Poster on Safety Characterization in Hybrid Inclusions Using Barrier Functions

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ABSTRACT

By this poster, we aim at presenting in a comprehensive manner our new results on safety characterization using barrier functions in the general context of hybrid systems. Roughly speaking, a dynamical system is said to be safe when the solutions starting from a given *initial set* never reach a given *unsafe set*. Barrier functions in this context constitute a qualitative methodological tool that avoid the computation of the system's solutions yet to determine if the safety property holds. According to literature, a barrier function *candidate* with respect to a given initial and unsafe sets is nonpositive on the initial set and strictly positive on the unsafe set. Such a barrier candidate becomes a *certificate* of safety provided that it satisfies some variational properties involving the system's dynamics at least in a specific region around its zero-sublevel set.

CCS CONCEPTS

- **Software and its engineering** → **Model checking;**
- **Computer systems organization** → **Embedded and cyber-physical systems;**

KEYWORDS

Safety, Forward invariance, Barrier functions, Hybrid inclusions.

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1 SUMMARY OF THE POSTER CONTENTS

1.1 Hybrid system model

We consider general hybrid inclusions of the form

$$\mathcal{H} : \begin{cases} x \in C & \dot{x} \in F(x) \\ x \in D & x^+ \in G(x), \end{cases} \quad (1)$$

with the state variable $x \in \mathbb{R}^n$, the flow set $C \subset \mathbb{R}^n$, the jump set $D \subset \mathbb{R}^n$, the flow and the jump set-valued maps, respectively, $F : O \rightrightarrows \mathbb{R}^n$ and $G : O \rightrightarrows O$ with $O \subset \mathbb{R}^n$ an open set such that $C \cup D \subset O$ and both C and D are closed.

A hybrid arc x is defined on a hybrid time domain denoted $\text{dom } x \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ where $\mathbb{R}_{\geq 0} := [0, \infty)$ and $\mathbb{N} := \{0, 1, \dots\}$. The forward hybrid arc x is parametrized by an ordinary time variable $t \in \mathbb{R}_{\geq 0}$ and a discrete jump variable $j \in \mathbb{N}$. Its domain of definition $\text{dom } x$ is such that for each $(T, J) \in \text{dom } x$, $\text{dom } x \cap ([0, T] \times \{0, 1, \dots, J\}) = \cup_{j=0}^J ([t_j, t_{j+1}], j)$ for a sequence $\{t_j\}_{j=0}^{J+1}$, such that $t_{j+1} \geq t_j$ and $t_0 = 0$; see [1].

DEFINITION 1. (Solution to $\mathcal{H}$) A hybrid arc $x : \text{dom } x \rightarrow \mathbb{R}^n$ defined on a hybrid time domain $\text{dom } x$ and such that, for each $j \in \mathbb{N}$, $t \mapsto x(t, j)$ is absolutely continuous is a forward solution to $\mathcal{H}$ if

(S0) $x(0, 0) \in C \cup D$;

(S1) for all $j \in \mathbb{N}$ such that $I^j := \{t : (t, j) \in \text{dom } x\}$ has nonempty interior

$$\begin{aligned} x(t, j) &\in C && \text{for all } t \in \text{int}(I^j), \\ \dot{x}(t, j) &\in F(x(t, j)) && \text{for almost all } t \in I^j; \end{aligned} \quad (2)$$

(S2) for all $(t, j) \in \text{dom } x$ such that $(t, j+1) \in \text{dom } x$,

$$x(t, j) \in D, \quad x(t, j+1) \in G(x(t, j)). \quad (3)$$

•

1.2 Basic notions

Given a hybrid system $\mathcal{H} = (C, F, D, G)$ and two sets $\chi_o \subset C \cup D$ and $\chi_u \subset \mathbb{R}^n$ such that $\mathbb{R}^n \setminus (C \cup D) \subset \chi_u$, we introduce a safety notion for general setting of hybrid inclusions.

DEFINITION 2. [Safety] The hybrid system $\mathcal{H}$ is said to be safe with respect to the initial set χ_o and the unsafe set χ_u , where $\chi_o \subset \mathbb{R}^n \setminus \chi_u$, if each solution x starting from $x_o \in \chi_o$ satisfies $x(t, j) \in \mathbb{R}^n \setminus \chi_u$ for all $(t, j) \in \text{dom } x$. •

A barrier function candidate with respect to the initial and the unsafe sets (χ_o, χ_u) is defined as follows.

