Mixed Linear and Non-linear Recursive Types

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We describe a type system with mixed linear and non-linear recursive types called LNL-FPC (the linear/non-linear fixpoint calculus). The type system supports linear typing which enhances the safety properties of programs, but also supports non-linear typing as well which makes the type system more convenient for programming. Just like in FPC, we show that LNL-FPC supports type-level recursion which in turn induces term-level recursion. We also provide sound and computationally adequate categorical models for LNL-FPC which describe the categorical structure of the substructural operations of Intuitionistic Linear Logic at all non-linear types, including the recursive ones. In order to do so, we describe a new technique for solving recursive domain equations within the category CPO by constructing the solutions over *pre-embeddings*. The type system also enjoys implicit weakening and contraction rules which we are able to model by identifying the canonical comonoid structure of all non-linear types. We also show that the requirements of our abstract model are reasonable by constructing a large class of concrete models that have found applications not only in classical functional programming, but also in emerging programming paradigms that incorporate linear types, such as quantum programming and circuit description programming languages.

CCS Concepts: • **Theory of computation** → **Linear logic; Program semantics; Denotational semantics; Operational semantics; Categorical semantics.**

Additional Key Words and Phrases: recursive types, intuitionistic linear logic, recursion, categorical semantics, algebraic compactness, computational adequacy, linear/non-linear model, FPC, monoidal adjunctions

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1 INTRODUCTION

The *Fixpoint Calculus* (FPC) is a type system that has been extensively studied as a foundation for functional programming languages with recursive types. Originally proposed by Plotkin [Plotkin 1985], FPC was the focus of Fiore's celebrated PhD thesis [Fiore 1994]. The seminal paper [Fiore and Plotkin 1994] gives a summary account of how axiomatic domain theory can be used to characterize sound and computationally adequate denotational models of FPC.

Girard's introduction of *linear logic* [Girard 1987] initiated a parallel line of research into the logics underpinning functional programming languages [Abramsky 1993], focusing on the analysis of intuitionistic logic in terms of how hypotheses are consumed as resources for proofs. Attempts to bring linear types into functional programming languages soon followed [Wadler 1990]. Recently, there has been a concrete proposal for extending Haskell with linear types [Bernardy et al. 2017].

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and new languages are proposed that provide both linear and non-linear types with convenience for programming a design goal [Morris 2016; Tov and Pucella 2011]. One of the main benefits of linear types is the enhanced safety properties of programs which results from the more fine-grained control of resources.

Linear types also appear in concurrent settings where session types can be used to model the π -calculus [Caires et al. 2016]; in quantum programming languages where they are used to ensure error-freeness by enforcing compliance with the laws of quantum mechanics [Pagani et al. 2014]; and in circuit description programming languages where they are used to ensure that wires do not split and do not remain unconnected [Lindenhovius et al. 2018; Rennela and Staton 2017].

In this paper we present a foundational treatment of mixed linear and non-linear recursive types. The primary focus of the paper is to demonstrate that recursive types behave very well syntactically (uniform syntax), operationally (type-level and term-level recursion) and denotationally (sound and adequate categorical model) in such a mixed linear and non-linear setting.

1.1 Overview and summary of results

To illustrate our results, we present a type system called *LNL-FPC* (the linear/non-linear fixpoint calculus). The syntax of our language (§2) can roughly be understood as adding linear features to FPC, or alternatively, as adding recursive types to a linear/non-linear lambda calculus (such as DILL [Barber 1996] or LNL [Benton 1995]). More precisely, it is an extension with recursive types of the circuit-free fragment of Proto-Quipper-M [Rios and Selinger 2017] which is referred to as the CLNL calculus in [Lindenhovius et al. 2018]. The non-linear terms, types and contexts form *subsets* of the terms, types and contexts of LNL-FPC. There is no strict separation between linear and non-linear primitives in LNL-FPC. Instead, we view the non-linear primitives as having the additional property of being well-behaved with respect to the substructural operations of ILL (Intuitionistic Linear Logic), and the ones which are not non-linear are necessarily treated linearly by the type system. Notably, the type system has *implicit* weakening and contraction rules: non-linear variables are automatically copied and discarded whenever necessary by the language and not by the users, ensuring greater convenience for programming. We also equip LNL-FPC with a call-by-value big-step operational semantics and show that type-level recursion induces term-level recursion (§3), thus recreating a well-known result from FPC.

The primary difficulties in designing LNL-FPC are on the denotational side. Our categorical model (§5) is a CPO-enriched linear/non-linear (LNL) model [Benton 1995] which has suitable

ω -colimits. It is given by a CPO-enriched symmetric monoidal adjunction
$$\text{CPO} \begin{array}{c} \xrightarrow{F} \\ \perp \\ \xleftarrow{G} \end{array} \mathbf{L} .$$

We show that the category $\mathbf{L}$ is CPO-algebraically compact in the sense of [Fiore and Plotkin 1994] and thus one can solve recursive domain equations in $\mathbf{L}$ by constructing limits and colimits over embedding-projection pairs using the famous limit-colimit coincidence theorem [Smyth and Plotkin 1982]. The same paper shows that any mixed-variance CPO-functor $T : \mathbf{L}^{\text{op}} \times \mathbf{L} \rightarrow \mathbf{L}$ may be seen as a *covariant* functor $T_e : \mathbf{L}_e \times \mathbf{L}_e \rightarrow \mathbf{L}_e$ on the subcategory $\mathbf{L}_e$ of embeddings and we use this as a basis for the interpretation of types.

However, since we work in a mixed linear/non-linear setting, we also have to explain how to solve certain recursive domain equations involving mixed-variance functors within the category CPO which is a more challenging problem. We do so by *reflecting* the solutions from $\mathbf{L}_e$ via the left adjoint F into the subcategory of *pre-embeddings* CPO_{pe} , where a pre-embedding $f \in \text{CPO}$ is a morphism, s.t. Ff is an embedding in $\mathbf{L}_e$. Unlike the subcategory of embeddings CPO_e , the subcategory of pre-embeddings CPO_{pe} has sufficient structure for constructing (parameterised) initial algebras,

which moreover satisfy important coherence properties with respect to the (parameterised) initial algebras constructed in L_e (§4).

Then, the (standard) interpretation of an (arbitrary) type $\Theta \vdash A$ is a covariant functor denoted $\llbracket \Theta \vdash A \rrbracket : L_e^{|\Theta|} \rightarrow L_e$. A non-linear type $\Theta \vdash P$ admits an *additional non-linear interpretation* as a covariant functor $\llbracket \Theta \vdash P \rrbracket : \mathbf{CPO}_{pe}^{|\Theta|} \rightarrow \mathbf{CPO}_{pe}$ which is strongly related to its standard interpretation via a natural isomorphism $\alpha^{\Theta \vdash P} : \llbracket \Theta \vdash P \rrbracket \circ F_{pe}^{\times|\Theta|} \Rightarrow F_{pe} \circ \llbracket \Theta \vdash P \rrbracket : \mathbf{CPO}_{pe}^{|\Theta|} \rightarrow L_e$, where F_{pe} is the restriction of F to $\mathbf{CPO}_{pe}$. By exploiting this natural isomorphism and the strong coherence properties that it enjoys, we provide a coherent interpretation of the substructural operations of ILL at all *non-linear* types, including the recursive ones. This then allows us to characterise the canonical comonoid structure of non-linear (recursive) types and we prove our semantics sound (§6). We emphasise that even though our recursive type expressions allow use of $\multimap$, the reason we can interpret these types as *covariant* functors is because we have identified suitable well-behaved subcategories of $\mathbf{CPO}$ and $\mathbf{L}$ in our model.

We also show that the requirements of our abstract model are reasonable by constructing a large class of concrete models which have been used in different programming paradigms ranging from classical to quantum (§5.4). In addition, we present a computational adequacy result for a (non-empty) class of models that satisfy certain additional conditions (§7).

2 SYNTAX OF LNL-FPC

We begin by describing the syntax of LNL-FPC. The type-level syntax is very similar to that of FPC [Abadi and Fiore 1996] and the term-level syntax can be seen as an extension with recursive types of either the circuit-free fragment of Proto-Quipper-M [Rios and Selinger 2017], or the CLNL calculus in [Lindenhovius et al. 2018].

2.1 LNL-FPC Types

We use X, Y, Z to range over *type variables*, which are needed in order to form recursive types. We use Θ to range over *type contexts* (see Figure 1). A type context $\Theta = X_1, \dots, X_n$ is simply a list of type variables. A type context Θ is *well-formed*, denoted $\vdash \Theta$, if it can be derived from the rules:

$$\frac{}{\vdash \cdot} \quad \frac{\vdash \Theta}{\vdash \Theta, X} X \notin \Theta,$$

that is, if all the variables of Θ are distinct.

We use A, B, C to range over (arbitrary) types of our language (see Figure 1 for their grammar). A type A is well-formed in type context Θ , denoted $\Theta \vdash A$, if the judgement can be derived according to the following rules:

$$\frac{\vdash \Theta}{\Theta \vdash \Theta_i} 1 \leq i \leq |\Theta| \quad \frac{\Theta \vdash A}{\Theta \vdash !A} \quad \frac{\Theta \vdash A \quad \Theta \vdash B}{\Theta \vdash A \star B} \star \in \{+, \otimes, \multimap\} \quad \frac{\Theta, X \vdash A}{\Theta \vdash \mu X.A}$$

Note that if $\Theta \vdash A$ is derivable, then so is $\vdash \Theta$. A well-formed type A is *closed* if $\cdot \vdash A$.

The *non-linear types* form a subset of our types and we use P, R to range over them (see Figure 1 for their grammar). Of course, they are governed by the same formation rules as those for (arbitrary) types. A type which is not non-linear is called *linear*. Note that we do not provide a separate grammar for linear types. This is because we will mostly be working with arbitrary types (which we do not assume we may copy/discard/promote) and with non-linear types (which we may always copy/discard/promote). In particular, depending on the context, it is possible for a non-linear type to be treated as arbitrary, but not vice versa (which is done to increase the convenience of the type system for its *users*). The next subsection explains this in greater detail.

Type variables	X, Y, Z	
Term variables	x, y, z	
Types	A, B, C	$::= X \mid A + B \mid A \otimes B \mid A \multimap B \mid !A \mid \mu X. A$
Non-linear types	P, R	$::= X \mid P + R \mid P \otimes R \mid !A \mid \mu X. P$
Type contexts	Θ	$::= X_1, X_2, \dots, X_n$
Term contexts	Γ, Σ	$::= x_1 : A_1, x_2 : A_2, \dots, x_n : A_n$
Non-linear term contexts	Φ	$::= x_1 : P_1, x_2 : P_2, \dots, x_n : P_n$
Terms	m, n, p	$::= x \mid \text{left}_{A,B} m \mid \text{right}_{A,B} m$ $\mid \text{case } m \text{ of } \{\text{left } x \multimap n \text{ right } y \multimap p\}$ $\mid \langle m, n \rangle \mid \text{let } \langle x, y \rangle = m \text{ in } n \mid \lambda x^A. m \mid mn$ $\mid \text{lift } m \mid \text{force } m \mid \text{fold}_{\mu X. A} m \mid \text{unfold } m$
Values	v, w	$::= x \mid \text{left}_{A,B} v \mid \text{right}_{A,B} v \mid \langle v, w \rangle \mid \lambda x^A. m$ $\mid \text{lift } m \mid \text{fold}_{\mu X. A} v$

Fig. 1. Syntax of the LNL-FPC Calculus.

