 1}\}$ be an antichain of sets of size at most ℓ . Let t_i be an integer such that $F_i \subseteq \mathcal{L}_{t_i}$ for $i = 1, \dots, (2c)^\ell + 1$. We assume that the sets F_i are enumerated in such a way that $t_i \leq t_j$ whenever $i \leq j$. The set F_1 is incomparable to all sets F_i with $i > 1$, hence $F_i \cap \mathcal{L}_{t_1+1} \not\subseteq F_1$ for all $i > 1$. In particular it is non-empty, hence all sets F_i , for $i = 1, \dots, (2c)^\ell + 1$, have an edge in $\mathcal{L}_{t_1+1}$. The sets F_i such that $t_i = t_1$ have an edge in $\mathcal{L}_{t_1}$ by definition. The sets F_i such that $t_i > t_1$ have an edge in $\mathcal{L}_{t_1+1}$ as $\mathcal{L}_{t_i} \cap \mathcal{L}_{t_1+1} \subseteq \mathcal{L}_{t_1+1}$, by definition of the sets $\mathcal{L}_i$. Since the size of $\mathcal{L}_{t_1} \cup \mathcal{L}_{t_1+1}$ is at most $2c$, there exists a subset B of A of size at least $(2c)^{\ell-1} + 1$ of sets F_i sharing a same edge $e \in \mathcal{L}_{t_1} \cup \mathcal{L}_{t_1+1}$. The set $\{F \setminus e \mid F \in B\}$ has size $|B| \geq (2c)^{\ell-1} + 1$ and is an antichain of sets of size at most $\ell - 1$ by definition of $\preceq$. This contradicts the induction hypothesis.

By Dilworth's Theorem, there exists a partition of $\mathcal{L}$ into $(2c)^k$ sets totally ordered by $\preceq$, which can be refined according to the cardinality to obtain a partition into $k(2c)^k$ correlated sets. Such a partition can be found in FPT time. $\square$

Let us now consider such a partition $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_{k(2c)^k}$ of $\mathcal{L}$ into correlated sets. Observe that by Lemma 23 we can restrict our search to multicuts of the following type in G_1 :

- A backbone edge $v_i v_{i+1}$.
- Other edges in the lemon M_i , which separate v_i from v_{i+1} in M_i .
- Edges in $\mathcal{L}_i$.
- Edges in $\mathcal{R}_i$, which is defined analogously to $\mathcal{L}_i$, with the roles of vertices x and y reversed.

Due to space restrictions, the two following proofs are omitted. These proofs are quite technical and we refer the curious reader to a full version of the paper.

LEMMA 25. *We can assume that there are no cherries C_i . Moreover, if a multicut of size at most k exists, there exists one which contains only edges in one lemon M_i .*

THEOREM 26. *We can assume that G_1 only consists of the backbone P_1 .*

5.7 Reducing to 2-SAT

We are left with instances in which the Y -components with two attachment vertices consist of backbones. We now reduce the last components.

LEMMA 27. *We can assume that there is no component with one attachment vertex.*

PROOF. Let $Y = \{y_1, \dots, y_p\}$ and k be the number of free edges in our multicut. A vertex $y_i \in Y$ is *safe* if for each pair of components attached only to y_i there are no internal requests. If y_i is not safe then there is a request (u, y_i, v) in the union of the two components attached to y_i , hence y_i must be disconnected from u or from v by the solution. We explore one branch where u is added to Y , and one branch where v is added to Y . This creates a component with two

attachment vertices. This component has a backbone, and then the number of free edges decreases.

Hence we can assume that all the vertices of Y are safe. The y_i -cherry is the union of all the components attached to y_i . We branch over all possible integer partitions of k into a sum $k_1 + k_2 + \dots + k_p = k$. In each branch, we require that k_i edges are deleted in the y_i -cherry for $i = 1, \dots, p$. By Lemma 10, the y_i -cherry has a bounded active set $\mathcal{L}_i$, hence in the y_i -cherry we can consider only a bounded number of cuts of size k_i : all subsets of $\mathcal{L}_i$ of size k_i . We then branch over these different choices. In a given branch, we delete a particular set of edges F_i in the y_i -cherry. Thus, we delete the vertices of the y_i -cherry isolated from y_i by F_i , and contract the other vertices of the y_i -cherry to y_i . Finally, no Y -cherry remains. $\square$

THEOREM 28. *Multicut is FPT.*

PROOF. By Lemma 27, we are only left with a graph G which is a subdivision of a graph with at most k edges, and a multicut must consist of exactly one edge in each subdivided edge. Let us consider a half-request (v_i, x, v'_j) . Assume without loss of generality that $v_i \in G_1$, $v'_j \in G_2$, and x belongs to G_1 and G_2 (if x does not belong to G_1 or G_2 , then splitting Y automatically results in cutting the half-request (v_i, x, v'_j)). For simplicity, we assume that the edges of both P_1 and P_2 are enumerated in increasing order from x . We add to $\mathcal{C}$ the clauses $x_1 \geq i \Rightarrow x_2 \leq j - 1$ and $x_2 \geq j \Rightarrow x_1 \leq i - 1$. We transform all the half requests in this way. Hence we are only left with a set of clauses which we have to satisfy.

We add all the relations $x_i \geq a \Rightarrow x_i \geq a - 1$ and $x_i \leq a \Rightarrow x_i \leq a + 1$ and $x_i \geq a \Rightarrow \neg(x_i \leq a - 1)$ and $x_i \leq a \Rightarrow \neg(x_i \geq a + 1)$. We now have a 2-SAT instance which is equivalent to the original multicut instance. As 2-SAT is solvable in polynomial time, this shows that BACKBONE MULTICUT is FPT. Hence the simpler COMPONENT MULTICUT problem is FPT. Together with Theorem 12 which reduces MULTICUT to COMPONENT MULTICUT, this concludes the proof of Theorem 28. $\square$

We do not detail the running time of the overall algorithm, but it is not hard to see that it is single exponential. The branching tree has a polynomial depth as the invariants are polynomially bounded, and the amount of work at each node and the branching degree are bounded by single exponentials in k .

We think that it should be possible to adapt our proof for edge-multicut in terms of vertex-multicut.

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