DEFINITION 3. A function $B : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be a barrier function candidate for safety with respect to (χ_o, χ_u) if

$$\begin{cases} B(x) > 0 & \forall x \in \chi_u \cap (C \cup D) \\ B(x) \leq 0 & \forall x \in \chi_o. \end{cases}$$

Furthermore, we introduce the the zero-sublevel set of the barrier candidate B as

$$K_e := \{x \in \mathbb{R}^n : B(x) \leq 0\}.$$

1.3 Results

- Our initial results in [2, 4, 5] provide the tightest possible sufficient conditions guaranteeing different forward invariance notions of the zero-sublevel set K_e of the barrier candidate introduced in Definition 3. Some of these invariance notions automatically imply safety.
- In subsequent research, we study converse results for the safety problem, which is less explored and not completely solved. More precisely, given a safe system with respect to a given initial and unsafe sets (χ_o, χ_u) , we find conditions on the system's dynamics and the sets guaranteeing the existence of a (smooth enough) barrier function certifying safety. The few existing solutions in this direction concern particular classes of continuous-time systems: smooth and nonoscillatory systems, and smooth systems on compact manifolds. In our recent work [6], starting from the particular case of continuous-time systems, the converse safety problem is studied. Via Example 1, we showed the lack of existence of a continuous barrier function depending only on the system's variables and certifying safety.

EXAMPLE 1. Consider the two dimensional system

$$\begin{aligned} \dot{x}_1 &= -x_2 + rx_1 \sin(1/r) \\ \dot{x}_2 &= x_1 + rx_2 \sin(1/r), \quad r = |x|. \end{aligned}$$

In the polar coordinates

$$\dot{r} = r^2 \sin(1/r), \quad \dot{\theta} = 1.$$

The system in the original coordinates is safe with respect to the sets

$$\chi_o := \{0\}, \quad \chi_u := \mathbb{R}^2 \setminus \chi_o.$$

Indeed, the safety property in this case is equivalent to forward invariance of the origin (which coincides with χ_o). However, we were able to show that it is not possible to find a continuous barrier candidate B function only of x which is non increasing along the system's solutions and at the same time having a value at the origin which is strictly smaller than all the values elsewhere. $\square$

As a consequence, we show that only time-varying barrier functions certificates exist; in fact, their existence is shown to be equivalent to safety (under mild requirements on the system's dynamics) as in the following statement:

THEOREM 1.1. Consider the system $\dot{x} = f(x)$ with f continuously differentiable and every maximal solution is complete. Safety with respect to the sets (χ_o, χ_u) is equivalent to the existence of a continuous function $B : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \rightarrow \mathbb{R}$ of class C^1 on the set $(\mathbb{R}_{\geq 0} \times \mathbb{R}^n) \setminus K$ such that

$$\begin{cases} B(t, x) > 0 & \forall x \in \chi_u \quad \forall t \geq 0, \\ B(t, x) \leq 0 & \forall x \in \chi_o \quad \forall t \geq 0, \\ \frac{\partial B}{\partial t}(t, x) + \frac{\partial B}{\partial x}(t, x)f(x) \leq 0 & \forall (t, x) \in (\mathbb{R}_{\geq 0} \times \mathbb{R}^n) \setminus K, \end{cases}$$

where

$$K := \{(t, x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n : B(t, x) \leq 0\}.$$

- The latter study has been successfully extended to hybrid inclusions in our recent work [3] which is to be presented at this conference. In [3], we introduce an adequate definition of safety for hybrid inclusions. Furthermore, nonautonomous barrier functions and sufficient conditions for safety that are also necessary are proposed. Specifically, under mild assumptions on the data defining the hybrid system, we show that the safety property is equivalent to the existence of a lower semicontinuous and nonautonomous barrier function certifying safety.

Our results uniquely contribute to both areas of safety and hybrid systems. Furthermore, they provide the foundations to characterize combined safety and stability in hybrid systems. We believe that presenting a poster summarizing our recent developments will inspire future research directions and create fruitful collaborations.

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