Example 2.1.1. We list some important closed types which are definable in LNL-FPC. The *empty* type is $0 \equiv \mu X. X$, which is non-linear. Another non-linear type is the *unit* type which is defined by $I \equiv !(0 \multimap 0)$. The type of natural numbers is $\text{Nat} \equiv \mu X. I + X$, which is also non-linear. Given a closed type A , then lists of type A are defined by $\text{List } A \equiv \mu X. I + A \otimes X$. The type $\text{List } A$ is non-linear iff A is non-linear. *Lazy* datatypes can be defined by making use of the $!$ and the $\multimap$ connectives. For instance, streams of type A can be defined by $\text{Stream } A \equiv \mu X. A \otimes !X$, which is a non-linear type iff A is non-linear. The ability to form recursive types using both $\multimap$ and $!$ is a powerful feature, because it allows us to *derive* a term-level recursion operator (see §2.3).

For any types A and B and type variable X , we denote with $A[B/X]$ the type where all free occurrences of X in A are replaced by B (which is defined in the standard way).

Lemma 2.1.2. *If $\Theta, X \vdash A$ and $\Theta \vdash B$, then $\Theta \vdash A[B/X]$. Moreover, if A and B are non-linear, then so is $A[B/X]$.*

Remark 2.1.3. Just like FPC, our language also allows *nested type recursion* (e.g., $\mu X. \mu Y. I + (X \otimes Y)$).

2.2 LNL-FPC Terms

We use x, y, z to range over *term variables*. We use Γ, Σ to range over (arbitrary) *term contexts*. A term context is a list of term variables with types, written as $\Gamma = x_1 : A_1, \dots, x_n : A_n$. A term context is *well-formed* in type context Θ , denoted $\Theta \vdash \Gamma$, if the judgement can be derived from the rules:

$$\frac{\vdash \Theta}{\Theta \vdash \cdot} \quad \frac{\Theta \vdash \Gamma \quad \Theta \vdash A}{\Theta \vdash \Gamma, x : A} \quad x \notin \Gamma,$$

that is, if Γ is a list of distinct variables with well-formed types. A *non-linear term context* is a term context whose types are all non-linear. We use Φ to range over non-linear term contexts.

The terms and values of our language are defined in Figure 1. A *term judgement* has the form $\Theta; \Gamma \vdash m : A$ and indicates that m is a well-formed term of type A in type context Θ and term context Γ . The formation rules are shown in Figure 2, under the condition that the Γ and Σ contexts do not have any variables in common (one can deduce $\Phi \cap \Gamma = \emptyset = \Phi \cap \Sigma$). The formation rules for fold and unfold are the same as in FPC (cf. [Abadi and Fiore 1996]). Observe that if $\Theta; \Gamma \vdash m : A$, then also $\Theta \vdash \Gamma$ and $\Theta \vdash A$.

$$\begin{array}{c}
\frac{\Theta \vdash \Phi, x : A}{\Theta; \Phi, x : A \vdash x : A} \quad \frac{\Theta; \Gamma \vdash m : A \quad \Theta \vdash B}{\Theta; \Gamma \vdash \text{left}_{A,B} m : A + B} \quad \frac{\Theta; \Gamma \vdash m : B \quad \Theta \vdash A}{\Theta; \Gamma \vdash \text{right}_{A,B} m : A + B} \\
\\
\frac{\Theta; \Phi, \Gamma \vdash m : A + B \quad \Theta; \Phi, \Sigma, x : A \vdash n : C \quad \Theta; \Phi, \Sigma, y : B \vdash p : C}{\Theta; \Phi, \Gamma, \Sigma \vdash \text{case } m \text{ of } \{\text{left } x \rightarrow n \mid \text{right } y \rightarrow p\} : C} \\
\\
\frac{\Theta; \Phi, \Gamma \vdash m : A \quad \Theta; \Phi, \Sigma \vdash n : B}{\Theta; \Phi, \Gamma, \Sigma \vdash \langle m, n \rangle : A \otimes B} \quad \frac{\Theta; \Phi, \Gamma \vdash m : A \otimes B \quad \Theta; \Phi, \Sigma, x : A, y : B \vdash n : C}{\Theta; \Phi, \Gamma, \Sigma \vdash \text{let } \langle x, y \rangle = m \text{ in } n : C} \\
\\
\frac{\Theta; \Gamma, x : A \vdash m : B}{\Theta; \Gamma \vdash \lambda x^A. m : A \multimap B} \quad \frac{\Theta; \Phi, \Gamma \vdash m : A \multimap B \quad \Theta; \Phi, \Sigma \vdash n : A}{\Theta; \Phi, \Gamma, \Sigma \vdash mn : B} \quad \frac{\Theta; \Phi \vdash m : A}{\Theta; \Phi \vdash \text{lift } m : !A} \\
\\
\frac{\Theta; \Gamma \vdash m : !A}{\Theta; \Gamma \vdash \text{force } m : A} \quad \frac{\Theta; \Gamma \vdash m : A[\mu X. A/X] \quad \Theta, X \vdash A}{\Theta; \Gamma \vdash \text{fold}_{\mu X. A} m : \mu X. A} \quad \frac{\Theta; \Gamma \vdash m : \mu X. A}{\Theta; \Gamma \vdash \text{unfold } m : A[\mu X. A/X]}
\end{array}$$

where $\Gamma \cap \Sigma = \emptyset$.

Fig. 2. Formation rules for LNL-FPC terms.

The type system enforces that a linear variable is used exactly once, whereas a non-linear variable may be used any number of times, including zero. Since linear and non-linear variables live on equal footing, we say our type system is *mixed linear/non-linear*.

Of course, type assignment to terms is unique. If $\Theta; \Gamma \vdash m : A$ and $\Theta; \Gamma \vdash m : B$, then $A \equiv B$. However, derivations of term judgements in LNL-FPC are in general not unique, because non-linear variables may be part of an arbitrary context Γ , that is, our type system allows for non-linear variables to be treated as if they were linear (but not vice-versa). For example, if $\Theta \vdash P_1$ and $\Theta \vdash P_2$ are non-linear types, then:

$$\frac{\Theta; x : P_1 \vdash x : P_1 \quad \Theta; y : P_2 \vdash y : P_2}{\Theta; x : P_1, y : P_2 \vdash \langle x, y \rangle : P_1 \otimes P_2} \quad \frac{\Theta; x : P_1 \vdash x : P_1 \quad \Theta; x : P_1, y : P_2 \vdash y : P_2}{\Theta; x : P_1, y : P_2 \vdash \langle x, y \rangle : P_1 \otimes P_2}$$

are two different derivations of the same judgement. This non-uniqueness is a result of the design choice to have only one kind of pairing, one kind of conditional branching, etc. (instead of having separate linear and non-linear ones). This results in a more convenient syntax for the user and allows us to make the weakening and contraction rules (of intuitionistic linear logic) implicit, rather than explicit, i.e., the user is not required to explicitly indicate when variables should be copied or discarded. Instead, the type system keeps track of that and ensures that only non-linear variables may be copied or discarded, without inconveniencing the user. Moreover, this design choice also allows for interesting language extensions [Lindenhovius et al. 2018; Rios and Selinger 2017]. Also, the interpretation of any two derivations of the same judgement are equal (see Theorem 6.3.4).

A *term of closed type* is a term m , such that $\cdot; \Gamma \vdash m : A$ for some type A and context Γ . In such a situation we simply write $\Gamma \vdash m : A$. Naturally, we are primarily interested in these terms (observe that the term formation rules are invariant with respect to the type context). A *program* is a term p , such that $\cdot; \cdot \vdash p : A$ for some type A and we simply write $p : A$ to indicate this.

Example 2.2.1. We list some important programs. We define $*$ $\equiv$ $\text{lift } \lambda x^0. x : I$, which is the canonical value of unit type. The zero natural number is defined by $\text{zero} \equiv \text{fold}_{\text{Nat}} \text{left}_{I, \text{Nat}} * : \text{Nat}$. The successor function is defined by the program $\text{succ} \equiv \lambda n^{\text{Nat}}. \text{fold}_{\text{Nat}} \text{right}_{I, \text{Nat}} n : \text{Nat} \multimap \text{Nat}$.

Given terms m, n and a variable x , we denote with $m[n/x]$ the term obtained from m by replacing all free occurrences of x with n (which is defined in the standard way).

Lemma 2.2.2 (Substitution). *If $\Theta; \Phi, \Gamma, x : A \vdash m : B$ and $\Theta; \Phi, \Sigma \vdash n : A$ and $\Gamma \cap \Sigma = \emptyset$, then $\Theta; \Phi, \Gamma, \Sigma \vdash m[n/x] : B$.*

We say a value $\Theta; \Gamma \vdash v : P$ is *non-linear* whenever the type P is non-linear.

Lemma 2.2.3. *If $\Theta; \Gamma \vdash v : P$ is a non-linear value, then Γ is also non-linear.*

2.3 Term Recursion in LNL-FPC

Recall that in FPC, general recursion on terms may be implemented using the fold and unfold terms. The same is also true for LNL-FPC. Moreover, the derived term for recursion has exactly the same syntax as the one in [Lindenhovius et al. 2018], where the authors showed how to extend a mixed linear/non-linear type system with recursion. In the next section we will show it has the same operational behaviour as well.

Theorem 2.3.1. *The rule $\frac{\Theta; \Phi, z : !A \vdash m : A}{\Theta; \Phi \vdash \text{rec } z^{!A}.m : A}$ is derivable in LNL-FPC, where Φ is non-linear,*

$$\begin{aligned} \text{rec } z^{!A}.m &\equiv (\text{unfold force } \alpha_m^z) \alpha_m^z \quad \text{and} \\ \alpha_m^z &\equiv \text{lift fold } \lambda x^{! \mu X. (!X \multimap A)}. (\lambda z^{!A}.m) (\text{lift (unfold force } x) x), \end{aligned}$$

such that $X \notin \Theta$ and $x \notin \Phi$.

