Crossing Patterns in Nonplanar Road Networks

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ABSTRACT

We define the *crossing graph* of a given embedded graph (such as a road network) to be a graph with a vertex for each edge of the embedding, with two crossing graph vertices adjacent when the corresponding two edges of the embedding cross each other. In this paper, we study the sparsity properties of crossing graphs of real-world road networks. We show that, in large road networks (the Urban Road Network Dataset), the crossing graphs have connected components that are primarily trees, and that the remaining non-tree components are typically sparse (technically, that they have bounded degeneracy). We prove theoretically that when an embedded graph has a sparse crossing graph, it has other desirable properties that lead to fast algorithms for shortest paths and other algorithms important in geographic information systems. Notably, these graphs have *polynomial expansion*, meaning that they and all their subgraphs have small separators.

CCS CONCEPTS

• Information systems → Geographic information systems;

KEYWORDS

road network; crossings; nonplanar graphs; sparsity

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1 INTRODUCTION

Road networks are often modeled graph-theoretically, by placing a graph vertex at each intersection or terminus of roads, and connecting vertices by edges that represent each segment of road between two vertices. Thus, each vertex is naturally associated with a coordinate on the earth's surface.

Much past work on algorithms for road networks has either assumed that these networks are planar (that is, that no two roads cross without forming an intersection at their crossing point) or

has added artificial intersection points to roads that cross without intersection, to force these networks to be planar. Planar graphs have many convenient properties, including planar graph duality and planar graph separator theorems [4, 21] that allow natural and important problems on these networks to be solved more quickly. For instance, for planar graphs, it is known how to compute shortest paths in linear time, based on the planar separator theorem [19], in contrast to the situation for general graphs where shortest paths are slower by a logarithmic factor. Unfortunately, as Eppstein et al. [8] observed, the available data for real-world road networks shows that these networks are not actually planar: they include many crossings. This discovery naturally raises the question of how to model nonplanar road networks, in a way that allows efficient algorithms to be based on their properties.

In this context, one would like a model of road networks that is in some sense near-planar (after all, road networks have few points where roads cross without intersecting, although their number is not zero), that is realistic (accurately modeling real-world road networks), and that is useful (leading to efficient algorithms).

One clear property of road networks is that they are *sparse*: the number of road segments exceeds the number of road intersections by only a small factor, half of the average number of segments that meet at an intersection. Since the vast majority of intersections are the meeting point of three or four road segments, this means that the number of road segments should be between 1.5 and 2 times the number of intersections. Researchers in graph algorithms and graph theory have developed a sophisticated hierarchy of classifications of sparse graph families centered around the intuitive notion of sparseness. Many of these types of sparseness imply general algorithmic meta-theorems about the properties that can be computed efficiently for graphs in the given family. In particular, many of the known algorithms for planar graphs can be extended to the class of graphs of *polynomial expansion*, a property that was originally defined using graph minor theory but that has a more natural equivalent definition (for classes of graphs closed under taking subgraphs) in terms of the existence of sublinear-size separators [7, 23]. Graphs of polynomial expansion support efficient separator-based divide-and-conquer algorithms, as well as more sophisticated pattern matching algorithms based on their graph minor properties. We would like to show that road networks, too, have small separators, and therefore that they can support all of these algorithms.

In this paper, we provide a mathematical model of non-planar road networks in terms of the sparseness of their *crossing graphs*, graphs representing pairs of road segments that cross in the network. We analyze the Urban Road Network Data set and show that, indeed, it is a good fit for the model. Additionally, we prove that networks within this model have polynomial expansion, from which it

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follows that the linear-time planar shortest path algorithm of Klein et al [19] can be adapted to work on these networks, despite their non-planarities.

2 PAST WORK

2.1 Nonplanar road networks

The past work by Eppstein et al. [8–10] has attempted to model nonplanarities in planar road networks in two different ways. In [8] the authors posited that road networks are subgraphs of the intersection graphs of systems of disks (the disks centered at each intersection of roads with radius equal to half the length of the longest segment of roads meeting at that intersection) and that, with a small number of exceptional high-radius disks, these disks have low *ply* (at most a constant number of non-exceptional disks cover any point of the Earth's surface). They performed an empirical analysis of road network data showing that this model fits actual road networks reasonably well, and they used the assumption to develop efficient road network algorithms. Unfortunately, the class of networks defined by this model does not fit well into the theory of sparse networks, because of its handling of the exceptional disks. Because these disks could be arbitrary, they could in principle heavily overlap each other, producing dense subgraphs that prevent the graphs defined in this way from having polynomial expansion or other good sparseness properties. This misbehavior seems unlikely to happen in actual road networks, but this mismatch between theory (where these graphs can have dense subgraphs) and practice (where dense subgraphs are unlikely) indicates that it should be possible to replace their model of road networks by another model that more accurately matches the sparsity properties of real-world road networks.

