Synthesis of Fast and Collision-free Folding of Polyhedral Nets

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Figure 1: An optimized unfolding (top) created using our method and an arbitrary unfolding (bottom) for the fish mesh with 150 triangles (left). Each row shows the folding sequence by linearly interpolating the initial and target configurations. Self-intersecting faces, shown in red at bottom, result in failed folding. Additional results, foldable nets produced by the proposed method and an accompanied video are available on <http://masc.cs.gmu.edu/wiki/LinearlyFoldableNets>.

ABSTRACT

A predominant issue in the design and fabrication of highly non-convex polyhedral structures through self-folding, has been the collision of surfaces due to inadequate controls and the computational complexity of folding-path planning. We propose a method that creates *linearly foldable* polyhedral nets, a kind of unfoldings with linear collision-free folding paths. We combine the topological and geometric features of polyhedral nets into a hypothesis fitness function for a genetic-based unfold and use it to map the polyhedral nets into a low dimensional space. An efficient learning strategy is used to optimize the fitness function to produce the optimal nets. We experimentally demonstrate that the proposed method can find linearly foldable nets for highly non-convex polyhedra with substantial complexity. The technique presented in the

paper will provide a powerful tool to enable designers, materials engineers, roboticists, to name just a few, to make physically conceivable structures through self-assembly by eliminating the common self-collision issue. It also simplifies the design of the control mechanisms when making deployable shape morphing devices. Additionally, our approach makes foldable papercraft more accessible to younger children and provides chances to enrich their education experiences.

CCS CONCEPTS

• **Theory of computation** → **Computational geometry**; *Random search heuristics*; • **Computing methodologies** → *Mesh models*;

KEYWORDS

Polyhedral nets, foldability analysis, optimization, motion planning, papercraft, self-folding robot, education

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1 INTRODUCTION

For centuries, it is known that generating unfoldings from a 3D polyhedron, called *polyhedral net* [Dürer 1977], is a difficult problem [O'Rourke 2008]. Finding a net that can be folded rigidly along the crease lines from the unfolded 2D state to the target 3D state, while avoiding self-intersections, is more challenging even for simple shapes. For example, in a recently research led by An et al. [2014], self-folded 3D surfaces by uniform heating, researchers encountered the issue of collisions in some of their designs, which need a lot of further optimization.

1.1 Not All Nets are Created Equal

It is due to these difficulties, current practices treat net design and folding motion as two independent steps. However, there are numerous examples showing that the foldability of the nets created from a single polyhedron can vary significantly. For example, Fig. 1 shows two nets of a fish model. From the shapes of these nets, it is not possible to identify the better solution with respect to avoiding self-collisions until the model is simulated for folding. These nets also have substantial variations in computing folding paths. For example, in Fig. 1, it is about 350 times slower to finding a collision-free folding motion for an arbitrary net than the optimized net.

1.2 Challenges

Recently, researchers have made progress in both generating valid nets and planning continuous foldings. However, given a polyhedron mesh, it remains computationally infeasible to enumerate all of its unfoldings and evaluate their foldability [Prautzsch and Straub 2005]. Existing methods either generate nets without considering foldability [Prautzsch and Straub 2005; Schlickerrieder 1997; Takahashi et al. 2011; Xi et al. 2016], plan motion on given nets ignoring the fact that net structures have significant impact on foldability [Demaine et al. 2004; Song and Amato 2004; Tachi 2009; Xi and Lien 2015], or require convexity to guarantee unfoldability and foldability [Kim et al. 2017]. The challenge of considering both net design and foldability remains significant for highly non-convex structures.

1.3 Contributions

This paper presents the first work that designs nets by optimizing its foldability as well as its *quality of foldability*. Our approach aims to create nets that are linearly foldable and *uniformly foldable*, if such nets exist. A net is linearly foldable if there exists a collision-free straight line linearly interpolating the flat unfolded 2D state to the fully folded 3D state. Furthermore, a uniformly foldable net can fold under the constraint that their hinges must rotate at the same speed. To overcome high computational complexity in foldability analysis, we combine the topological and geometric features of polyhedral nets into a fitness function and use it to approximate the computationally expensive foldability metric. This synthesis of the domain knowledge with learning technique relieves us from the problem of the complex hypothesis model design and acquisition of large sample data that the traditional evolution control method faces.

