On a Class of Stochastic Multilayer Networks

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ABSTRACT

In this paper, we introduce a new class of stochastic multilayer networks. A stochastic multilayer network is the aggregation of M networks (one per layer) where each is a subgraph of a foundational network G . Each layer network is the result of probabilistically removing links and nodes from G . The resulting network includes any link that appears in at least K layers. This model is an instance of a non-standard site-bond percolation model. Two sets of results are obtained: first, we derive the probability distribution that the M -layer network is in a given configuration for some particular graph structures (explicit results are provided for a line and an algorithm is provided for a tree), where a configuration is the collective state of all links (each either active or inactive). Next, we show that for appropriate scalings of the node and link selection processes in a layer, links are asymptotically independent as the number of layers goes to infinity, and follow Poisson distributions. Numerical results are provided to highlight the impact of having several layers on some metrics of interest (including expected size of the cluster a node belongs to in the case of the line). This model finds applications in wireless communication networks with multichannel radios, multiple social networks with overlapping memberships, transportation networks, and, more generally, in any scenario where a common set of nodes can be linked via co-existing means of connectivity.

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1 INTRODUCTION

There is an increasing need to understand how different networks interact with each other. One means of such interaction arises when users (nodes) belong to two or more networks (layers). In recent years, there has been a surge of interest in such *multilayer networks* [2, 6] due to their relevance in problems stemming in varied fields. Various models of multilayer networks (also termed multiplex networks and composite networks in the literature) relevant to different application scenarios have been proposed, in particular stochastic multilayer networks whose constructions can be described by one or more control parameters (such as probability of the presence of a node, edge or more complex attributes). For such networks, a wide variety of percolation formulations have been proposed and studied. However, even simple multilayer network models have proven extremely difficult to analyze exactly [4]. Consequently, most of the recent literature on properties of multilayer networks consists of numerical and heuristic analyses.

Our goal in this paper is to consider a simple model for a stochastic multilayer network and to attempt exact characterization of the joint probability distribution of the collective (on-off) configuration of the links of the multilayer network. Our model is as follows. A multilayer network consists of M co-existing networks $G^{(1)}, G^{(2)}, \dots, G^{(M)}$ connecting a common set of users. Each user is active in only a subset of these networks. A user active in both $G^{(1)}$ and $G^{(2)}$, for example, can help connect two other users that are active in $G^{(1)}$ alone, and in $G^{(2)}$ alone, respectively, by forming a bridge. Figure 1 illustrates an example with $M = 3$ networks (layers), where a path connecting u_1 and u_2 must traverse all three layers, and one such path is shown to go through the bridge nodes u_3 and u_4 , both of which belong to more than one layer. A *stochastic multilayer network* is a graph $G = (V, E)$ along with a random process by which each network layer is obtained from G by randomly removing links (called *link thinning*) and randomly deactivating nodes, and a process by which the M thinned layers are merged into a single graph. There are different ways of creating a multilayer network out of the M layers. One is simply to take the union of (the nodes and links of) all the layer graphs; this is illustrated by the three layer network in Figure 1. We consider a slightly more general process whereby all active nodes are included in the final graph and all links that appear in at least K layers.

Figure 1: Multilayer network with three layers.

Concrete examples of such multilayer networks are: (1) a network of cities connected via different airline companies where each city is served only by a subset of all the airlines [1, 3], (2) a network of users with accounts on multiple online social networks [7], and (3) a communication network of units equipped with radios that can listen and transmit simultaneously on a subset of multiple frequencies [8].

With a more liberal interpretation of “co-existence”, such multilayer networks may also arise from taking snapshots of a single network at different time epochs. For example, consider a duty-cycled wireless sensor network where each sensor is active or dormant according to a random periodic schedule and each period is divided into M slots. The m -th layer $G^{(m)}$ then consists of sensors that are active in the m -th slot.