3 OPERATIONAL SEMANTICS

The operational semantics of LNL-FPC is standard. We use a big-step call-by-value evaluation relation whose rules are shown in Figure 3.

$$\begin{array}{c} \frac{}{x \Downarrow x} \quad \frac{m \Downarrow v}{\text{left } m \Downarrow \text{left } v} \quad \frac{m \Downarrow v}{\text{right } m \Downarrow \text{right } v} \\[10pt] \frac{m \Downarrow \text{left } v \quad n[v/x] \Downarrow w}{\text{case } m \text{ of } \{\text{left } x \rightarrow n \mid \text{right } y \rightarrow p\} \Downarrow w} \quad \frac{m \Downarrow \text{right } v \quad p[v/y] \Downarrow w}{\text{case } m \text{ of } \{\text{left } x \rightarrow n \mid \text{right } y \rightarrow p\} \Downarrow w} \\[10pt] \frac{m \Downarrow v \quad n \Downarrow w}{\langle m, n \rangle \Downarrow \langle v, w \rangle} \quad \frac{m \Downarrow \langle v, v' \rangle \quad n[v/x, v'/y] \Downarrow w}{\text{let } \langle x, y \rangle = m \text{ in } n \Downarrow w} \\[10pt] \frac{}{\lambda x. m \Downarrow \lambda x. m} \quad \frac{m \Downarrow \lambda x. m' \quad n \Downarrow v \quad m'[v/x] \Downarrow w}{mn \Downarrow w} \\[10pt] \frac{}{\text{lift } m \Downarrow \text{lift } m} \quad \frac{m \Downarrow \text{lift } m' \quad m' \Downarrow v}{\text{force } m \Downarrow v} \quad \frac{m \Downarrow v}{\text{fold } m \Downarrow \text{fold } v} \quad \frac{m \Downarrow \text{fold } v}{\text{unfold } m \Downarrow v} \end{array}$$

Fig. 3. Operational semantics of the LNL-FPC calculus

As usual, the *values* are the terms v such that $v \Downarrow v$ (see Figure 1). The evaluation relation $(- \Downarrow -)$ is, in fact, a *partial* function from terms to values. Of course, it is not a total function, because the language supports a general recursion operator, as we show next.

Theorem 3.0.1. *The evaluation rule $\frac{m[\text{lift rec } z^{!A}.m / z] \Downarrow v}{\text{rec } z^{!A}.m \Downarrow v}$ is derivable within LNL-FPC, where $\text{rec } z^{!A}.m$ is defined as in Theorem 2.3.1.*

This theorem shows that our *derived* general recursion operator is exactly the same as the one in [Lindenhovius et al. 2018], where it was added as an axiom. A term m is said to *terminate*, denoted by $m \Downarrow$, if there exists a value v , such that $m \Downarrow v$. As an example, the simplest non-terminating program of type A is given by $\text{rec } z^{!A}.\text{force } z : A$.

We conclude the section with the expected subject reduction property.

Theorem 3.0.2 (Subject reduction). *If $\Theta; \Gamma \vdash m : A$ and $m \Downarrow v$, then $\Theta; \Gamma \vdash v : A$.*

Assumption 3.0.3. Throughout the remainder of the paper, we implicitly assume that all types, contexts and terms are well-formed.

4 ω -CATEGORIES AND (PARAMETERISED) INITIAL ALGEBRAS

In this section we recall the theory of ω -categories introduced in [Lehmann and Smyth 1981] and develop new results of our own. In §4.1, we introduce some notation for operations on natural transformations which we use throughout the paper. In §4.2, we recall how initial algebras are constructed in ω -categories. In §4.3, we show how to construct parameterised initial algebras in ω -categories which we use to model recursive types that may potentially be defined by nested recursion. In §4.4, we present new results which show how (parameterised) initial algebras of functors acting on different categories may be related to one another, provided there exist suitable mediating functors between the two categories. The type-level semantics makes heavy use of this relationship in order to present *coherent* non-linear type interpretations which are strongly related to the standard type interpretations.

4.1 Operations on natural transformations

Given natural transformations τ and σ and a functor F , we denote: vertical composition by $\sigma \circ \tau$; horizontal composition by $\sigma * \tau$; whiskering by $F\tau$ and τF , whenever these operations are admissible.

If $\tau : T \Rightarrow H : C \rightarrow C$ is a natural transformation between two endofunctors, then we define $\tau^{*n} := \tau * \dots * \tau : T^n \Rightarrow H^n : C \rightarrow C$ to be the n -fold horizontal composition of τ with itself (if $n = 0$, then τ^{*0} is the identity $\text{id} : \text{Id} \Rightarrow \text{Id} : C \rightarrow C$).

If $\tau : T \Rightarrow T' : A \rightarrow B$ and $\sigma : H \Rightarrow H' : C \rightarrow D$ are natural transformations, we define a natural transformation $\tau \times \sigma : T \times H \Rightarrow T' \times H' : A \times C \rightarrow B \times D$ by $(\tau \times \sigma)_{(A,C)} := (\tau_A, \sigma_C)$.

If $\tau : T \Rightarrow T' : A \rightarrow B$ and $\sigma : H \Rightarrow H' : A \rightarrow C$ are natural transformations, we define a natural transformation $\langle \tau, \sigma \rangle : \langle T, H \rangle \Rightarrow \langle T', H' \rangle : A \rightarrow B \times C$ by $\langle \tau, \sigma \rangle_A := (\tau_A, \sigma_A)$.

We denote with ω the poset of natural numbers when viewed as a category. A functor $D : \omega \rightarrow C$ is then an ω -diagram. We let $[A, B]$ denote the functor category from A to B . Given a functor $M : A \rightarrow B$, we define a functor $M \triangleright - : [\omega, A] \rightarrow [\omega, B]$ by:

$$M \triangleright D = M \circ D \quad \text{and} \quad M \triangleright \tau = M\tau.$$

So, the functor $M \triangleright -$ is just whiskering with M .

4.2 Initial algebras in ω -categories

We now recall some definitions and facts about ω -categories and ω -functors which are stated in [Lehmann and Smyth 1981]. A functor $F : A \rightarrow C$ is an ω -functor if F preserves all existing colimits of ω -diagrams. If, in addition, A has an initial object and F preserves it, then we say that F is a *strict* ω -functor. Of course, ω -functors are closed under composition and pairing, that is, if F and G are ω -functors, then so are $F \circ G$ and $\langle F, G \rangle$ whenever composition and pairing are admissible.

A category $\mathbf{C}$ is an ω -category if it has an initial object and all ω -colimits. We denote with $[\mathbf{A} \rightarrow_{\omega} \mathbf{C}]$ the full subcategory of $[\mathbf{A}, \mathbf{C}]$ consisting of ω -functors. If $\mathbf{C}$ is an ω -category, then so are $[\mathbf{A}, \mathbf{C}]$ and $[\mathbf{A} \rightarrow_{\omega} \mathbf{C}]$.

Definition 4.2.1 ([Lehmann and Smyth 1981]). Given an ω -category $\mathbf{C}$ with an initial object $\emptyset$, we define a functor $S : [\mathbf{C} \rightarrow_{\omega} \mathbf{C}] \rightarrow [\omega, \mathbf{C}]$ in the following way:

- Given an ω -functor $T : \mathbf{C} \rightarrow \mathbf{C}$, then $S(T)$ is the ω -diagram $\emptyset \xrightarrow{\iota} T\emptyset \xrightarrow{T\iota} T^2\emptyset \xrightarrow{T^2\iota} \dots$. More specifically, $S(T)(n) := T^n\emptyset$ and $S(T)(n \leq n+1) := T^n\iota$, where $\iota : \emptyset \rightarrow T\emptyset$ is the initial map.
- Given $\tau : T \Rightarrow H : \mathbf{C} \rightarrow \mathbf{C}$, then $S(\tau) : S(T) \Rightarrow S(H) : \omega \rightarrow \mathbf{C}$ is given by

$$S(\tau)_n := (\tau^{*n})_{\emptyset} : T^n\emptyset \rightarrow H^n\emptyset.$$

We also define a functor $Y := \text{colim} \circ S : [\mathbf{C} \rightarrow_{\omega} \mathbf{C}] \rightarrow \mathbf{C}$, where $\text{colim} : [\omega, \mathbf{C}] \rightarrow \mathbf{C}$ is the colimiting functor, which is the left-adjoint of the ω -ary diagonal functor $\Delta : \mathbf{C} \rightarrow [\omega, \mathbf{C}]$.

Theorem 4.2.2. [Lehmann and Smyth 1981] If $\mathbf{C}$ is an ω -category, then $S : [\mathbf{C} \rightarrow_{\omega} \mathbf{C}] \rightarrow [\omega, \mathbf{C}]$ and $Y : [\mathbf{C} \rightarrow_{\omega} \mathbf{C}] \rightarrow \mathbf{C}$ are both ω -functors.

Therefore, for an ω -functor $T : \mathbf{C} \rightarrow \mathbf{C}$ on an ω -category $\mathbf{C}$, its initial sequence is given by $S(T)$ and the carrier of its initial algebra is given by $Y(T)$, thanks to a famous result in [Adámek 1974]. To describe its initial algebra structure we need an additional definition.

Theorem 4.2.3 ([Adámek et al. 2018]). Let $T : \mathbf{C} \rightarrow \mathbf{C}$ be an ω -endofunctor on an ω -category $\mathbf{C}$. We define a natural transformation $s^T : S(T) \Rightarrow T \circ S(T) : \omega \rightarrow \mathbf{C}$ by $(s^T)_n := T^n(\iota_{T\emptyset})$, where $\iota_{T\emptyset} : \emptyset \rightarrow T\emptyset$ is the initial morphism. Then one can define an isomorphism $y^T : T(Y(T)) \rightarrow Y(T)$ by:

$$y^T := \left(TY(T) = T\text{colim}(S(T)) = \text{colim}(TS(T)) \xrightarrow{(\text{colim}(s^T))^{-1}} \text{colim}(S(T)) = Y(T) \right)$$

and the pair $(Y(T), y^T)$ forms the initial T -algebra.

4.3 Parameterised initial algebras

Initial algebras can be used to model recursive types where the type recursion is done over a single type variable. In order to model recursive types defined by *nested recursion*, one has to allow recursive types to depend on several type variables. Modeling these more general recursive types requires a more general notion, namely *parameterised initial algebras*, which we introduce next.

Lemma 4.3.1 ([Lehmann and Smyth 1981]). Given an ω -category $\mathbf{C}$ and an ω -functor $T : \mathbf{A} \times \mathbf{B} \rightarrow \mathbf{C}$, we define a functor $\lambda B.T(-, B) : \mathbf{A} \rightarrow [\mathbf{B} \rightarrow_{\omega} \mathbf{C}]$ by:

$$\begin{aligned} \lambda B.T(A, B) &:= T(A, -) : \mathbf{B} \rightarrow \mathbf{C}, & \text{for } A \in \text{Ob}(\mathbf{A}); \\ \lambda B.T(f, B) &:= T(f, -) : T(A_1, -) \Rightarrow T(A_2, -) : \mathbf{B} \rightarrow \mathbf{C}, & \text{for } f : A_1 \rightarrow A_2. \end{aligned}$$

Then $\lambda B.T(-, B) : \mathbf{A} \rightarrow [\mathbf{B} \rightarrow_{\omega} \mathbf{C}]$ is an ω -functor.