We will demonstrate that the polyhedral nets generated using our method are, on average, up to 87 times faster for the planner

to fold. For some non-convex shapes, our algorithm can converge to linearly foldable nets if such net exists, in which cases, the folding planning time is orders of magnitude faster than arbitrarily generated nets. We will also show that the linear foldable nets can facilitate papercraft fabrication in our user study experiment, and our technique provides low cost, easily foldable 3D models to younger children and offers chances to enrich their learning experiences

2 BACKGROUND

This section reviews recent progress in generating valid unfoldings, planning folding motion, and the concept of fitness approximation and evolution control.

The concept of self-folding machines has recently inspired many innovations [Davis et al. 2015; Felton et al. 2014; Liu et al. 2012]. Especially for robots made from low costs 2D active material that can be folded into 3D shape using light or heat [An et al. 2014; Shigemune et al. 2015]. However, most of these self-folding structures cannot perform complex folding motions, e.g., can only uniformly rotate their hinges [Sundaram et al. 2017]. Even for self-folding robots equipped with mechanical hinges [An et al. 2011; Felton et al. 2014], it is highly desirable to reduce the complexity of folding motion. These limitations and constraints raise a problem: given a target 3D shape, how can we design a 2D pattern that can be rigidly folded into the desired 3D shape without stretching or self-collision by following a folding motion that is as simple as possible? In the existing literature, this designing problem has only been addressed for convex shapes. Demaine et al. [2011] presented a polynomial time algorithm that can unfold and fold all convex polyhedra without self-intersection using source unfolding [Miller and Pak 2009].

In the rest of this paper, we use *unfolding* to denote the process of creating a net without considering motion and use *folding* exclusively to address the motion aspect of the problem.

2.1 Polyhedra Unfolding

Given a polyhedron P , the problem of polyhedra unfolding seeks a function $u(P)$ that transforms the faces of P rigidly to produce a net N . In this work, we will focus on edge unfolding that cuts along the edges of the 3D meshes. Finding an edge unfolding of P is to find a spanning tree in the dual graph of P . The difficulty is to ensure the flattened faces in a spanning tree do not overlap, in which case the spanning tree is a *net*. The unfolding problems are much easier if P is convex. Heuristic methods [Prautzsch and Straub 2005; Schlickerrieder 1997] have been developed to assign weights to each dual edge such that the minimum spanning tree of the dual graph has a high probability of becoming a net.

Finding nets of non-convex shapes is a much more challenging task. Recently, genetic-based approaches [Takahashi et al. 2011; Xi et al. 2016] were proposed to evolve the unfoldings by mutating the cut edges until a net with zero overlaps is found. Their unfolding function $u(P, f)$ is parameterized by a fitness function $f : N \rightarrow \mathbb{R}$ that evaluates an unfolding N . An examples of $f(N)$ used in [Xi et al. 2016] is defined as:

$$f(N) = -(\lambda_0 \delta_0 + \lambda_1 \delta_1), \quad (1)$$

where δ_0 is the number of overlaps in the unfolding N , δ_1 is the number of hyperbolic vertices that cause local overlaps in N , and λ_0 and λ_1 are user parameters. The value $f(N)$ of a net (i.e., an unfolding without overlaps) N is 0, however, it is not guaranteed that N can be folded back to P without self-intersection [Biedl et al. 2005].

2.2 Folding Motion

Planning folding motion is nontrivial. Early research focused on folding origami. Demaine et al. [2004] proved that a continuous folding motion always exists if the faces need not remain rigid. Tachi et al. [2009] proposed a heuristic method to continuously fold rigid origami via gradient descent. Origami usually have many faces but low Degrees of Freedom (DoF) due to the closure constraints. On the other hand, polyhedral nets have high DoF that is as large as the number of faces. The motion of these nets is typically found by a motion planner that models the net as a tree-like articulated robot. For example, Song and Amato [2004] proposed a PRM-based planner for nets generated from non-convex polyhedra. Recently, another motion planner has been developed by Xi and Lien [2015] that focuses on finding implicit folding order of a net (Fig. 2). Although these works provide effective algorithms to fold a given unfolding (net or origami), the structure of the unfoldings remains fixed in these approaches.