Denote the configuration of a multilayer network by the collection of states of all links in the underlying graph G after the layers have been merged. Here the state of each link is either active (1) or inactive (0). We are interested in characterizing the configuration probability distribution of the multilayer network under the assumption that thinning and deactivation operations occur as independent events. Such a characterization can be useful for computing quantities such as the distribution of the sizes of connected components and average path lengths. Our contributions are:

- We present a new model of a stochastic multilayer network based on link thinning and node deactivation, and show that in general it is a difficult problem to compute probabilities of multilayer network configurations and it remains difficult even to approximate these probabilities.
- We develop efficient algorithms for computing multilayer network configuration probabilities for tree topologies.
- We consider a setting where the number of layers M goes to infinity and where link thinning probabilities and node deactivation probabilities are functions of M . We provide conditions for link existence events to be asymptotically independent.

The paper is organized as follows. Section 2 presents our stochastic multilayer model. The hardness of the problem of computing multilayer network configuration probabilities is addressed in Section 3. Exact results and efficient computational algorithms are presented in Section 4 and the asymptotic independence of the link states as $M \rightarrow \infty$ is found in Section 5. Details are found in [5].

2 MODEL

Let $G = (V, E)$ denote the underlying connectivity network, where V is the set of nodes and $E \subset V \times V$ is the set of links that represent all possible connections between pairs of nodes in V . We assume network G is connected.

Consider an M -layer network whose layers are sub-networks of G obtained by randomly removing links (called *link thinning*) and deactivating nodes. When a node is deactivated on a layer, all links incident on it are removed from the same layer, including those that have survived the independent *link thinning* process. More precisely, the M -layer network is obtained from G as follows. Let $\mathcal{M} = \{1, 2, \dots, M\}$ be the index set for layers. Let $Y_{\mathcal{M}, E} = \{Y_{m, \ell} : \ell \in E, m \in \mathcal{M}\}$ and $Z_{\mathcal{M}, V} = \{Z_{m, i} : i \in V, m \in \mathcal{M}\}$ be two mutually independent sets of independent Bernoulli random variables. For the m -th layer $G^{(m)} = (V, E^{(m)})$, node i is active if and only if $Z_{m, i} = 1$, and the link set $E^{(m)}$ is given by $E^{(m)} = \{\ell \in E : W_{m, \ell} = 1\}$, where $W_{m, \ell} = Y_{m, (i, j)} Z_{m, i} Z_{m, j}$ for link $\ell = (i, j)$. We assume that the link thinning probabilities and the node activation probabilities are the same across different layers but may depend on individual links and nodes, i.e. $p_\ell = \mathbb{E}[Y_{m, \ell}]$ and $q_i = \mathbb{E}[Z_{m, i}]$ for all $m \in \mathcal{M}$, $\ell \in E$ and $i \in V$. We also assume that all p_ℓ 's and q_i 's are strictly positive.

Let $W_\ell \in \mathbb{N}$ denote the number of layers in which link ℓ is present, i.e. $W_\ell = \sum_{m \in \mathcal{M}} W_{m, \ell}$. Given some threshold $K \in \mathbb{N}$, let $X_\ell = \mathbb{1}_{\{W_\ell \geq K\}}$, where $\mathbb{1}_A$ is the indicator of event A . We say that link ℓ is *active* (*inactive*) in the multilayer network if $X_\ell = 1$ ($X_\ell = 0$). We obtain a merged network $\mathcal{G} = (V, \mathcal{E})$, where $\mathcal{E} = \{\ell \in E : X_\ell = 1\}$ is the set of active links; we say the multilayer network has link configuration $\mathcal{E}$, or equivalently, configuration $X_E \triangleq \{X_\ell : \ell \in E\}$. More generally, we call any vector x with component $x_\ell \in \{0, 1\}$ for $\ell \in E$ a link configuration of the multilayer network, and we call it a feasible link configuration if $\mathbb{P}[X_\ell = x_\ell, \forall \ell \in E] > 0$.

3 HARDNESS RESULTS

In this section, we show that it is very hard to compute the probability of given link configurations in arbitrary multilayer networks. We assume $K = 1$ in this section.