Later work of the same authors [10] attempted to justify the low number of crossings in road networks by showing that randomly chosen lines (modeling, for instance, a highway cutting across an older city grid) typically have a sublinear number of crossings with other roads. Another paper [9] used this observation of few crossings per line as the basis for a very weak assumption about road networks, that the total number of crossings is smaller than the number of intersections by a sufficiently large non-constant factor. This assumption allows some algorithms to be performed efficiently; notably, the crossings themselves can all be found in linear time. However, it is not strong enough to imply the existence of small graph separators for arbitrary subgraphs of road networks, a property that is required by fast separator-based graph algorithms. Additionally, this paper's assumptions about the sparsity of crossings are on dubious ground from an empirical point of view: what reason is there to believe that the ratio of intersections to crossings in large street networks is non-constant rather than a large constant?

2.2 Nearly-planar graphs

The graph theory literature includes many natural generalizations of planar graphs that we may choose from. Among these, the k -apex graphs have been defined as the graphs that can be made planar by the removal of k vertices [18]; however, in road networks, the number of vertices that would need to be removed would typically be proportional to the number of crossings, a large enough

Figure 1: A 1-planar graph: each edge is crossed at most once. Although more general than planar graphs, this class of graphs does not adequately model real-world road networks.

number that it is not reasonable to treat it as a constant. Similarly, the k -genus graphs are the graphs that can be embedded without crossings into a surface of genus at most k [5]. A road network can be embedded without crossings on a surface with a handle for each overpass or tunnel, and this surface would have genus proportional to the number of overpasses and tunnels, but again this number would be too large to treat as a constant. These two graph families form minor-free graph families: families of graphs that, like the planar graphs, have some forbidden graphs that cannot be formed from contractions or deletions of graphs in their family. However, there is no reason to expect any particular forbidden minors in road networks.

Among the many generalizations of planar graphs that have been studied in the graph theory literature, another one seems more promising as a model for road networks: the 1-planar graphs [26] or more generally k -planar graphs [6, 15, 25]. A 1-planar graph is a graph in which every road segment has at most one crossing (Figure 1). More generally a k -planar graph is a graph in which every road segment has at most k crossings. Many of the sparsity properties of these graphs follow directly from planarization: if one replaces each crossing with a vertex, one obtains a planar graph in which the number of vertices has been blown up only by a factor of $O(k)$. Based on this principle, it is known that these form sparse families of graphs, and in particular they obey a separator theorem like that for planar graphs but with a dependence on k as well as on the number of vertices in the size of the separator [6]. Although k -planar graphs are NP-hard to recognize from their graph structure alone [2, 15], that is not problematic for their application to road networks, because in this case an embedding with few crossings would already be known: the actual embedding of the roads on the surface of the earth.

Therefore, it is tempting to model road networks as 1-planar or k -planar graphs. However, the restriction on the number of crossings per edge may be too restrictive to model real-world road network graphs. As we show, the assumption that road networks are 1-planar does not fit the actual data, because real-world road network data includes road segments that have many crossings. In particular, a long segment of highway may have many crossings between interchanges. However, despite the poor fit of this model to

Figure 2: A drawing of a graph with crossings (top) and its crossing graph (bottom). The eight isolated vertices of the crossing graph correspond to the eight uncrossed edges in the upper graph.

the data, we may take inspiration from 1-planar graphs in finding a more general class of graphs with few crossings per edge in some more general sense, that still maintains the other desirable properties of this graph class and that allows the algorithms from the theory of 1-planar graphs to be applied to road networks.

3 OVERVIEW OF NEW RESULTS

3.1 The crossing graph

The main new idea of this paper is to study crossings in road networks by introducing a new auxiliary graph, the *crossing graph* of the road network.