Figure 2: Folding a non-linearly foldable net requires an implicit ordering; the planner must uncover this order and take two folding steps in correct sequence to avoid self-intersections. The computational cost for such planning can be tremendous for large nets.

2.3 Evolution-based Designs and Evolution Control

Evolution algorithms are widely found in many applications involving morphology designs of linked structures. Lund et. al. [1997] use genetic algorithms to encode the body parameters of robot hardware into a morphological space. These parameters, e.g. robot wheel size, are then evolved through evaluating of the robot’s performance on a specific task in the simulation. The idea of evolving virtual creatures, originally attempted by Sims [1994], where the robot morphology is represented in directed graphs. This method evolves new topologies but lacks improvements in complexity and does not scale to large parameter spaces. Some recent researches were able to evolve more complex robots using compositional pattern-producing networks [Auerbach and Bongard 2010, 2012; Cheney et al. 2013]. Even with the advanced encoding methods, evolving folding robots is still computationally infeasible with constraint kinematics of

folding robots. Our work seeks to combine the domain knowledge of polyhedral nets to facilitate the encoding.

A naive unfolders $u(P, f^*)$ uses f^* as a fitness function in the genetic-based algorithm. However, evaluating the foldability using motion planning is extremely time consuming; a single call of motion planner involves a large amount of sampling and collision checking and can take minutes to hours to complete. Additionally, as many planners are sampling-based algorithm, the estimation of motion planning time is stochastic. In order to obtain a stable estimation of foldability landscape, one can either sample the fitness function multiple times, or average over more individuals. Both methods will, again, dramatically increase the computational cost.

The fitness approximation methods [Jin 2005] have been widely applied to similar problems. Specifically, the functional approximation methods can be used in our problem domain. In functional approximation, an alternate and explicit expression is constructed for the fitness function. Take the most popular polynomial approximation for example, the constructed fitness function has the following form: $f' = w_0 + \sum_i w_i x_i + \sum_{i < j} w_{i+j} x_i x_j$, where $\{x_i\}$ is a set of criteria used to approximate f^* and they are combined by coefficients $W = \{w_i\}$. The coefficients W are then optimized using *evolution control* [Jin 2005] so that $|f' - f^*|$ is minimized.

The success of fitness approximation methods stems from the sparse evaluation of the computationally expensive objective f^* ; only once in each iteration of evolution control. The functional approximation method is essentially a supervised regression learner embedded in the evolution control framework. However, in polyhedron unfolding, it is difficult to find a good hypothesis fitness function f' , even using sophisticated non-linear models, to substitute the folding path planning time f^* . Minimizing $|f' - f^*|$ relies on a large number of samples through evaluation of f^* , which remains extremely time-consuming to obtain.

Fortunately, because our goal is finding the foldable nets, obtaining a perfect approximation that minimizes the error of $|f^* - f'|$ is unnecessary. In this paper, we take a new approach that obtains the optimal net without finding a perfect fitness approximation. This idea is essential to the success of our method.

3 FOLDABILITY OF POLYHEDRAL NETS

If a polyhedral net N is modeled as a tree-like articulated robot, the configuration space (C-space) $\mathbb{C}$ of N is the set of valid configurations $c = \{\theta_i\}$, where θ_i is the folding angle of each hinge. The free configuration space (C-free) is a subset of $\mathbb{C}$ where N has no self-intersection. For self-folding machines made of active materials, the folding motion is more constrained, as most hinges can only rotate monotonically or uniformly at the same speed. Due to a lack of coverage in the existing literature, we provide the following definitions to better describe the foldability of nets under these constraints. Note that dynamics are often not considered in the context of computational folding, and in our discussion the folding motion is considered quasi-static.

Continuously Foldable. Continuous foldable means a net N can be folded back to P without self-collision regardless the complexity of the folding motion.

Monotonically Foldable. The monotonic foldability is a special case in continuous foldability, where the angular velocity must remain positive $\dot{\theta}_i \geq 0$ or negative $\dot{\theta}_i \leq 0$ throughout the folding, as many active material driven hinges cannot reverse the direction.