Now we can define the parameterised fixpoint operator for ω -functors.

Theorem 4.3.2. Let $\mathbf{B}$ be an ω -category and let $T : \mathbf{A} \times \mathbf{B} \rightarrow \mathbf{B}$ be an ω -functor. Then:

- (1) The functor $T^{\dagger} := Y \circ \lambda B.T(-, B) : \mathbf{A} \rightarrow \mathbf{B}$ is an ω -functor.
- (2) There exists a natural isomorphism $\phi^T : T \circ \langle \text{Id}, T^{\dagger} \rangle \Rightarrow T^{\dagger} : \mathbf{A} \rightarrow \mathbf{B}$ given by:

$$\phi_A^T := \left(T(A, T^{\dagger}A) = T(A, Y(T(A, -))) \xrightarrow{y^{T(A, -)}} Y(T(A, -)) = T^{\dagger}A \right).$$

We also simply write ϕ for the above natural isomorphism when T is clear from the context.

Definition 4.3.3. In the situation of Theorem 4.3.2, the pair $(T^\dagger, \phi^T)$ is called the *parameterised initial algebra* of T (see [Fiore 1994, Section 6.1] for a more detailed and abstract exposition).

A parameterised initial algebra $(T^\dagger, \phi^T)$ is characterised by the property that $(T^\dagger A, \phi_A^T)$ is the initial $T(A, -)$ -algebra. Notice that by trivialising the category $\mathbf{A}$, we recover the usual notion of initial algebra. Next, we prove an important proposition for proving the type substitution lemma.

Proposition 4.3.4. *Let $\mathbf{C}$ be an ω -category, $T : \mathbf{B} \times \mathbf{C} \rightarrow \mathbf{C}$ an ω -functor and $H : \mathbf{A} \rightarrow \mathbf{B}$ an ω -functor. Then $(T \circ (H \times \text{Id}))^\dagger = T^\dagger \circ H : \mathbf{A} \rightarrow \mathbf{C}$.*

4.4 Coherence properties of (parameterised) initial algebras

We model (arbitrary) recursive types within a linear category that is algebraically compact in a strong sense and which allows us to model recursive types involving the $\multimap$ connective. The non-linear recursive types form a subset of our types for which we have to, in addition, provide categorical structure that allows for them to be copied, discarded and promoted (the substructural rules of intuitionistic linear logic). We do this by providing a non-linear interpretation of these types within a cartesian category. The two interpretations live in different categories, but they are strongly related to each other via suitable mediating functors and a natural isomorphism. In order to achieve this, we first explain how parameterised initial algebras can be related to each other in such a strong sense.

Assumption 4.4.1. Throughout the remainder of the section, we assume we are given the following data. Let $\mathbf{A}$ and $\mathbf{C}$ be categories and let $\mathbf{B}$ and $\mathbf{D}$ be ω -categories with initial objects $\emptyset$ and 0 , respectively. Let $\alpha : T \circ (N \times M) \Rightarrow M \circ H$ be a natural isomorphism, as in:

$$\begin{array}{ccc} \mathbf{A} \times \mathbf{B} & \xrightarrow{N \times M} & \mathbf{C} \times \mathbf{D} \\ \downarrow H & \swarrow \alpha & \downarrow T \\ \mathbf{B} & \xrightarrow{M} & \mathbf{D} \end{array}$$

where H and T are ω -functors and where M is a strict ω -functor with $z : 0 \rightarrow M\emptyset$ the required (unique) isomorphism. The functor N need not be an ω -functor.

Theorem 4.4.2. *The natural isomorphism α induces a natural isomorphism*

$$\alpha^\dagger : T^\dagger \circ N \Rightarrow M \circ H^\dagger : \mathbf{A} \rightarrow \mathbf{D}.$$

PROOF. For a functor $F : \mathbf{A} \times \mathbf{B} \rightarrow \mathbf{B}$, define a functor:

$$F^* := S \circ \lambda B. F(-, B) : \mathbf{A} \rightarrow [\omega, \mathbf{B}]$$

so that $F^\dagger = \text{colim} \circ F^*$. Next, define a natural isomorphism

$$\alpha^* : T^* \circ N \Rightarrow (M \triangleright -) \circ H^* : \mathbf{A} \rightarrow [\omega, \mathbf{D}] \quad \text{given by}$$

$$\begin{aligned} (\alpha_A^*)_0 &:= \left(0 \xrightarrow{z} M\emptyset \right) \\ (\alpha_A^*)_{n+1} &:= \left(T(NA, -)^{n+1} 0 \xrightarrow{T(NA, (\alpha_A^*)_n)} T(NA, MH(A, -)^n 0) \xrightarrow{\alpha_{A, H(A, -)^n 0}} MH(A, -)^{n+1} 0 \right). \end{aligned}$$

The naturality of α^* is straightforward to check. Define $\alpha^\dagger : T^\dagger \circ N \Rightarrow M \circ H^\dagger : \mathbf{A} \rightarrow \mathbf{D}$ by

$$\alpha_A^\dagger := \left(T^\dagger NA = \text{colim}(T^* NA) \xrightarrow{\text{colim}(\alpha_A^*)} \text{colim}(MH^* A) = M \text{colim}(H^* A) = MH^\dagger A \right).$$

Since α_A^* is an isomorphism, then so too is $\alpha_A^\dagger$. Naturality of $\alpha^\dagger$ follows from:

$$\begin{array}{ccccccc}
 T^\dagger NA & \xlongequal{\quad} & \text{colim}(T^* NA) & \xrightarrow{\text{colim}(\alpha_A^*)} & \text{colim}(MH^* A) & \xlongequal{\quad} & M\text{colim}(H^* A) \xlongequal{\quad} MH^\dagger A \\
 \downarrow T^\dagger Nf & & \downarrow \text{colim}(T^* Nf) & & \downarrow \text{colim}(MH^* f) & & \downarrow M\text{colim}(H^* f) \\
 (1) & & (2) & & (3) & & (4) \\
 T^\dagger NB & \xlongequal{\quad} & \text{colim}(T^* NB) & \xrightarrow{\text{colim}(\alpha_B^*)} & \text{colim}(MH^* B) & \xlongequal{\quad} & M\text{colim}(H^* B) \xlongequal{\quad} MH^\dagger B \\
 & & & & \downarrow MH^\dagger f & &
 \end{array}$$

where (1) and (4) commute by definition, (2) commutes by naturality of α^* and functoriality of colim , and (3) commutes because M preserves ω -colimits. $\square$

Remark 4.4.3. The above construction generalises the operation $(-)^{\dagger}$ from [Fiore 1994, Corollary 7.3.13] to the context of ω -categories.

Corollary 4.4.4. If $A = B^n$, $C = D^n$, $N = M^{\times n}$ and $\alpha : T \circ M^{\times n+1} \Rightarrow M \circ H : B^{n+1} \rightarrow D$ is a natural isomorphism, then so is $\alpha^\dagger : T^\dagger \circ M^{\times n} \Rightarrow M \circ H^\dagger : B^n \rightarrow D$.

By reading off the proof of Theorem 4.4.2, we obtain another corollary which we use in our adequacy proof.

Corollary 4.4.5. In the special case where $\alpha = \text{id}$ and $z = \text{id}$, then $\alpha^\dagger = \text{id}$ and thus $T^\dagger \circ N = M \circ H^\dagger$.

Lemma 4.4.6. The operation $(-)^{\dagger}$ enjoys the following properties:

(1) If $\beta : H \Rightarrow H' : A \times B \rightarrow B$ is a natural isomorphism, then:

$$(M\beta \circ \alpha)^\dagger = M\beta^\dagger \circ \alpha^\dagger$$

(2) Given functors $P : E \rightarrow C$ and $Q : E \rightarrow A$ and a natural isomorphism $\gamma : P \Rightarrow N \circ Q : E \rightarrow C$, then:

$$(\alpha(Q \times \text{Id}) \circ T(\gamma \times M))^\dagger = \alpha^\dagger Q \circ T^\dagger \gamma$$

Remark 4.4.7. As a special case, if one takes N and M to be identity functors, then given a natural isomorphism $\alpha : T \Rightarrow H : A \times B \rightarrow B$, it follows $\alpha^\dagger : T^\dagger \Rightarrow H^\dagger : A \rightarrow B$ is also a natural isomorphism and Lemma 4.4.6 (1) shows the operation $(-)^{\dagger}$ is functorial.

Theorem 4.4.8. The following diagram of natural isomorphisms:

$$\begin{array}{ccc}
 T \circ \langle N, T^\dagger \circ N \rangle & \xrightarrow{T\langle N, \alpha^\dagger \rangle} & T \circ \langle N, M \circ H^\dagger \rangle \xlongequal{\quad} T \circ (N \times M) \circ \langle \text{Id}, H^\dagger \rangle \\
 \parallel & & \downarrow \alpha\langle \text{Id}, H^\dagger \rangle \\
 T \circ \langle \text{Id}, T^\dagger \rangle \circ N & & M \circ H \circ \langle \text{Id}, H^\dagger \rangle \\
 \downarrow \phi^T N & & \downarrow M\phi^H \\
 T^\dagger \circ N & \xrightarrow{\alpha^\dagger} & M \circ H^\dagger
 \end{array}$$

commutes.

By combining this theorem together with Corollary 4.4.5, we get another corollary which is important for the adequacy proof.

Corollary 4.4.9. *In the special case where $\alpha = \text{id}$ and $z = \text{id}$, the following (2-categorical) diagram*

$$\begin{array}{ccc}
 A & \xrightarrow{N} & C \\
 \downarrow \scriptstyle H \circ \langle \text{Id}, H^\dagger \rangle & \scriptstyle \phi^H \Rightarrow & \downarrow \scriptstyle T \circ \langle \text{Id}, T^\dagger \rangle \\
 B & \xrightarrow{M} & D \\
 \uparrow & & \uparrow
 \end{array}$$

commutes, i.e. $\phi^T N = M \phi^H$.

5 CATEGORICAL MODEL

In this section we introduce our categorical model for LNL-FPC and describe its categorical properties. A CPO-LNL model is a CPO-enriched model of intuitionistic linear logic (also known as linear/non-linear model [Benton 1995; Benton and Wadler 1996]) which has suitable ω -colimits.

5.1 CPO-categories and Algebraic Compactness

A *cpo* (complete partial order) is a poset which has suprema of all increasing chains. If, in addition, the cpo has a least element, then we say it is *pointed*. A monotone function between two cpo's is *Scott continuous* if it preserves all suprema of increasing chains. If, in addition, the two cpo's are pointed and the function preserves the least element, then we say the function is *strict*.