Definition 3.1. We define the *crossing graph* of an embedded graph G to be an undirected graph, different from G itself. Each edge of G becomes a vertex in the crossing graph. When two edges of G cross each other, we connect the two corresponding vertices of the crossing graph (the ones representing the two segments) by an edge representing the crossing.

This concept is illustrated in Figure 2. In particular, for a road network, each vertex of the crossing graph represents a segment of road, and each edge of the crossing graph represents two road segments that cross without meeting at an intersection.

This structure allows many natural properties of the underlying road network can be read off directly from the crossing graph. For instance, a segment of road is uncrossed if it corresponds to an isolated vertex (one without any incident edges) in the crossing graph. A road network is crossing-free (planar) if and only if all its segments are uncrossed, if and only if the vertices of the crossing graph are all isolated, if and only if the crossing graph itself is an independent set.

For a more complex example, a road network can be modeled as a 1-planar graph (each edge has at most one crossing) if and only if the crossing graph has maximum degree one; that is, if its crossing graph is a matching. Similarly, the road network is k -planar if its crossing graph has maximum degree k .

Our hypothesis is that the crossing graph is a sparse graph (although possibly not one with constant maximum degree) and that its structure can be used to investigate the graph-theoretic structure of the road network itself. We study this question both empirically (by computing and examining the structure of crossing graphs for actual large-scale road networks) and theoretically (by proving that certain types of sparsity in the crossing graph imply the existence of small separators and efficient algorithms for the underlying road networks).

3.2 Empirical experiments

We investigate empirically the graph structure of the crossing graph, by constructing this graph for the 80 of the most populated urban areas in the world given by the Urban Road Network Dataset [17].

Our investigations show that the degree of the crossing graph is small, but not necessarily small enough to consider to be a constant: we found vertices of degree up to 166 (road segments with up to 166 other segments crossing them, none of the crossings forming an intersection). However, we found that the crossing graphs tend to have much smaller *degeneracy*, a number d such that every graph in a given family of graphs (closed under taking subgraphs) has at least one vertex of degree at most d . For instance, trees are exactly the connected graphs of degeneracy one, and our work found that many of the connected components of the crossing graph are trees. More generally, the maximum degeneracy that we found in the crossing graph of any road network was 6.

3.3 Theory of networks with sparse crossing graphs

Based on our empirical investigations, we undertook a theoretical study of the networks whose crossing graphs have bounded degeneracy. We prove theoretical results showing that these networks are closed under subgraphs and that (like the k -planar graphs) they always have small separators. Therefore, they form a new family of graphs of polynomial expansion.

4 PRELIMINARIES

Before detailing our experimental and theoretical results, we provide some necessary definitions.

4.1 Sparse graph properties

All graphs in this paper finite. Although road networks are typically directed (by the direction of traffic on one-way streets and divided highways), the direction of the edges does not matter for the crossing pattern and separator properties considered here. Therefore, we treat these graphs as undirected.

Definition 4.1. The *degree* of a vertex in a graph is the number of edges touching that vertex. The *minimum degree* $\delta(G)$ and *maximum degree* $\Delta(G)$ of a graph G are the minimum and maximum, respectively, of the degrees of the vertices in G . A family $\mathcal{F}$ of graphs has *bounded degree* if all graphs in $\mathcal{F}$ have maximum degree $O(1)$; that is, if there is an upper bound on the maximum degree that may depend on $\mathcal{F}$ itself but that does not depend on the choice of a graph within $\mathcal{F}$.

Figure 3: The Robert C. Levy tunnel in San Francisco, in which Broadway (orange) passes under seven other streets (L–R: Hyde, Cyrus Pl., Leavenworth, Jones, Taylor, Himmelmann Pl., and Mason) without intersecting them. CC-BY-SA image from OpenStreetMap.

Definition 4.2. The *degeneracy* of a graph G is the maximum, over subgraphs of G , of the minimum degree of the subgraph.

A concept equivalent to degeneracy (but differing from it by one) was originally called the coloring number by Erdős and Hajnal [11]. For instance, the graphs of degeneracy one are exactly the forests. A graph has degeneracy at most d if and only if its vertices can be ordered in such a way that every vertex has at most d later neighbors. For, given such an ordering, every subgraph of G has a vertex of degree at most d , namely the first vertex of the subgraph to appear in the ordering. Given a graph of degeneracy d , an ordering with this property can be found by greedily removing the minimum degree vertex from each remaining subgraph. This greedy removal process can be performed in linear time and allows the degeneracy to be computed in linear time [22].