Linearly Foldable. A net is linearly foldable if there exists a straight line connecting initial and target states in C-free. Linearly foldable net has the shortest folding path in C-free, and the length of the path is

$$L_l = \sqrt{\sum_{i=1}^n \theta_i^2} \quad (2)$$

Physically, a linearly foldable net can be folded under the constraint that all hinges rotate at a speed proportional to the folding angle.

Uniformly Foldable. Distinct from linear folding, uniformly foldable nets can fold under the constraint that their hinges must rotate at the same speed. The linear and uniform foldability are both a special case of monotonic foldability. The length of uniformly folding path is

$$L_u = \sqrt{n\theta_1^2} + \sum_{j=1}^{n-1} \sqrt{(n-j)(\theta_{j+1} - \theta_j)^2} \quad (3)$$

where $\theta_j \leq \theta_{j+1}$ for $j = 1, 2, \dots, n$. If all θ_j are equal, then $L_l = L_u$ and the linear folding path and uniform path is identical for a given net.

Note that L_u is upper bounded by $\sum_{i=1}^n \theta_i$, so $\sqrt{2} > \frac{L_u}{L_l} \geq 1$. This suggests that the uniformly folding path is longer than linear one by a constant, which suggests that the uniform foldability may be slightly harder to achieve.

To evaluate the foldability of a net, motion planning time can be a useful metric. Shorter planning time usually indicates an easier foldable net. Thus, we let $f^*(N) = T_{N,\Pi}/T_{N,A}$ be the foldability metric of a net N , where $T_{N,\Pi}$ is the time needed to check the validity of the path Π , and $T_{N,A}$ is the time needed for a planner A to generate a collision-free path Π . If a net N is continuously foldable, $0 < f^*(N) \leq 1$. And, if N is linear or uniform foldable, $f^*(N)$ is 1. Our objective is to design an unfold $u(P, f^*)$ that maps a polyhedron P to a net N with $f^*(N)$ as close to 1 as possible.

4 OUR METHOD

Inspired by the ideas of *fitness approximation* and *evolution control* [Jin 2005], we use learning strategy (shown in Fig. 5) to overcome two major obstacles in both net design and robot design: large number of design parameters and expensive evaluation function. We will show that our approach is computationally efficient and requires significantly fewer evaluations in each iteration.

4.1 Geometric and Topological Features of Nets

To construct a good hypothesis fitness function, we observe that different unfoldings for a polyhedron may have distinct values in geometric and topological features. For example, Fig. 3 shows four distinct unfoldings of the spherical polygon which meshed by 320 triangles, and their corresponding feature values are listed in Table 1.

Figure 3: The spherical polyhedron model with 320 triangular faces (a) and some its representative nets (b)-(e).

Table 1: Geometric and topological features of the nets of a spherical polyhedron with 320 triangular faces shown in Fig. 3.

Characteristic	Net			
	(b)	(c)	(d)	(e)
Avg. Branch Length	11.83	5.36	11.27	1.94
Avg. Geodesic Branch Length	2.02	0.91	1.92	0.32
Avg. Node Distance	19.62	22.31	51.34	25.60
Avg. Node Geodesic Distance	3.39	3.85	8.84	4.51
No. of Leaf Nodes	24	44	18	104
No. of One-Degree Nodes	274	233	286	114
No. of Two-Degree Nodes	21	43	15	101
Diameter	40	45	124	66
Cut Length	48.26	47.87	47.92	45.62
Border Cuts Length	3.11	13.41	33.59	6.02
Hull Area	23.77	19.94	44.02	19.03
Avg. Node Degree	1.07	1.14	1.05	1.47
Total Vertex Travel Distance	103	120	246	119

These features are insufficient to uniquely define a net, however, we presume that there are underlying correlations between these characteristics and shape of the nets. Our first goal is to map a polyhedral net N into a *feature space* $\mathbb{X}$, in which the foldability can be evaluated for each point in $\mathbb{X}$. Three criteria are considered on selecting characteristics to construct our fitness function:

- (1) There is a trade-off between the model complexity and training time. More characteristics generally lead to more robust model as the nets can be sparsely mapped into the feature space. However, it results in higher dimensional feature space and longer training time.