We denote with **CPO** the category of cpo's and Scott continuous functions and with **CPO_!** the category of pointed cpo's and strict Scott continuous functions. Both categories are complete and cocomplete, **CPO** is cartesian closed and **CPO_!** is symmetric monoidal closed when equipped with the smash product and the strict function space [Abramsky and Jung 1994].

Our categorical model makes heavy use of *enriched* category theory. In particular, we use **CPO**-enriched and **CPO_!**-enriched categories. We only provide a brief introduction to **CPO**-categories and we suggest the reader consults [Fiore 1994, Chapter 2] for a more detailed introduction.

A **CPO**-category **C** is a category **C** whose homsets have the additional structure of a cpo and composition of morphisms is a Scott continuous function (in both arguments). A **CPO**-functor $F : C \rightarrow D$ between **CPO**-categories is a functor whose action on hom-cpo's $F_{X,Y} : C(X, Y) \rightarrow D(FX, FY)$ is Scott continuous for each pair of objects $X, Y \in C$. A **CPO**-adjunction $\Phi : F \dashv G : D \rightarrow C$ is given by **CPO**-categories **C** and **D**, **CPO**-functors $F : C \rightarrow D$ and $G : D \rightarrow C$, together with a natural isomorphism $\Phi : C(-, G-) \cong D(F-, -) : C^{\text{op}} \times D \rightarrow \mathbf{CPO}$. A **CPO**-symmetric monoidal closed category $C(I, \otimes, \multimap)$ is a **CPO**-category **C** together with **CPO**-functors $\otimes : C \times C \rightarrow C$ and $\multimap : C^{\text{op}} \times C \rightarrow C$ which form a symmetric monoidal closed category in the usual sense (one can then conclude Currying is a **CPO**-adjunction).

Similarly, **CPO_!**-enriched versions of these notions may be defined by requiring the hom-cpo's to be pointed and Scott continuity to be strict.

Definition 5.1.1. In a **CPO**-category **C**, a morphism $e : A \rightarrow B$ is called an *embedding* if there is a morphism $p : B \rightarrow A$, called a *projection*, such that $p \circ e = \text{id}_A$ and $e \circ p \leq \text{id}_B$. The pair (e, p) is said to form an *embedding-projection* (*e-p*) *pair*.

It is well-known that one component of an e-p pair uniquely determines the other, so we shall write $e^\bullet$ to indicate the projection counterpart of an embedding e . Every isomorphism is an embedding and embeddings are closed under composition with $(e_1 \circ e_2)^\bullet = e_2^\bullet \circ e_1^\bullet$, which allows us to make the next definition.

Definition 5.1.2. Given a CPO-category $\mathbf{C}$, its *subcategory of embeddings*, denoted $\mathbf{C}_e$, is the full-on-objects subcategory of $\mathbf{C}$ whose morphisms are exactly the embeddings of $\mathbf{C}$.

By duality, one can also define $\mathbf{C}_p$ to be the subcategory of projections. The notion of an e-p pair is of fundamental importance for establishing the famous limit-colimit coincidence theorem [Smyth and Plotkin 1982]. Before we can state it, we need to define a few additional notions.

Definition 5.1.3. In a CPO-category $\mathbf{C}$, an ω -*diagram over embeddings* is a diagram $D : \omega \rightarrow \mathbf{C}$, such that all $D(i \leq j)$ are embeddings. Given such a diagram, we define $D^\bullet : \omega^{\text{op}} \rightarrow \mathbf{C}$, to be the ω^{op} -diagram given by $D^\bullet(n) := D(n)$ and $D^\bullet(i \leq j) := D(i \leq j)^\bullet$. Given a cocone $\mu : D \rightarrow A$ in $\mathbf{C}$, we define a cone $\mu^\bullet : A \rightarrow D^\bullet$ in $\mathbf{C}$ by $(\mu^\bullet)_i := (\mu_i)^\bullet$. We shall say that $\mathbf{C}$ has all ω -colimits over embeddings if every ω -diagram over embeddings has a colimit in $\mathbf{C}$.

By combining several results from [Smyth and Plotkin 1982], one can show the next theorem.

Theorem 5.1.4 (Limit-colimit coincidence). *Let $\mathbf{C}$ be a CPO-category with all ω -colimits over embeddings. Let $D : \omega \rightarrow \mathbf{C}$ be an ω -diagram over embeddings and let $\mu : D \rightarrow A$ be a cocone of D . The following are equivalent:*

- (1) μ is a colimiting cocone of D in $\mathbf{C}$,
- (2) μ is a colimiting cocone of D in $\mathbf{C}_e$,
- (3) Each μ_i is an embedding, the sequence $(\mu_i \circ \mu_i^\bullet)_i$ is an increasing chain and $\bigvee_i \mu_i \circ \mu_i^\bullet = \text{id}_A$,
- (4) $\mu^\bullet$ is a limiting cone of $D^\bullet$ in $\mathbf{C}$,
- (5) $\mu^\bullet$ is a limiting cone of $D^\bullet$ in $\mathbf{C}_p$.

Remark 5.1.5. This theorem applies to both categories $\mathbf{CPO}$ and $\mathbf{CPO}_{\perp}$. However, when constructing an initial algebra for an endofunctor $T : \mathbf{CPO} \rightarrow \mathbf{CPO}$, the theorem is not useful, because the initial map $\iota : \emptyset \rightarrow T\emptyset$ is not an embedding (unless $T\emptyset = \emptyset$) and therefore the initial sequence of T is not computed over embeddings (unless its initial sequence is trivial). Also, observe that $\mathbf{CPO}_e$ has ω -colimits, but has no initial object and thus it is not a good setting for constructing initial algebras.

On the other hand, this theorem is very useful for constructing initial (and final) (co)algebras for many categories like $\mathbf{CPO}_{\perp}$ that have some additional structure, which we explain next.

Definition 5.1.6 ([Fiore 1994, Definition 7.1.1]). In a CPO-category, an *e-initial* object is an initial object such that every morphism with it as source is an embedding. The dual notion is called a *p-terminal* object. An object which is both e-initial and p-terminal is called an *ep-zero* object.

The category $\mathbf{CPO}$ has an initial object given by $\emptyset$, but it does not have an e-initial object. The category $\mathbf{CPO}_{\perp}$ has an ep-zero object which is given by the one-element cpo $\{\perp\}$.

An endofunctor $T : \mathbf{C} \rightarrow \mathbf{C}$ is *algebraically compact* if T has an initial T -algebra (Ω, τ) , such that (Ω, τ^{-1}) is a final T -coalgebra. In this situation we say (Ω, τ) is a *free* T -algebra. A CPO-category $\mathbf{C}$ is *CPO-algebraically compact* if every CPO-endofunctor $T : \mathbf{C} \rightarrow \mathbf{C}$ has a free T -algebra. CPO-algebraic compactness also implies the existence of *parameterised free algebras* [Fiore 1994].

Theorem 5.1.7 ([Fiore 1994, Corollary 7.2.4]). *A CPO-category with ep-zero and colimits of ω -chains over embeddings is CPO-algebraically compact.*

Therefore, categories which enjoy this structure are an ideal setting for solving recursive domain equations. In particular, algebraic compactness is crucial for interpreting isorecursive types involving $\multimap$. We will see that the linear category of our model is indeed such a category and that we can very easily interpret (arbitrary) types within it. Non-linear types, however, need to admit an additional non-linear interpretation within a cartesian category which does not enjoy such a strong property. To resolve this problem, our type interpretations will be based on the following theorem which combines results of [Smyth and Plotkin 1982].

Theorem 5.1.8. *Let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be CPO-categories where $\mathbf{A}$ and $\mathbf{B}$ have ω -colimits over embeddings. If $T : \mathbf{A}^{\text{op}} \times \mathbf{B} \rightarrow \mathbf{C}$ is a CPO-functor, then the covariant functor $T_e : \mathbf{A}_e \times \mathbf{B}_e \rightarrow \mathbf{C}_e$ given by*

$$T_e(A, B) = T(A, B) \quad \text{and} \quad T_e(e_1, e_2) = T((e_1^\bullet)^{\text{op}}, e_2)$$

is an ω -functor.

PROOF. This follows from [Smyth and Plotkin 1982, Theorem 3], because both categories $\mathbf{A}$ and $\mathbf{B}$ have locally determined ω -colimits of embeddings thanks to Theorem 5.1.4. $\square$

Therefore, this theorem allows us to consider mixed-variance CPO-functors on CPO-categories as covariant ones on subcategories of embeddings. By trivialising the category $\mathbf{A}$, we get:

Corollary 5.1.9. *Let $T : \mathbf{B} \rightarrow \mathbf{C}$ be a CPO-functor between CPO-categories $\mathbf{B}$ and $\mathbf{C}$ where $\mathbf{B}$ has ω -colimits over embeddings. Then T restricts to an ω -functor $T_e : \mathbf{B}_e \rightarrow \mathbf{C}_e$.*

5.2 Models of Intuitionistic Linear Logic

Our type system has both linear and non-linear features, so naturally, it should be interpreted within a model of Intuitionistic Linear Logic, also known as linear/non-linear (LNL) model (see [Benton 1995; Benton and Wadler 1996] for more details).

Definition 5.2.1. A model of Intuitionistic Linear Logic is given by the following data: a cartesian closed category with finite coproducts $(\mathbf{C}, \times, \rightarrow, 1, \amalg, \emptyset)$; a symmetric monoidal closed category

with finite coproducts $(\mathbf{L}, \otimes, \multimap, I, +, 0)$; and a symmetric monoidal adjunction $\mathbf{C} \begin{array}{c} \xrightarrow{F} \\ \perp \\ \xleftarrow{G} \end{array} \mathbf{L}$.

We also adopt the following notation. The comonad-endofunctor is $! := F \circ G : \mathbf{L} \rightarrow \mathbf{L}$; the unit and counit of the adjunction are $\eta : \text{Id} \Rightarrow G \circ F : \mathbf{C} \rightarrow \mathbf{C}$ and $\epsilon : ! \Rightarrow \text{Id} : \mathbf{L} \rightarrow \mathbf{L}$, respectively.

Moreover, in this situation, the left-adjoint F is both cocontinuous and strong symmetric monoidal. Thus, there exist isomorphisms $I \xrightarrow{u} F(1)$ and $0 \xrightarrow{z} F(\emptyset)$ in $\mathbf{L}$, together with natural isomorphisms

$$\otimes \circ (F \times F) \xrightarrow{m} F \circ \times : \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{L} \quad \text{and} \quad + \circ (F \times F) \xrightarrow{c} F \circ \amalg : \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{L}$$

that satisfy some coherence conditions, which we omit here. Observe, that the structure of the natural isomorphisms m and c is very similar. In fact, most of our propositions involving these natural isomorphisms can be proven in a uniform way, so we introduce some notation for brevity.