Definition 4.3. As with degree, we define a family $\mathcal{F}$ of graphs to have *bounded degeneracy* if all graphs in $\mathcal{F}$ have degeneracy $O(1)$.

That is, family $\mathcal{F}$ has bounded degeneracy if there is an upper bound on the degeneracy of the graphs in $\mathcal{F}$ that may depend on $\mathcal{F}$ itself but that does not depend on the choice of a graph within $\mathcal{F}$.

Definition 4.4. The *hop count* from a vertex u to vertex v in a graph is the minimum number of edges between them. The *radius* of a graph or subgraph is the smallest number r such that there exists a vertex within hop count of r of all other vertices. An *r -shallow minor* of a graph G is a graph obtained from G by possibly deleting some edges and/or vertices of G , and then contracting some radius- r subgraphs of the remaining graph into supervertices. A family $\mathcal{F}$ of graphs has *bounded expansion* if, for all choices of the parameter r , the r -shallow minors of the graphs in $\mathcal{F}$ have bounded edge/vertex ratio. More strongly, family $\mathcal{F}$ has *polynomial expansion* if this edge/vertex ratio is bounded by a polynomial in r .

In particular, for $r = 0$ the shallow minors are just the subgraphs, so a family of graphs with bounded expansion or polynomial expansion must have subgraphs with bounded edge/vertex ratio. This implies that they necessarily also have bounded degeneracy. The graphs of polynomial expansion include the k -apex graphs, k -genus graphs, and k -planar graphs [23], described in our earlier discussion of near-planar families of graphs (Section 2.2).

Graph families of polynomial expansion can also be characterized in terms of separators, small sets of vertices that partition the graph and form the basis of many divide-and-conquer graph algorithms.

Definition 4.5. For an n -vertex graph G and a constant $c < 1$ we define a c -separator to be a subset S of vertices of G such that every connected component of $G \setminus S$ (the subgraph formed by deleting S from G) has at most cn vertices. We say that a family $\mathcal{F}$ of graphs has *sublinear separators* if there exist constants c , d , and e , with $c < 1$ and $e < 1$, such that every n -vertex graph in $\mathcal{F}$ has a c -separator of size (number of vertices) at most dn^e .

For instance, the famous planar separator theorem states that planar graphs have $2/3$ -separators of size $O(\sqrt{n})$, so we can take $c = 2/3$, $d = O(1)$, and $e = 1/2$ [21]. A *separator hierarchy* is formed by taking separators recursively, until all remaining components have size $O(1)$; for planar graphs it can be constructed in linear time [14], and enables linear-time computation of shortest paths [19], among other problems.

Then, for a family $\mathcal{F}$ of graphs that is closed under taking subgraphs (every subgraph of a graph in $\mathcal{F}$ is also in $\mathcal{F}$), $\mathcal{F}$ has polynomial expansion if and only if $\mathcal{F}$ has sublinear separators [7]. The graphs of polynomial expansion have other important algorithmic properties, not directly deriving from their separators; for instance, for every fixed pattern graph H it is possible to test whether H is a subgraph of a graph in a family of bounded expansion (the subgraph isomorphism problem) in linear time. More generally, one can test any property that can be formulated in the first-order logic of graphs, for members of a family of graphs of bounded expansion, in linear time [23].

4.2 Classification of nonplanarities

We distinguish between two kinds of nonplanarity in a road network.

Definition 4.6. An *embedding* of a graph is a mapping from its vertices to points and its edges to curves, such that the vertices at the ends of each edge are mapped to the points at the ends of the corresponding curve. A *crossing* is a point where two edge curves intersect that is not a common endpoint of both curves. A *removable crossing* is a crossing between two edges in an embedding of a

Figure 4: High Five Interchange in Dallas, Texas. CC-BY image File:High Five.jpg by fatguyinalittlecoat from Wikimedia commons.

graph that can be removed (without introducing other crossings) by making only local reroutings in the embedding.

Such a crossing may occur, for instance, when an actual segment of road follows a curved path, passing between segments of other roads without crossing them, but the network segment representing it follows a different straight path that crosses nearby road segments. In such a case, re-routing the segment to follow the actual path of the road will cause these crossings to go away.

Definition 4.7. An *essential crossing* is a crossing between two edges in an embedding of a graph that represent disjoint (non-intersecting) road segments and that cannot be removed by local changes.