- (2) The selected features need to be efficiently computed, as they serve as a substitution for evaluating the foldability to save computing costs.
- (3) In order to learn these features in a combined fitness function, the values must be normalized so they are within range $[0, 1]$.

Based on these considerations, the features in bold font type in Table 1 are chosen for constructing our fitness function and their definitions are summarized as follow.

Node Degrees. The first property is a topological feature defined as the ratio between the number of leaf (degree one) nodes and the total number of nodes in a net. Recall that nets are spanning trees, and each node has 1, 2 or 3 neighbors in triangular meshes. As we can observe in a net, the number of leaf nodes equals to the number of branches and approximates the branch lengths of the net.

Cut Length. Edges of a net are either cut or crease edges. Given a polyhedron with $|F|$ faces and $|E|$ edges, all net of this polyhedron must have $\sigma = 2(|E| - |F| + 1)$ cut edges and $|F| - 1$ crease edges. The length of these σ cut edges affects foldability and is usually minimized [Prautzsch and Straub 2005; Takahashi et al. 2011]. For example, cutting two longest edges of a narrow triangle creates a spike and these spikes are likely to collide. The cut length is upper bounded and lower bounded by the total length of σ longest edges and σ shortest edges, respectively, thus is normalized by these bounds.

Border Cuts Length. Cut length does not vary a lot if the polyhedron is tessellated with (nearly) equilateral triangles. Thus, we introduce the concept of border cuts to capture additional features. The intuition behind this is that some cut edges can be easily “zipped” without collision. On the contrary, pairs of matching border cuts are often separated far away and harder to be joined back. The border cut length is normalized by the total cut length.

Hull Area. The convex hull bounding the flattened net provides information about the clearance between branches. If the hull area is close to the total area of the net surface, this would indicate that the net is tightly bounded. We use the ratio of the surface area with respect to the hull area as a normalized factor in our fitness function.

4.2 Enhanced Genetic-based Unfolder

In the previous subsection, we mapped nets into a low dimensional feature space $\mathbb{X}$. We now discuss how to generate nets that corresponds to a particular region in $\mathbb{X}$. Recall that in Section 2.1, we reviewed the state-of-art genetic-based unfoldier that generates valid unfoldings. We can enhance this genetic-based unfoldier to generate unfoldings that satisfy desired topological and geometric features, using the following fitness function

$$f' = \begin{cases} f & \text{in Eq. (1) if net is invalid} \\ \beta + \hat{f}(x_i) & \text{otherwise} \end{cases} \quad (4)$$

Firstly, the unfoldier utilizes f in Equation (1) to generate valid unfoldings without overlapping faces. Secondly, it uses $\hat{f}(x_i)$ to further evolve the nets, where x_i denotes the quantified features in $\mathbb{X}$. The constant β is used to ensure $\beta + \hat{f}(x_i) > 0 \geq f$.

Figure 4: An example of the enhanced genetic-based unfoldier that minimizes the Hull Area feature. (a) The original 3D model. (b) A valid unfolding with Hall Area = 31.6 (c) A further evolved unfolding with Hall Area = 19.2.

Shown in Fig. 4, the enhanced genetic-based unfoldier can minimize the hull area of the unfolding for the bunny model by setting $\hat{f}$ negatively correlated to the *Hull Area* feature. Similarly, we can construct a hypothesis fitness that combines all the selected features that can potentially evolve the nets towards the maximum foldability. Due to the page limits, in this paper, we focus on the design of the learning framework. Readers interested the design of the hypothesis fitness functions should refer to a companion paper [Hao et al. 2018] for the full technical details.

Figure 5: The framework of our learning strategy. In each iteration, multiple enhanced genetic-based unfolders run in parallel to generate polyhedral nets using the assigned fitness functions $\{f_i\}$. The best individuals are then evaluated by the foldability evaluator which is either a linear or an uniform motion planner. An external CMA-ES optimizer then updates the fitness functions $\{f_i\}$ according to their performance $\{f_i^*\}$, and generates a new set of fitnesses. This process is repeated until the evaluation converges.