One source of hardness is the generality of the underlying connectivity network G . Proposition 3.1 shows that it is hard to compute link configuration probabilities for general underlying connectivity network G even when there is only one layer, i.e. $M = 1$.

PROPOSITION 3.1. *Suppose $M = 1$, $p_\ell = 1$ for all $\ell \in E$, and $q_i = 1/2$ for all $i \in V$. It is #P-hard to compute the probability that the network is in the configuration with no active link. It is NP-hard to approximate this probability within a multiplicative factor of $2^{n^{1-\epsilon}}$ for any $\epsilon > 0$, where $n = |V|$ is the number of nodes.*

Hardness arises in yet another dimension. For the special case where the underlying network G is a clique, it is trivial to compute link configuration probabilities for a single layer, but for a large number of layers, Proposition 3.2 shows that it is hard even to test the feasibility of a configuration, which is a simpler problem, since a configuration is feasible if and only if its probability is nonzero.

PROPOSITION 3.2. *It is NP-complete to decide if a link configuration x is feasible for an M -layer network with G being a clique.*

4 EXACT RESULTS

In this section, we provide recursions for computing link configuration probabilities in the case of trees. A detailed discussion for lines is found in [5].

Pick any root r for T . For $v \in V$, let T_v denote the subtree rooted at v . With a slight abuse of notation, let $T_v(x)$ be the event that the links in T_v are configured according to $x : E \rightarrow \{0, 1\}$, i.e. $X_\ell = x_\ell$ for all $\ell \in T_v$. For $v, w \in V$, let A_v be the number of layers on which v is active, and A_{vw} the number of layers on which both v and w are active. For $m = 0, 1, \dots, M$, let $f_v^{(m)}(x) = \mathbb{P}[A_v = m, T_v(x)]$ and $g_{vw}^{(m)}(x) = \mathbb{P}[X_{(v,w)} = x_{(v,w)}, T_w(x) \mid A_v = m]$. Let $ch(v)$ denote the set of v 's children. Then $f_v^{(m)}(x)$ and $g_{vw}^{(m)}(x)$ satisfy the following recursions,

$$\begin{aligned} f_v^{(m)}(x) &= B(m; M, q_v) \prod_{w \in ch(v)} g_{vw}^{(m)}(x), \\ g_{vw}^{(m)}(x) &= \sum_{k=0}^M \min\{m, k\} f_w^{(m)}(x) \times H(j; M, m, k) \\ &\quad \times \left[(1 - x_{vw}) \sum_{\ell < k} B(\ell; j, p_{(v,w)}) + x_{vw} \sum_{\ell \geq k} B(\ell; j, p_{(v,w)}) \right], \end{aligned}$$

where $B(m; M, q_v) \triangleq \binom{M}{m} q_v^m (1 - q_v)^{M-m}$ is a binomial pmf, and $H(j; M, m, k) \triangleq \binom{M}{j} \binom{M-m}{k-j} / \binom{M}{k}$ a hypergeometric pmf. If $ch(v) = \emptyset$, we use the convention $\prod_{w \in ch(v)} g_{vw}^{(m)}(x) = 1$.

The probability of configuration x is then $\mathbb{P}[X_\ell = x_\ell, \forall \ell \in E] = \sum_{m=0}^M f_r^{(m)}(x)$. We can compute $\{f_v^{(m)}(x) : m = 0, 1, \dots, M\}$ for all $v \in V$ sequentially from leaf nodes up to the root. The complexity is linear in $n = |V|$. Since the pmfs of hypergeometric and binomial distributions can be computed in time polynomial in M , the probability of a configuration can be computed in time $O(n \cdot \text{poly}(M))$. This pseudopolynomial time algorithm relies on the assumption that the layers are i.i.d. Although a similar recursion can be developed when layers are independent but have non-identical distributions, the complexity will scale as $M!$, making it feasible only for small M .