Such a crossing may be caused, for instance, when one road follows a tunnel that crosses under several other roads (Figure 3), or by the multiple crossing segments of a highway interchange (Figure 4).

It is possible to remove the essential crossings of a road network by replacing each crossing by an artificial intersection point, one that exists in the graph but not in the actual road network that it represents. We call a planar network constructed in this way the *planarization* of the road network. Indeed, the TIGER data set, commonly used for experiments on road networks, has been planarized in this way. If the edges in this data set are drawn as straight line segments, instead of following the curves of the actual roads that the edges represent, some crossings may arise from the straightening, but the underlying graph of this data set is planar. For this reason, our experiments use a different data set, the Urban Road Network Dataset, that has not already been planarized [17].

However, planarized networks cannot be used to obtain correct results for many road network computations. For instance, using the planarization of a road network in a shortest path routing algorithm would potentially create routes that turn from one road to another at the artificial intersections added in planarization, which do not form usable routes in the real-world network. For this reason, it is problematic to use algorithms designed specifically to work in planar networks, such as the known linear-time planar graph shortest path algorithms [19], on road network data. Instead, we must show that despite their non-planarities, road networks have the underlying properties necessary to support generalized versions of these efficient algorithms.

5 EXPERIMENTS

In this section, we examine the sparsity of crossing graphs experimentally, on real-world road networks. We compute and analyze the crossing graphs for the 80 of the most populated urban areas in the world. Our analysis uses the Urban Road Network Dataset [17].

The Urban Road Network Dataset includes graphs with self-loops and parallel edges. We removed self-loops and parallel edges before processing the data. The data contains both essential and removable crossings.

To find the crossings in the Urban Road Network Dataset, we used a plane sweep algorithm for line segment intersection detection [3], as implemented in the CGAL computational geometry library [12]. Although theoretically-faster algorithms are known for the type of data considered here, in which the segments form a connected geometric graph with few crossings [9], CGAL’s plane sweep is practical and usable, and the slight superlinearity of its time bound is not problematic for the problem sizes we tested.

Because a perfect identification of essential crossings would require the solution of the NP-hard problem of minimizing the number of crossings in graph embeddings [13], our experiments determined whether a crossing is essential or removable heuristically, by using the fact that the data associated with each road segment in the Urban Road Network Dataset indicates whether it is a bridge or tunnel. Our heuristic is that when a crossing occurs between two road segments neither of which is a bridge or a tunnel, then it is removable. However, the Urban Road Network Dataset only includes this bridge and tunnel labeling for a subset of the cities that it covers. For this reason, we restricted our experiments to this subset.

We used the NetworkX Python package [16] to study the structure of the crossing graphs we constructed.

5.1 Hypothesis

Based on our intuitions concerning bridges and tunnels in real-world road networks, we expected the crossing graphs to include some vertices of moderate degree, but otherwise to be very sparse. For instance, we considered it to be possible that all of the connected components of the crossing graph would be trees.

5.2 Results

The results of our experiments can be seen in Table 1 and 2. We have given some of the key properties of the crossing graphs of road networks for 45 out of 80 cities in the table. The table columns

Table 1: Crossing Graphs (both essential and removable crossings)