4.3 Optimizing the Fitness Function

Recall that in Section 2.3, the naive unfoldier $u(P, f^*)$ is infeasible because the foldability evaluator f^* is computationally expensive. In practice, the genetic-based unfoldier typically uses a population of 400 unfoldings, and takes about 300 evolutions in the enhancement

Figure 6: First row: the models used in our experiments. Second row: the intermediate configuration on the linear folding path of the nets in the third row. The star shape has no linearly foldable net. Third row: one of the most continuously foldable nets for the models in the first row, generated using our proposed method.

step. In this case, calling f^* for 120,000 times is extremely time-consuming even for simple nets. Similarly, the traditional evolution control and function approximation method require even more computation to fully approximate f^* with f' using the regression learner. Therefore, we propose a new learning strategy as shown in Fig. 5.

This learning strategy is a sampling based optimizer. In every iteration, we use a set of enhanced genetic-based unfolders $\{u(P, \hat{f}_i)\}$, each is assigned with a slightly different $\hat{f}_i$. Unlike the evolution control method, we do not evaluate all the individuals in the final generation of the evolution. Instead, we only evaluate the best individual (the individual with highest value of fitness) produced by each unfolders. In experiments, we typically run 40 unfolders in parallel, and the linear foldable nets often emerge within 20 iterations for simple models. Thus, we only need 800 calls of the evaluator. Due to the non-convexity of f^* , we use the Covariance Matrix Adaptation Evolution Strategy (CMA-ES) algorithm [Hansen et al. 2003] as an external optimizer to update the fitnesses. The initial guess $\{\hat{f}_i\}_{init}$ is typically generated at random. However, we can reuse the $\hat{f}_{opt}$ obtained from previous training on simple models to accelerate the training of complex 3D models. This iterative relaxation technique is discussed in the experiment section.

5 EXPERIMENTS AND RESULTS

In this section, we evaluate foldability of our optimized polyhedral nets in terms of path planning time (Sections 5.1). To further increase training efficiency of the proposed learning framework, we will discuss an optional optimization step in Section 5.2. We then

demonstrate the uniformly folding with simulated hinge controller in Bullet Physics environment in Section 5.3. Finally, a user study is conducted to show the linearly foldable nets are easier for human to fold (Section 5.4).

5.1 Optimized Polyhedral Nets

We first show that the optimized unfolders generate more foldable polyhedral nets than arbitrary nets produced by an existing unfolders discussed in Section 2.2. We evaluate the foldability using the Average Path Planning Time **APPT**. The metric APPT is defined as follows: (1) For a polyhedron, we unfold it into 40 nets; (2) For each net, we plan the folding path 10 times using a given motion planner; (3) APPT is the average planning time for the 400 paths (40 nets $\times$ 10 paths/net).

All experimental results reported in this section are set up as follows. To learn the fitness function, we run 40 genetic-based unfolders and the maximum training iteration is 50. The population size within the genetic-based unfolders is 400, and the maximum number of generations is 200 for polyhedra with less than 50 faces and 300 for over 50 faces. In each iteration, 40 best individuals are produced, and we use motion planner to evaluate each net 10 times and report the average. Parallelization is used in most stages. All data are collected from a C++ implementation on a workstation with two 2.3GHz Intel CPUs.

The 3D models in Figs. 6 are used in our experiment. Among these models, only the star is not linearly foldable. Our algorithm uncovers the linearly foldable nets if such nets exist. Table 2 clearly shows that the optimized nets have much smaller APPTs, thus are easier to fold.

Figure 7: An optimal fitness $\hat{f}_{128}$ for Bunny-128 is obtained from $\hat{f}_{64}$ trained on Bunny-64, which is used as initial fitness for the learning of $\hat{f}_{128}$. If we only simplify it to Bunny-96, it is still a relatively complex model. However, if we reduce it to Bunny-48, it results in loss of details and non-convexity of the 3D model (e.g. one of the ears of the bunny is lost), the learning converges to a local optimal and $\hat{f}_{48}$ cannot be transferred to the original model.