Notation 5.2.2. We write $\odot$ for either the $\otimes$ or $+$ bifunctor; $\square$ for the bifunctor $\times$ or $\amalg$ respectively; β for the natural isomorphisms m or c , respectively. Then, we have a natural isomorphism:

$$\odot \circ (F \times F) \xrightarrow{\beta} F \circ \square : \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{L}.$$

5.3 Models of LNL-FPC

We can now describe our categorical model by combining results from the previous two subsections.

Definition 5.3.1. A CPO-LNL model is given by the following data:

1. A CPO-symmetric monoidal closed category $(\mathbf{L}, \otimes, \multimap, I)$, such that:
 - 1a. $\mathbf{L}$ has an e-initial object 0 and all ω -colimits over embeddings;
 - 1b. $\mathbf{L}$ has finite CPO-coproducts, where $(- + -) : \mathbf{L} \times \mathbf{L} \rightarrow \mathbf{L}$ is the coproduct functor.

2. A CPO-symmetric monoidal adjunction $\mathbf{CPO} \begin{array}{c} \xrightarrow{F} \\ \perp \\ \xleftarrow{G} \end{array} \mathbf{L}$.

Our categorical model is a model of Intuitionistic Linear Logic with some additional structure for solving recursive domain equations. Because of that, we shall use the same notation for it as introduced in the previous subsection.

Assumption 5.3.2. Throughout the rest of the paper we assume we are working with an arbitrary, but fixed, CPO-LNL model as in Definition 5.3.1.

While not immediately obvious, $\mathbf{L}$ is CPO-algebraically compact. We reason as follows.

Theorem 5.3.3. *In every CPO-LNL model:*

- (1) *The initial object 0 is an ep-zero object;*
- (2) *Each zero morphism $\perp_{A,B}$ is least in $\mathbf{L}(A, B)$;*
- (3) *$\mathbf{L}$ is CPO- $\perp_!$ -enriched;*
- (4) *$\mathbf{L}$ is CPO-algebraically compact.*

PROOF. (1) For every object A , define a morphism $\perp_A := I \xrightarrow{e_I^\bullet} 0 \xrightarrow{e_A} A$, where $e_X : 0 \rightarrow X$ is unique and $e_X^\bullet : X \rightarrow 0$ is its projection counterpart. Then, for every $h : A \rightarrow B$, we have $h \circ \perp_A = \perp_B$. This means the category is *weakly pointed* in the sense of [Braüner 1997, Definition 9]. One can now define morphisms

$$\perp_{A,B} := A \xrightarrow{\lambda_A^{-1}} I \otimes A \xrightarrow{\text{uncurry}(\perp_{A \multimap B})} B$$

and prove that:

$$\begin{array}{lll} g \circ \perp_{A,B} = \perp_{A,C} & \text{for } g : B \rightarrow C & [\text{Braüner 1997, Proposition 11}] \\ \perp_{B,C} \circ f = \perp_{A,C} & \text{for } f : A \rightarrow B & [\text{Braüner 1997, Proposition 12}] \end{array}$$

In the presence of an initial object, this then implies that 0 is a zero object; $\perp_A = \perp_{I,A}$ and that the morphisms $\perp_{A,B}$ are zero morphisms [Lindenhovius et al. 2018, Lemma 4.8]. Therefore 0 is an ep-zero, because $(e_X, e_X^\bullet)$ forms an e-p pair for every object X and also $\perp_{A,B} = A \xrightarrow{e_A^\bullet} 0 \xrightarrow{e_B} B$.

(2) For any $f : A \rightarrow B$, we have $\perp_{A,B} = \perp_{B,B} \circ f = e_B \circ e_B^\bullet \circ f \leq \text{id}_B \circ f = f$.

(3) is now immediate from (2) and (4) follows by Theorem 5.1.7. $\square$

The next concept we introduce is that of a *pre-embedding*, which is novel (to the best of our knowledge). Pre-embeddings are of fundamental importance for defining the non-linear interpretations of non-linear types.

Definition 5.3.4. Let $T : \mathbf{A} \rightarrow \mathbf{B}$ be a CPO-functor between CPO-categories $\mathbf{A}$ and $\mathbf{B}$. Then a morphism f in $\mathbf{A}$ is called a *pre-embedding with respect to T* if $T(f)$ is an embedding in $\mathbf{B}$.

Every embedding in $\mathbf{A}$ is a pre-embedding (because CPO-functors preserve embeddings) and pre-embeddings are closed under composition, so one can form the full-on-objects subcategory of $\mathbf{A}$ of all pre-embeddings. We define such a subcategory within our model.

Definition 5.3.5. Let $\mathbf{CPO}_{pe}$ be the full-on-objects subcategory of CPO of all cpo's with pre-embeddings with respect to the functor F .

Note that the subcategory $\mathbf{CPO}_{pe}$ depends on the choice of CPO-LNL model. The importance of this subcategory is that it allows us to *reflect* solutions of certain recursive domain equations from the category $\mathbf{L}_e$ into the category $\mathbf{CPO}_{pe}$. As we have previously explained, the construction of initial algebras of endofunctors on CPO is not done over embeddings, in general, because the initial maps $\iota : \emptyset \rightarrow X$ are not embeddings. Observe, however, that they are pre-embeddings with respect to F . In fact, we shall see that the initial algebras for *all* non-linear type interpretations are constructed over initial sequences consisting of pre-embeddings.

Theorem 5.3.6. *In every CPO-LNL model:*

- (1) $\mathbf{L}_e$ is an ω -category, and the subcategory inclusion $\mathbf{L}_e \hookrightarrow \mathbf{L}$ is a strict ω -functor which also reflects ω -colimits.
- (2) $\mathbf{CPO}_{pe}$ is an ω -category and the subcategory inclusion $\mathbf{CPO}_{pe} \hookrightarrow \mathbf{CPO}$ is a strict ω -functor which also reflects ω -colimits.
- (3) The subcategory inclusion $\mathbf{CPO}_e \hookrightarrow \mathbf{CPO}_{pe}$ preserves and reflects ω -colimits ($\mathbf{CPO}_e$ has no initial object).

By definition of $\mathbf{CPO}_{pe}$, the functor $F : \mathbf{CPO} \rightarrow \mathbf{L}$ restricts to a functor $F_{pe} : \mathbf{CPO}_{pe} \rightarrow \mathbf{L}_e$. By Corollary 5.1.9 the functor $G : \mathbf{L} \rightarrow \mathbf{CPO}$ restricts to an ω -functor $G_e : \mathbf{L}_e \rightarrow \mathbf{CPO}_e$. Let $G_{pe} : \mathbf{L}_e \rightarrow \mathbf{CPO}_{pe}$ be the functor given by $G_{pe} := \left(\mathbf{L}_e \xrightarrow{G_e} \mathbf{CPO}_e \hookrightarrow \mathbf{CPO}_{pe} \right)$.

Theorem 5.3.7. *In any CPO-LNL model, $F_{pe} : \mathbf{CPO}_{pe} \rightarrow \mathbf{L}_e$ is a strict ω -functor. In addition, $G_{pe} : \mathbf{L}_e \rightarrow \mathbf{CPO}_{pe}$ is an ω -functor.*

Recall that by Notation 5.2.2, we have a natural isomorphism $\odot \circ (F \times F) \xRightarrow{\beta} F \circ \square : \mathbf{CPO} \times \mathbf{CPO} \rightarrow \mathbf{L}$ and by Corollary 5.1.9, we see that $\odot : \mathbf{L} \times \mathbf{L} \rightarrow \mathbf{L}$ restricts to an ω -functor $\odot_e : \mathbf{L}_e \times \mathbf{L}_e \rightarrow \mathbf{L}_e$.

Theorem 5.3.8. *In any CPO-LNL model, the bifunctor $\square : \mathbf{CPO} \times \mathbf{CPO} \rightarrow \mathbf{CPO}$ restricts to an ω -bifunctor $\square_{pe} : \mathbf{CPO}_{pe} \times \mathbf{CPO}_{pe} \rightarrow \mathbf{CPO}_{pe}$. Moreover, the natural isomorphism β restricts to a natural isomorphism $\odot_e \circ (F_{pe} \times F_{pe}) \xRightarrow{\beta^{pe}} F_{pe} \circ \square_{pe} : \mathbf{CPO}_{pe} \times \mathbf{CPO}_{pe} \rightarrow \mathbf{L}_e$.*

5.4 Concrete CPO-LNL Models

We now describe some concrete CPO-LNL models. We begin with the simplest non-trivial model.

Theorem 5.4.1. *The adjunction $\mathbf{CPO} \xrightleftharpoons[\underset{U}{\perp}]{\overset{(-)_{\perp}}{\dashv}}} \mathbf{CPO}_{\perp!}$, where the left adjoint is given by (domain-theoretic) lifting and the right adjoint U is the forgetful functor, is a CPO-LNL model.*

We will also show this model is computationally adequate in §7. This concrete model actually arises from a more general class of CPO-LNL models, as we shall now explain.

Let $\mathbf{M}$ be a small $\mathbf{CPO}_{\perp!}$ -symmetric monoidal category and let $\widehat{\mathbf{M}}$ be the category of $\mathbf{CPO}_{\perp!}$ -presheafs, that is, the category whose objects are the $\mathbf{CPO}_{\perp!}$ -functors $T : \mathbf{M}^{\text{op}} \rightarrow \mathbf{CPO}_{\perp!}$ and whose morphisms are natural transformations between them (the notion of a $\mathbf{CPO}_{\perp!}$ -natural transformation coincides with the ordinary notion). Then, by the enriched Yoneda Lemma, the category $\widehat{\mathbf{M}}$ is complete and cocomplete and it has a $\mathbf{CPO}_{\perp!}$ -symmetric monoidal closed structure when equipped with the Day convolution [Day 1974]. In this situation there exists an adjunction

$\mathbf{CPO}_{\perp!} \xrightleftharpoons[\underset{\widehat{\mathbf{M}}(I, -)}{\perp}]{\overset{- \odot I}{\dashv}}} \widehat{\mathbf{M}}$, where the left adjoint is given by taking the $\mathbf{CPO}_{\perp!}$ -copower with the

tensor unit I and the right adjoint is the representable functor (see [Borceux 1994, §6]).