City	Roads	Crossings	Uncrossed roads	Degeneracy	Degree	Components	Trees	Non-trees
Abidjan	46423	699	45565	3	11	214	193	21
Ahmedabad	18327	703	17567	3	64	149	122	27
Ankara	111054	2713	107943	3	16	769	643	126
Atlanta	381649	7505	372135	4	33	2737	2501	236
Bangdung	26154	537	25555	2	23	156	140	16
Bangkok	188069	9940	178225	5	38	1991	1637	354
Barcelona	296969	13218	281762	4	38	3792	3309	483
Beijing	110115	13491	98339	5	71	2440	1915	525
BeloHorizonte	94207	2797	91221	4	23	632	531	101
Bengaluru	200828	3102	197273	4	36	946	846	100
Bogota	199846	5929	194621	5	166	1070	960	110
Boston	470360	9324	458955	5	25	3125	2830	295
BuenosAires	435039	5344	429340	6	60	1284	1083	201
Calcutta	90283	991	88995	3	18	395	374	21
Chengdu	25847	3278	23076	4	100	530	412	118
Chongqing	27251	3899	23207	4	18	873	658	215
Dalian	18017	1169	16638	3	21	352	292	60
Dallas	548721	15287	532025	4	52	3882	3136	746
Delhi	76240	1853	74240	3	24	458	364	94
Detroit	390597	8163	381056	4	34	2374	2070	304
Dhaka	19922	477	19445	2	20	128	120	8
Dongguan	11076	1773	9557	4	28	279	183	96
Fuzhou	16093	1948	14496	4	23	330	247	83
Guangzhou	69410	9765	60238	4	26	1804	1234	570
Hangzhou	28270	3433	25238	4	26	623	459	164
Harbin	14805	1383	13465	3	38	271	195	76
HoChiMinh	110953	1627	109087	3	13	519	469	50
Houston	632289	13337	616864	4	28	3888	3255	633
Hyderabad	172822	2461	169948	3	166	737	701	36
Istambul	380466	12002	368090	4	56	2480	2059	421
Jacarta	105318	4811	100998	4	159	746	620	126
Johannesburg	267193	6715	259424	4	38	1957	1710	247
Karachi	55362	1091	54236	3	38	235	206	29
Lahore	41115	1506	39689	3	61	223	188	35
London	376437	9015	365645	4	23	2988	2757	231
LosAngeles	552690	17905	532472	5	50	4731	4140	591
Madrid	511910	23482	487090	5	49	5139	4349	790
Manila	328623	5518	321984	4	26	1786	1664	122
Medellin	33310	1118	32174	4	20	271	225	46
Miami	312082	6396	304945	5	41	1668	1387	281
Milan	262466	8512	252789	4	25	2536	2227	309
Moscow	940251	20469	916398	4	40	6647	6185	462
Mumbai	61299	2085	58976	3	20	580	514	66
Nanjing	29947	3224	27018	5	21	623	467	156
Naples	169589	7089	161590	4	49	2242	2006	236

labeled “Roads” and “Crossings” give the total number of nodes and edges, respectively, in each crossing graph; that is, the numbers of road segments and crossings in the original road network. The column giving the number of uncrossed roads lists the number of nodes in the crossing graph which have no incident edges; that is,

the number of uncrossed road segments. These isolated nodes constitute the vast majority of crossing graph nodes. Table 1 captures both essential and removable crossings whereas Table 2 captures only essential crossings.

Table 2: Crossing Graphs (essential crossings only)

City	Roads	Crossings	Uncrossed roads	Degeneracy	Degree	Components	Trees	Non-trees
Abidjan	46423	175	46218	2	5	52	41	11
Ahmedabad	18327	299	18056	3	10	46	24	22
Ankara	111054	1041	110053	3	9	233	144	89
Atlanta	381649	2081	379231	4	14	664	533	131
Bangdung	26154	175	25976	2	21	45	37	8
Bangkok	188069	6902	181482	5	35	1252	956	296
Barcelona	296969	5048	291548	4	36	1380	1118	262
Beijing	110115	10772	101255	5	71	1764	1314	450
BeloHorizonte	94207	1047	93155	3	11	241	187	54
Bengaluru	200828	1158	199700	3	35	249	190	59
Bogota	199846	3270	197517	5	18	422	357	65
Boston	470360	3668	466144	5	23	1098	916	182
BuenosAires	435039	3217	431874	6	60	715	559	156
Calcutta	90283	363	89894	2	15	85	70	15
Chengdu	25847	2385	23951	4	99	346	257	89
Chongqing	27251	2163	25020	4	11	522	368	154
Dalian	18017	757	17126	3	21	230	186	44
Dallas	548721	7694	541247	4	40	1553	966	587
Delhi	76240	836	75437	3	10	171	100	71
Detroit	390597	3123	387062	4	10	891	709	182
Dhaka	19922	247	19711	2	7	61	56	5
Dongguan	11076	1162	10133	4	26	171	101	70
Fuzhou	16093	1228	15134	3	14	194	130	64
Guangzhou	69410	6817	63105	4	25	1241	752	489
Hangzhou	28270	2462	26166	3	23	438	305	133
Harbin	14805	896	13973	3	38	164	113	51
HoChiMinh	110953	616	110325	3	12	149	116	33
Houston	632289	6769	625518	4	19	1431	894	537
Hyderabad	172822	998	171967	2	166	103	89	14
Istambul	380466	4391	376409	4	31	845	573	272
Jakarta	105318	2338	103588	4	158	247	171	76
Johannesburg	267193	1871	265257	3	38	509	379	130
Karachi	55362	460	54931	3	28	97	76	21
Lahore	41115	523	40632	3	57	71	49	22
London	376437	4946	370890	4	23	1497	1350	147
LosAngeles	552690	8941	543170	5	16	2095	1683	412
Madrid	511910	8734	503015	5	47	1981	1598	383
Manila	328623	2276	326260	4	19	577	505	72
Medellin	33310	635	32765	3	12	114	82	32
Miami	312082	2707	309459	5	13	529	294	235
Milan	262466	4297	258068	4	25	1103	887	216
Moscow	940251	9686	930967	4	37	2249	1947	302
Mumbai	61299	1218	60081	3	20	274	224	50
Nanjing	29947	2184	28039	5	19	391	266	125
Naples	169589	3682	165822	4	48	1024	889	135