Table 2: Comparison of APPT of Arbitrarily Generated Nets and Nets Generated Using Optimized Unfolder

Model	DoF	APPT (s)		Speedup
		Arbitrary	Optimized	
L-28	27	0.892	0.021	42.5x
Star-24	23	17.587	1.195	14.7x
Moneybox-48	47	59.546	1.211	48.2x
Bunny-64	63	100.223	1.152	87.0x
Fish-96	95	82.418	2.923	28.1x

Table 3: Comparison of Training Time (min.) for the first 20 Iterations on the Simplified Models with 48, 64, and 96 Faces, and the Original Model with or without Transfer Learning (TL).

Simplified Model	Num. of Faces			Original Model	w/o TL	w/ TL
	48	64	96			
Bunny	39	47	152	Bunny-128	427	76
Fish	35	44	95	Fish-150	521	102
Moneybox	46	71	162	Moneybox-128	502	77

5.2 Iterative Relaxation

Conducting trainings on polyhedra with 100 faces or more remains time consuming. We employ transfer learning [Pan and Yang 2010] to reduce the complexity of the problem by remeshing the polyhedron with fewer faces. The simplified mesh, though may lose details, retain most geometric characteristics. Therefore, the fitness function trained from the simplified model is used as the initial fitness function to train the original model. Table 3 shows that the training time rises dramatically as the DoF increases, e.g., from 39 mins to 427 mins when the bunny increases from 48 faces to 128 faces. Table 3 also shows that transfer learning can significantly accelerate the training, e.g. only 76 mins comparing to 427 mins for the bunny model. Fig. 7 illustrates the transfer learning of the bunny.

Fig. 8 shows results for more complex models obtained using the transfer learning strategy. Finding a linear folding path for these nets is almost instantaneous, while an arbitrarily generated net needs half an hour of path planning time.

5.3 Simulated Hinge Controller

We also implement our polyhedral nets as hinge robots in Bullet Physics simulation to show that it is possible to design a realistic hinge controller for uniformly foldable nets. Shown in Fig. 9, we apply the same impulse to the hinges of the L shape, and each triangular face in the net is a rigid body with small but non-negligible thickness and mass. For this particular shape, the target joint angles are either $\frac{\pi}{2}$ for mountain folds, $-\frac{\pi}{2}$ for valley folds, or 0 for non-folding creases, making it both linearly and uniformly foldable. Shown in Fig 10, the motion is not entirely uniform, due to the variable inertia of linked faces of different lengths, i.e. it is more difficult for a joint to lift heavier linked faces. Although the equal speed condition is not fully satisfied, the optimized net still have large clearance between its linked faces and folds perfectly into the 3D shape under Bullet’s a widely acknowledged collision checker.

Figure 8: The folding sequences of optimal nets for complex models created using transfer learning. First row: linearly folding Moneybox-128; Second row: linearly folding Bunny-128; Third row: uniformly folding Bunny-128.

Figure 9: The uniformly foldable L net folding in Bullet Physics simulation.

5.4 User Study

Polyhedral nets make great papercraft due to low cost and easy to manufacture and can be a great enjoyment for family members of all ages to fold. However folding complex nets can be frustrating due to a large amount of human work and difficulties, especially for children. Some earlier researches mainly focus on segmenting the nets [Xi et al. 2016] and divide the work for multiple people, while the approach may not produce desired artwork or causes sharing problems. One of the early motivation of this work is to find nets easy for a human to fold without reducing the complexity of the 3D mesh. We found the linearly foldable nets have three advantages over the arbitrarily generated nets in terms of folding: 1) the folding is relatively straightforward and does not involve ordering; 2) the

Figure 10: The hinge folding angles of uniformly foldable L shape during simulation.

linearly foldable net tree has larger clearance between branches and subsequently less colliding faces; and 3) it is easy to make video instruction to facilitate human folding by showing the folding process generated by computer motion planner.

We carried out a user study experiment to physically validate our ideas in a classroom setting. The lesson focuses on the essential organs of the human body including a pair of lungs, a heart, a stomach, a pair of kidneys, a liver. Through an experience of putting together and making a 3-D puzzle by hand, this lesson can effectively provide students with a realistic and profound learning of their own bodily organs rather than the superficial perception, as well as increase students' interests of the organs. The lesson consists of a series of learning of the major bodily organs in order to recognize the importance of and understand the functions of them. In this lesson based on muscle memory, students will enhance their spatial sensibility and the activity fabricating 3-D organ puzzles will effectively help students develop their brain. Moreover, students will get to know that human body organs are parts of the body and work properly in specific areas of the body, and they will understand, if they do not keep their body in good shape and treat themselves right, that they will suffer from a crop of troubles such as illness. As a result, through this lesson, students will feel the actual size of their organs and understand there are the limitations of the human body, and subsequently, cultivate a good habit protecting their bodily organs.