Theorem 5.4.2. *Composing the two adjunctions $\mathbf{CPO} \xrightleftharpoons[\underset{U}{\perp}]{\overset{(-)_{\perp}}{\dashv}}} \mathbf{CPO}_{\perp!} \xrightleftharpoons[\underset{\widehat{\mathbf{M}}(I, -)}{\perp}]{\overset{- \odot I}{\dashv}}} \widehat{\mathbf{M}}$ yields a CPO-LNL model.*

Observe that any $\mathbf{CPO}_{\perp!}$ -functor necessarily preserves least morphisms and therefore also ep-zero objects. Thus, if $\mathbf{M} = \mathbf{1}$, then $\widehat{\mathbf{M}} \simeq \mathbf{1}$ and we recover the trivial (or completely degenerate)

CPO-LNL model. If $\mathbf{M} = \mathbf{1}_\perp$ (the category with a unique object and two morphisms id and $\perp$, where $\perp$ is least), then $\widehat{\mathbf{M}} \cong \text{CPO}_{\perp!}$ and we recover the CPO-LNL model from Theorem 5.4.1. Choosing $\mathbf{M}$ to be a suitable category of quantum computation, we recover a model of Proto-Quipper-M, a quantum programming language [Rios and Selinger 2017]. Choosing $\mathbf{M}$ to be a suitable category of string diagrams, we recover a model of ECLNL, a programming language for string diagrams [Lindenhovius et al. 2018]. Similar presheaf models have been used to model Ewire, a circuit description language [Rennela and Staton 2017], and the quantum lambda calculus [Malherbe et al. 2013]. All of these languages have mixed linear/non-linear features and therefore our results presented here could help with design decisions on introducing recursive (or inductive) datatypes.

6 DENOTATIONAL SEMANTICS

In this section we describe the denotational semantics of LNL-FPC. Every type $\Theta \vdash A$ admits a (standard) interpretation as an ω -functor $\llbracket \Theta \vdash A \rrbracket : \mathbf{L}_e^{|\Theta|} \rightarrow \mathbf{L}_e$. Every *non-linear* type $\Theta \vdash P$ admits an *additional* non-linear interpretation as an ω -functor $\langle \Theta \vdash P \rangle : \text{CPO}_{pe}^{|\Theta|} \rightarrow \text{CPO}_{pe}$ which is related to its standard interpretation via a natural isomorphism (see Figure 5)

$$\alpha^{\Theta \vdash P} : \llbracket \Theta \vdash P \rrbracket \circ F_{pe}^{\times|\Theta|} \Rightarrow F_{pe} \circ \langle \Theta \vdash P \rangle : \text{CPO}_{pe}^{|\Theta|} \rightarrow \mathbf{L}_e.$$

It is precisely the existence of this natural isomorphism that allows us to define the substructural operations (copying, discarding and promotion) on non-linear types.

Just like in FPC, the interpretation of a term $\Theta; \Gamma \vdash m : A$ is a *family* of morphisms of $\mathbf{L}$

$$\llbracket \Theta; \Gamma \vdash m : A \rrbracket = \left\{ \llbracket \Theta; \Gamma \vdash m : A \rrbracket_{\vec{Z}} : \llbracket \Theta \vdash \Gamma \rrbracket (F^{\times|\Theta|} \vec{Z}) \rightarrow \llbracket \Theta \vdash A \rrbracket (F^{\times|\Theta|} \vec{Z}) \mid \vec{Z} \in \text{Ob}(\text{CPO}^{|\Theta|}) \right\}$$

indexed by objects $\vec{Z}$ of $\text{CPO}^{|\Theta|}$. In the special case when $\Theta = \cdot$ the interpretation can be seen simply as a morphism of $\mathbf{L}$. In order to show the semantics is sound, we also provide a non-linear interpretation of non-linear values which is compatible with the substructural operations.

We remark that types are interpreted as functors on $\mathbf{L}_e$, which is a subcategory of $\mathbf{L}$, the category in which terms are interpreted. Doing so is crucial for defining the semantics, as we shall explain.

6.1 Interpretation of LNL-FPC Types

The (standard) interpretation of a well-formed type $\Theta \vdash A$ is a functor $\llbracket \Theta \vdash A \rrbracket : \mathbf{L}_e^{|\Theta|} \rightarrow \mathbf{L}_e$ defined by induction on the derivation of $\Theta \vdash A$ (Figure 4, left). The non-linear interpretation of a well-formed non-linear type $\Theta \vdash P$ is a functor $\langle \Theta \vdash P \rangle : \text{CPO}_{pe}^{|\Theta|} \rightarrow \text{CPO}_{pe}$ defined by induction on the derivation of $\Theta \vdash P$ (Figure 4, right). Note that the non-linear interpretation of non-linear types depends on the standard interpretation, so the latter needs to be defined first, as we have done. Both type interpretations are functors defined on ω -categories. Next, we show that the interpretations are moreover ω -functors, so that we may construct (parameterised) initial algebras that we need to interpret recursive types.

Theorem 6.1.1. *For any well-formed types $\Theta \vdash A$ and $\Theta \vdash P$, where P is non-linear:*

- (1) *The functor $\llbracket \Theta \vdash A \rrbracket : \mathbf{L}_e^{|\Theta|} \rightarrow \mathbf{L}_e$ is an ω -functor.*
- (2) *The functor $\langle \Theta \vdash P \rangle : \text{CPO}_{pe}^{|\Theta|} \rightarrow \text{CPO}_{pe}$ is an ω -functor.*

PROOF. Both statements are proved by induction one after the other.

The functor Π_i is obviously an ω -functor. The functors $!_e, +_e, \otimes_e, \multimap_e$ are ω -functors because of Theorem 5.1.8. The functors $G_{pe}, F_{pe}, \times_{pe}, \Pi_{pe}$ are ω -functors (Theorems 5.3.7 and 5.3.8). Moreover ω -functors are closed under composition and tupling. Finally, the μ -case follows from Theorem 4.3.2 for both statements. $\square$

$$\begin{array}{ll}
\llbracket \Theta \vdash A \rrbracket : \mathbf{L}_e^{|\Theta|} \rightarrow \mathbf{L}_e & \langle \Theta \vdash P \rangle : \mathbf{CPO}_{pe}^{|\Theta|} \rightarrow \mathbf{CPO}_{pe} \\
\llbracket \Theta \vdash \Theta_i \rrbracket := \Pi_i & \langle \Theta \vdash \Theta_i \rangle := \Pi_i \\
\llbracket \Theta \vdash !A \rrbracket := !_e \circ \llbracket \Theta \vdash A \rrbracket & \langle \Theta \vdash !A \rangle := G_{pe} \circ \llbracket \Theta \vdash A \rrbracket \circ F_{pe}^{\times|\Theta|} \\
\llbracket \Theta \vdash A + B \rrbracket := +_e \circ \langle \llbracket \Theta \vdash A \rrbracket, \llbracket \Theta \vdash B \rrbracket \rangle & \langle \Theta \vdash P + Q \rangle := \Pi_{pe} \circ \langle \langle \Theta \vdash P \rangle, \langle \Theta \vdash Q \rangle \rangle \\
\llbracket \Theta \vdash A \otimes B \rrbracket := \otimes_e \circ \langle \llbracket \Theta \vdash A \rrbracket, \llbracket \Theta \vdash B \rrbracket \rangle & \langle \Theta \vdash P \otimes Q \rangle := \chi_{pe} \circ \langle \langle \Theta \vdash P \rangle, \langle \Theta \vdash Q \rangle \rangle \\
\llbracket \Theta \vdash A \multimap B \rrbracket := \multimap_e \circ \langle \llbracket \Theta \vdash A \rrbracket, \llbracket \Theta \vdash B \rrbracket \rangle & \langle \Theta \vdash \mu X.P \rangle := \langle \Theta, X \vdash P \rangle^\dagger \\
\llbracket \Theta \vdash \mu X.A \rrbracket := \llbracket \Theta, X \vdash A \rrbracket^\dagger &
\end{array}$$

Fig. 4. Standard interpretation of types (left) and non-linear interpretation of non-linear types (right).

$$\begin{array}{ccc}
\mathbf{CPO}_{pe}^{|\Theta|} & \xrightarrow{F_{pe}^{\times|\Theta|}} & \mathbf{L}_e^{|\Theta|} \\
\langle \Theta \vdash P \rangle \downarrow & \swarrow \alpha & \downarrow \llbracket \Theta \vdash P \rrbracket \\
\mathbf{CPO}_{pe} & \xrightarrow{F_{pe}} & \mathbf{L}_e
\end{array}$$

Fig. 5. Relationship between the standard and non-linear interpretations of non-linear types.

The next theorem shows that the standard and non-linear interpretations of non-linear types are strongly related via a natural isomorphism.

Theorem 6.1.2. *For any non-linear type $\Theta \vdash P$, there exists a natural isomorphism (see Figure 5)*

$$\alpha^{\Theta \vdash P} : \llbracket \Theta \vdash P \rrbracket \circ F_{pe}^{\times|\Theta|} \Rightarrow F_{pe} \circ \langle \Theta \vdash P \rangle : \mathbf{CPO}_{pe}^{|\Theta|} \rightarrow \mathbf{L}_e$$

which is given by:

$$\begin{aligned}
\alpha^{\Theta \vdash \Theta_i} &:= id \\
\alpha^{\Theta \vdash !A} &:= id \\
\alpha^{\Theta \vdash P \odot Q} &:= \beta^{pe} \langle \langle \Theta \vdash P \rangle, \langle \Theta \vdash Q \rangle \rangle \circ \odot_e \langle \alpha^{\Theta \vdash P}, \alpha^{\Theta \vdash Q} \rangle && \text{(see Theorem 5.3.8)} \\
\alpha^{\Theta \vdash \mu X.P} &:= (\alpha^{\Theta, X \vdash P})^\dagger && \text{(see Theorem 4.4.2).}
\end{aligned}$$

In FPC, one has to prove a substitution lemma which shows that the interpretation of types behaves well with respect to type substitution. In LNL-FPC, we have to prove three such lemmas: one for the standard interpretation $\llbracket \Theta \vdash A \rrbracket$ of (arbitrary) types, one for the non-linear interpretation $\langle \Theta \vdash P \rangle$ of non-linear types, and one for the natural isomorphism $\alpha^{\Theta \vdash P}$ which relates them. In order to do this, one first has to show a permutation and a contraction lemma, as we do now.

Lemma 6.1.3 (Permutation). *Given types $\Theta, X, Y, \Theta' \vdash A$ and $\Theta, X, Y, \Theta' \vdash P$, with P non-linear:*

- (1) $\llbracket \Theta, Y, X, \Theta' \vdash A \rrbracket = \llbracket \Theta, X, Y, \Theta' \vdash A \rrbracket \circ \text{swap}_{m, m'}$
- (2) $\langle \Theta, Y, X, \Theta' \vdash P \rangle = \langle \Theta, X, Y, \Theta' \vdash P \rangle \circ \text{swap}_{m, m'}$
- (3) $\alpha^{\Theta, Y, X, \Theta' \vdash P} = \alpha^{\Theta, X, Y, \Theta' \vdash P} \circ \text{swap}_{m, m'}$

where $|\Theta| = m$, $|\Theta'| = m'$ and $\text{swap}_{m, m'} = \langle \Pi_1, \dots, \Pi_m, \Pi_{m+2}, \Pi_{m+1}, \Pi_{m+3}, \dots, \Pi_{m+m'+2} \rangle$.