5 ASYMPTOTIC RESULTS

In this section, we derive the link configuration distribution in the limit as the number of layers, M , goes to infinity and the probabilities $\{p_\ell\}_{\ell \in E}$ and $\{q_i\}_{i \in V}$ decrease as functions of M . This case is especially relevant when the multilayer network arises from snapshots of a single network as in duty-cycled wireless sensor networks, where M can easily become very large. Even for moderate M , the asymptotics may already provide good approximations as shown in [4]. The results can be extended beyond the model of Section 2 to the case of non-identical layers [5].

For marginal distributions of W_ℓ , we have

THEOREM 5.1. *Suppose $\lim_{M \rightarrow \infty} M p_\ell q_i q_j = \lambda_\ell \in [0, \infty]$ exists for link $\ell = (i, j)$. Then, as $M \rightarrow \infty$, the limiting distribution of the multiplicity W_ℓ of ℓ is Poisson with parameter λ_ℓ , i.e.*

$$\lim_{M \rightarrow \infty} \mathbb{P}[W_\ell = w] = \pi(w; \lambda_\ell) \triangleq \frac{\lambda_\ell^w}{w!} e^{-\lambda_\ell}. \quad (1)$$

The joint distribution of the W_ℓ 's may have a complicated correlation structure. However, when p_ℓ and q_i scale with M appropriately, the W_ℓ 's become asymptotically independent. We consider the case that p_ℓ and q_i scale with M as follows,

$$p_\ell \sim c_\ell M^{-\alpha_\ell}, \quad \text{for } \ell \in E; \quad q_i \sim d_i M^{-\beta_i}, \quad \text{for } i \in V, \quad (2)$$

where $c_\ell, d_i > 0$ and $\alpha_\ell, \beta_i \geq 0$. For a link $\ell = (i, j)$, the parameter λ_ℓ defined in (1) is then given by

$$\lambda_\ell = \begin{cases} 0, & \text{if } \alpha_\ell + \beta_i + \beta_j > 1; \\ c_\ell d_i d_j, & \text{if } \alpha_\ell + \beta_i + \beta_j = 1; \\ +\infty, & \text{if } \alpha_\ell + \beta_i + \beta_j < 1. \end{cases} \quad (3)$$

For node k , let $\mathcal{N}_k = \{i \in V : (k, i) \in E, \alpha_{(k,i)} + \beta_k + \beta_i = 1\}$. We also assume the following condition,

$$\beta_k < 1, \text{ if } |\mathcal{N}_k| \geq 2. \quad (4)$$

THEOREM 5.2. *Under the conditions (2) and (4), the collection of random variables $\{W_\ell : \ell \in E\}$ become asymptotically independent as $M \rightarrow \infty$, i.e. for any $\{w_\ell : \ell \in E\} \in \mathbb{N}^E$,*

$$\lim_{M \rightarrow \infty} \mathbb{P}[W_\ell = w_\ell, \forall \ell \in E] = \prod_{\ell \in E} \pi(w_\ell; \lambda_\ell) = \prod_{\ell \in E} e^{-\lambda_\ell} \frac{\lambda_\ell^{w_\ell}}{w_\ell!}, \quad (5)$$

where λ_ℓ is given by (3).

For $K = 1$, the following corollary yields Theorem 1 of [4] if G is a tree and the conjecture therein if G is a general graph.

COROLLARY 5.3. *Suppose $p_\ell = 1$ for all $\ell \in E$, $q_i = dM^{-1/2}$ for all $i \in V$, and $K = 1$. Then*

$$\lim_{M \rightarrow \infty} \mathbb{P}[X_\ell = x_\ell, \forall \ell \in E] = e^{-d^2(|E| - |\mathcal{E}|)} (1 - e^{-d^2})^{|\mathcal{E}|}. \quad (6)$$

In the limit $M \rightarrow \infty$, the merged network $\mathcal{G}$ has a giant component if d exceeds the threshold $\sqrt{-\log(1 - p_c)}$, where p_c is the bond-percolation threshold of G .

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