Figure 11: Our easily foldable nets used in biology classes.

Figure 12: First row: the 3D models used in our human folding experiments; Second row: the arbitrary unfoldings of the first row; Third row: the linearly foldable nets for the models in the first row, generated using our proposed method; Fourth row: handmade papercraft created by folding the nets.

We use the 3D shapes for six human organs: brain, heart, liver, a pair of lungs, a pair of kidneys and stomach, each of which has over 200 triangular faces. Both linearly foldable and arbitrary nets of these shapes are generated and a cutting plotter Cricut is used to cut these nets on thick paper. Then the nets are given to elementary school students in an age of around 11 years old. There are a total of 127 students in six classes, with a population of 20 to 22 in each class, participated in this experiment. Three classes (64 students) will only fold linearly foldable nets and the other three (63 students) will fold arbitrary nets. In each class, about five to six students will be in a group and each student chooses an organ and are given 50 to 60 minutes to finish. We compare the time they use to fold these nets and how far they are from complete within given time limit.

Table 4 shows the results of our experiment and Fig. 12 shows the models and nets the students folded. Due the limited class time, many students were not able to finish. We evaluate how far they are from finishing by visually inspect the percentage of completion by the end of the class, with 0 being the initially unfolded nets and 1 being the fully folded status. The collected data show that linearly foldable nets take a shorter time to fold in all cases except the heart.

Table 4: Human Folding Experiment Results, Avg. Folding Time Shown in Minutes, Avg. Completion Shown in Percentage, Lin = Linearly Foldable Nets, Arb = Arbitrary Nets

Model	Faces	Avg. Folding Time			Avg. Completion		
		Arb	Lin	Diff	Arb	Lin	Diff
Brain	210	57.50	51.83	5.67	0.66	0.77	0.11
Heart	215	50.50	56.67	-6.17	0.68	0.72	0.04
Kidneys	172x2	56.25	44.00	12.25	0.98	0.88	-0.10
Liver	274	56.67	51.50	5.17	0.60	0.68	0.08
Lungs	144x2	51.60	47.22	4.13	0.84	0.96	0.13
Stomach	200	51.00	48.86	2.14	0.77	0.83	0.06

6 CONCLUSION AND DISCUSSION

In this paper, we proposed a computational design of fast and collision-free folding of polyhedral nets. Algorithmically creating a 2D crease pattern to realize the desired 3D structure involves two steps: (1) generating unfoldings (i.e., polyhedral net), and (2) creating collision-free folding motions that take polyhedral net back to the 3D shape. It is the current practice that net design and folding motion are decoupled and treated as two independent steps. This creates a major challenge in creating self-folding machines

because, given a polyhedron P , the foldability of P 's nets can vary significantly. Certain nets may not be foldable even using the most sophisticated folding motion planners.

Our research presents the seemingly counterintuitive results: there actually exist polyhedral nets with collision-free linearly path throughout the folding motion for those highly complex and highly non-convex geometries. The linearly and uniformly foldable nets open up new possibilities for manufacturing and fabrication that feature low-cost planar active material and self-folding into complex 3D structures. We also see great potential of the proposed work in educational fields. Similar to the pioneer research of HyperGami [Eisenberg and Nishioka 1997] in mathematical education, our work provides opportunities for students to learn abstract geometries in a tangible way.

For future work, we intend to use our algorithm in real-world folding-based structure fabrications, which may involve additional motion constraints. For example, in a recent work involving cutting and folding from planar metallic material using lasers [Lazarus and Smith 2017; Lazarus et al. 2018], the laser beam must not intersect with the unintended part of the folding shape during fabrication. This requires the net to satisfy a certain collision-free path other than linear or uniform folding, which raises new challenges in computationally generating such nets and paths.

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