The columns labeled “Degeneracy” and “Degree” give the degeneracy and maximum degree, respectively, of the crossing graph, as defined in Section 4.1. The remaining columns detail the total

number of nontrivial connected components, the number of components that are trees, and the number of components that are not trees, respectively, in the crossing graph.

5.3 Analysis

The table show that indeed these graphs are sparse. More specifically, our hypothesis that the degeneracy would be significantly smaller than the maximum degree held up in the experiments.

Although it is not true that (as we hypothesized) all components of the crossing graphs are trees, most of them are. The remaining non-tree components all have low degeneracy (at most 6).

6 THEORETICAL ANALYSIS OF GRAPHS WITH SPARSE CROSSINGS

Both our experimental results, and our consideration of nonplanar tunnels and interchanges forming essential crossings in (non-planarized) real road networks, motivate the following question. Suppose that, as the experiments appear to show, road networks have sparse crossing graphs. More precisely, suppose that their crossing graphs have bounded degeneracy, but not necessarily bounded degree. What does this imply about the graph-theoretic properties of road networks? Do they have bounded degeneracy? Polynomial expansion? Here we give positive answers to both of these questions.

First, let us formalize the notion of a graph with a sparse crossing graph.

Definition 6.1. We define a *nice embedding* to be a mapping of the vertices of the graph to points in the plane, and the edges to curves, such that the following conditions are all met:

- Each edge is mapped to a Jordan arc (a non-self-intersecting curve) whose endpoints are the images of the endpoints of the edge.
- If an edge and a vertex are disjoint in the graph, their images in the plane are disjoint.
- If two edges are mapped to curves that intersect, then that intersection consists of a single point, and is either a shared endpoint of both edges or a point where their two curves cross.
- No three edges have curves that all cross at the same point.

Let $\mathcal{C}_d$ denote the family of embedded graphs with nice embeddings, such that the crossing graphs of these embeddings have degeneracy at most d .

Definition 6.2. We say that a graph G is *d -crossing-degenerate* if it belongs to $\mathcal{C}_d$.

In order to apply the equivalence between separator theorems and polynomial expansion of Dvořák and Norin [7], we need to verify that the class of graphs we care about is closed under subgraphs.

LEMMA 6.3. *Every subgraph of a graph in $\mathcal{C}_d$ also belongs to $\mathcal{C}_d$.*

PROOF. Let G be in $\mathcal{C}_d$ and let H be a subgraph of G . Embed H by deleting the edges and vertices of $G \setminus H$ from the embedding of G . Then the crossing graph of the resulting embedding of H is an induced subgraph of the crossing graph of the embedding of G . Since taking induced subgraphs cannot increase the degeneracy, the degeneracy of the crossing graph of H is at most d . Therefore, H belongs to $\mathcal{C}_d$. $\square$

Next, we examine the number of crossings that a graph in this family can have. As we show below, a linear bound on the crossing number follows directly from the assumption of low crossing graph degeneracy.

LEMMA 6.4. *For the graphs in $\mathcal{C}_d$, a graph with m edges has at most dm crossings.*

PROOF. We can reduce any graph in $\mathcal{C}_d$ to one with no edges (and no crossings) by repeatedly removing an edge that is crossed by at most d other edges, using the assumption that the crossing graph is d -degenerate and therefore that there exists an edge that is crossed at most d times. This process eliminates at most d crossings per step and takes m steps, so there are at most dm crossings in the given graph. $\square$

In contrast, graphs with many edges are known to have many crossings per edge. We use the following well-known *crossing number inequality* of Ajtai, Chvátal, Newborn, Szemerédi, and Leighton [1, 20, 24]:

LEMMA 6.5. *Let G be an embedded graph with n vertices and m edges, with $m \geq 4n$. Then G has $\Omega(m^3/n^2)$ crossings.*

This allows us to show that the d -crossing-degenerate graphs are sparse.

LEMMA 6.6. *Every n -vertex embedded graph G in $\mathcal{C}_d$ has $O(n\sqrt{d})$ edges.*

PROOF. Let the number of edges in G be $\gamma n\sqrt{d}$, for some parameter γ . Then by Lemma 6.5 the number of crossings is $\Omega(n(\gamma\sqrt{d})^3)$ and the number of crossings per edge is $\Omega((\gamma\sqrt{d})^2) = \Omega(\gamma^2 d)$. However, by Lemma 6.4 the number of crossings per edge is also at most d . For these two things to both be true, it must be the case that $\gamma = O(1)$, so the number of edges in G is $O(n\sqrt{d})$. $\square$

Definition 6.7. Given a nicely embedded graph G , we define the *planarization* $P(G)$ to be the planar graph that has a vertex for each vertex or crossing point of G , and an edge for each maximal segment of an edge curve of G that does not contain a crossing point.

It follows from our previous lemmas that d -crossing-degenerate graphs have small planarizations.

LEMMA 6.8. *For the graphs in $\mathcal{C}_d$, every n -vertex graph G has a planarization with $O(nd^{3/2})$ vertices and edges.*

PROOF. This follows immediately from the already-proven facts that G itself has $O(n\sqrt{d})$ edges (Lemma 6.6) and an average of at most d crossings per edge (Lemma 6.4). $\square$

Using these lemmas, we can prove our main result, that these graphs have sublinear separators.

THEOREM 6.9. *The graphs in $\mathcal{C}_d$ have sublinear separators and polynomial expansion. A sublinear separator hierarchy for these graphs can be constructed from their planarizations in linear time.*

PROOF. To prove the existence of sublinear separators, we apply the planar separator theorem [21] to the planarization $P(G)$ of a graph G in $\mathcal{C}_d$. By Lemma 6.8 $P(G)$ is larger than G by at most

a factor depending only on d , so (treating d as a constant for the purposes of O -notation) $P(G)$ has separators of size $O(\sqrt{n})$. One application of the separator theorem may produce components of $P(G)$ that are larger than the number n of vertices in G , but recursively applying the separator theorem a bounded number of times will produce components that are smaller than n by a constant factor. The resulting separators have vertices in $P(G)$, corresponding to crossings in G ; they can be transformed into separators in G itself by replacing each crossing vertex in the separator by the set of four endpoints of the corresponding crossing edges in G . This replacement only increases the separator size by a constant factor.

Since this method uses only planarization and planar separators, we may apply the linear time method for constructing planar separator hierarchies to $P(G)$ [14], to get a separator hierarchy to G as well.

The fact that these graphs have polynomial expansion follows from the existence of sublinear separators, and from the fact that $\mathcal{C}_d$ is closed under subgraphs (Lemma 6.3). $\square$

By applying the method of [19], we obtain:

COROLLARY 6.10. *If we are given the planarization of a graph G in $\mathcal{C}_d$, we can compute shortest paths in G itself in linear time.*

In conjunction with known fast algorithms for finding planarizations [9], this leads to a linear-time algorithm for shortest paths whenever the number of crossings is sufficiently smaller than the overall number of road segments (as it was in our experiments).

7 CONCLUSIONS

We have performed a computational study of the removable crossings in large-scale planarized road network data. Our study shows that these crossings form a crossing graph that has high degree vertices (up to degree 166), but that most connected components of the crossing graph are trees and that the few remaining components have maximum degeneracy six.

Based on our study, we developed a model of nearly-planar graph, the d -crossing-degenerate graphs, consisting of the graphs that can be embedded with d -degenerate crossing graphs. We showed that this family of graphs is closed under the operation of taking subgraphs. In addition, for constant values of d , these graphs have a linear number of crossings, a linear number of edges, and separators of size proportional to the square root of the number of vertices. In addition, a separator hierarchy for these graphs can be constructed in linear time, and applied in separator-based divide and conquer algorithms for shortest paths and other computational problems on road networks.

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