PROOF. Essentially the same as [Fiore 1994, Lemma C.0.1]. □

$$\begin{array}{ccc}
\llbracket \Theta \vdash P[R/X] \rrbracket \circ F_{pe}^{\times|\Theta|} & \xRightarrow{\alpha} & F_{pe} \circ \langle \Theta \vdash P[R/X] \rangle \\
\parallel & & \uparrow F_{pe}\gamma^{-1} \\
\llbracket \Theta, X \vdash P \rrbracket \circ \langle Id, \llbracket \Theta \vdash R \rrbracket \rangle \circ F_{pe}^{\times|\Theta|} & & F_{pe} \circ \langle \Theta, X \vdash P \rangle \circ \langle Id, \langle \Theta \vdash R \rangle \rangle \\
\parallel & & \uparrow \alpha \langle Id, \langle \Theta \vdash R \rangle \rangle \\
\llbracket \Theta, X \vdash P \rrbracket \circ \langle F_{pe}^{\times|\Theta|}, \llbracket \Theta \vdash R \rrbracket \circ F_{pe}^{\times|\Theta|} \rangle & & \\
\parallel & & \\
\llbracket \Theta, X \vdash P \rrbracket \langle F_{pe}^{\times|\Theta|}, \alpha \rangle & & \\
\downarrow & & \\
\llbracket \Theta, X \vdash P \rrbracket \circ \langle F_{pe}^{\times|\Theta|}, F_{pe} \circ \langle \Theta \vdash R \rangle \rangle & \xlongequal{\quad} & \llbracket \Theta, X \vdash P \rrbracket \circ F_{pe}^{\times(|\Theta|+1)} \circ \langle Id, \langle \Theta \vdash R \rangle \rangle
\end{array}$$

Fig. 6. The commuting diagram of natural isomorphisms for Lemma 6.1.5 (3).

Lemma 6.1.4 (Contraction). *Given types $\Theta, \Theta' \vdash A$ and $\Theta, \Theta' \vdash P$, with P non-linear:*

- (1) $\llbracket \Theta, X, \Theta' \vdash A \rrbracket = \llbracket \Theta, \Theta' \vdash A \rrbracket \circ \text{drop}_{m,m'}$
- (2) $\langle \Theta, X, \Theta' \vdash P \rangle = \langle \Theta, \Theta' \vdash P \rangle \circ \text{drop}_{m,m'}$
- (3) $\alpha^{\Theta, X, \Theta' \vdash P} = \alpha^{\Theta, \Theta' \vdash P} \text{drop}_{m,m'}$

where $X \notin \Theta \cup \Theta'$, $|\Theta| = m$, $|\Theta'| = m'$ and $\text{drop}_{m,m'} = \langle \Pi_1, \dots, \Pi_m, \Pi_{m+2}, \dots, \Pi_{m+m'+1} \rangle$.

PROOF. Essentially the same as [Fiore 1994, Lemma C.0.2]. $\square$

In all models of FPC, the substitution lemma (and also the permutation and contraction lemmas) hold up to isomorphism. However, in models where the equality of Proposition 4.3.4 holds, this can be strengthened to an equality (cf. [Fiore 1994, pp. 181]). Indeed, in LNL-FPC the same is true for the standard substitution lemma, but the non-linear substitution lemma holds only up to isomorphism, which is induced by the isomorphism α .

Lemma 6.1.5 (Substitution). *Given types $\Theta, X \vdash A$ and $\Theta \vdash B$ and $\Theta, X \vdash P$ and $\Theta \vdash R$ with P and R non-linear:*

- (1) $\llbracket \Theta \vdash A[B/X] \rrbracket = \llbracket \Theta, X \vdash A \rrbracket \circ \langle Id, \llbracket \Theta \vdash B \rrbracket \rangle$
- (2) $\langle \Theta \vdash P[R/X] \rangle \cong \langle \Theta, X \vdash P \rangle \circ \langle Id, \langle \Theta \vdash R \rangle \rangle$, where the natural isomorphism, denoted $\gamma^{\Theta \vdash P[R/X]}$ is given by:

$$\begin{aligned}
\gamma^{\Theta \vdash \Theta_i[R/X]} &= id \\
\gamma^{\Theta \vdash X[R/X]} &= id \\
\gamma^{\Theta \vdash !A[R/X]} &= G_{pe} \llbracket \Theta, X \vdash A \rrbracket \langle F_{pe}^{\times|\Theta|}, \alpha^{\Theta \vdash R} \rangle \\
\gamma^{\Theta \vdash (P \odot Q)[R/X]} &= \sqsubset_{pe} \langle \gamma^{\Theta \vdash P[R/X]}, \gamma^{\Theta \vdash Q[R/X]} \rangle \\
\gamma^{\Theta \vdash \mu Y. P[R/X]} &= (\gamma^{\Theta, Y \vdash P[R/X]})^\dagger
\end{aligned}$$

- (3) $\alpha^{\Theta \vdash P[R/X]} = (F_{pe} \gamma^{\Theta \vdash P[R/X]})^{-1} \circ \alpha^{\Theta, X \vdash P} \langle Id, \langle \Theta \vdash R \rangle \rangle \circ \llbracket \Theta, X \vdash P \rrbracket \langle F_{pe}^{\times|\Theta|}, \alpha^{\Theta \vdash R} \rangle$, (see Figure 6).

The most important type isomorphism in FPC is the folding and unfolding of recursive types. Denotationally, this is modelled using the properties of parameterised initial algebras and the substitution lemma. The same is true in LNL-FPC, but, in addition to the standard folding and

unfolding of recursive types, we also have a non-linear interpretation for folding and unfolding of non-linear recursive types.

Definition 6.1.6. Given types $\Theta \vdash \mu X.A$ and $\Theta \vdash \mu X.P$, with P non-linear, let

$$\begin{aligned} \text{fold}^{\Theta \vdash \mu X.A} : \llbracket \Theta \vdash A[\mu X.A/X] \rrbracket &\Rightarrow \llbracket \Theta \vdash \mu X.A \rrbracket : \mathbf{L}_e^{|\Theta|} \rightarrow \mathbf{L}_e \\ \text{fold}^{\Theta \vdash \mu X.P} : \langle \Theta \vdash P[\mu X.P/X] \rangle &\Rightarrow \langle \Theta \vdash \mu X.P \rangle : \mathbf{CPO}_{pe}^{|\Theta|} \rightarrow \mathbf{CPO}_{pe} \end{aligned}$$

be the natural isomorphisms given by

$$\begin{aligned} \text{fold}^{\Theta \vdash \mu X.A} &:= \left(\llbracket \Theta \vdash A[\mu X.A/X] \rrbracket = \llbracket \Theta, X \vdash A \rrbracket \circ \langle \text{Id}, \llbracket \Theta \vdash \mu X.A \rrbracket \rangle \xRightarrow{\phi} \llbracket \Theta \vdash \mu X.A \rrbracket \right) \\ \text{fold}^{\Theta \vdash \mu X.P} &:= \left(\langle \Theta \vdash P[\mu X.P/X] \rangle \xRightarrow{\gamma} \langle \Theta, X \vdash P \rangle \circ \langle \text{Id}, \langle \Theta \vdash \mu X.P \rangle \rangle \xRightarrow{\phi} \langle \Theta \vdash \mu X.P \rangle \right). \end{aligned}$$

We denote their inverses by $\text{unfold}^{\Theta \vdash \mu X.A}$ and $\text{unfold}^{\Theta \vdash \mu X.P}$, respectively.

We now show that given a non-linear recursive type, the standard and non-linear interpretations of its folding and unfolding are also strongly related. We shall use this theorem later in order to prove that values $\Theta; \Gamma \vdash \text{fold } v : P$ with P non-linear admit a non-linear interpretation.

Theorem 6.1.7. Let $\Theta \vdash \mu X.P$ be a non-linear type. Then the diagram of natural isomorphisms

$$\begin{array}{ccc} \llbracket \Theta \vdash P[\mu X.P/X] \rrbracket \circ F_{pe}^{\times|\Theta|} & \xRightarrow{\alpha} & F_{pe} \circ \langle \Theta \vdash P[\mu X.P/X] \rangle \\ \downarrow \text{fold} F_{pe}^{\times|\Theta|} & & \downarrow F_{pe} \text{fold} \\ \llbracket \Theta \vdash \mu X.P \rrbracket \circ F_{pe}^{\times|\Theta|} & \xRightarrow{\alpha} & F_{pe} \circ \langle \Theta \vdash \mu X.P \rangle \end{array}$$

commutes.

PROOF.

$$\begin{aligned} &F_{pe} \text{fold}^{\Theta \vdash \mu X.P} \circ \alpha^{\Theta \vdash P[\mu X.P/X]} \\ &= F_{pe} \phi \circ F_{pe} \gamma^{\Theta \vdash P[\mu X.P/X]} \circ \alpha^{\Theta \vdash P[\mu X.P/X]} && \text{(Definition)} \\ &= F_{pe} \phi \circ \alpha^{\Theta, X \vdash P} \langle \text{Id}, \langle \Theta \vdash \mu X.P \rangle \rangle \circ \llbracket \Theta, X \vdash P \rrbracket \langle F_{pe}^{\times|\Theta|}, \alpha^{\Theta \vdash \mu X.P} \rangle && \text{(Lemma 6.1.5 (3))} \\ &= F_{pe} \phi \circ \alpha^{\Theta, X \vdash P} \langle \text{Id}, \langle \Theta, X \vdash P \rangle^\dagger \rangle \circ \llbracket \Theta, X \vdash P \rrbracket \langle F_{pe}^{\times|\Theta|}, (\alpha^{\Theta, X \vdash P})^\dagger \rangle && \text{(Definition)} \\ &= (\alpha^{\Theta, X \vdash P})^\dagger \circ \phi F_{pe}^{\times|\Theta|} && \text{(Theorem 4.4.8)} \\ &= \alpha^{\Theta \vdash \mu X.P} \circ \text{fold}^{\Theta \vdash \mu X.P} F_{pe}^{\times|\Theta|} && \text{(Definition)} \end{aligned}$$

□

6.2 Interpretation of LNL-FPC Term Contexts

The interpretation of term contexts is straightforward. The interpretation of an (arbitrary) term context $\Theta \vdash \Gamma$ is an ω -functor $\llbracket \Theta \vdash \Gamma \rrbracket : \mathbf{L}_e^{|\Theta|} \rightarrow \mathbf{L}_e$ (see Figure 7, left). A non-linear term context $\Theta \vdash \Phi$ admits an additional non-linear interpretation as an ω -functor $\langle \Theta \vdash \Phi \rangle : \mathbf{CPO}_{pe}^{|\Theta|} \rightarrow \mathbf{CPO}_{pe}$ (see Figure 7, right). Just like with the interpretation of types, the two are strongly related.

$$\begin{aligned}
\llbracket \Theta \vdash \Gamma \rrbracket &: \mathbf{L}_e^{|\Theta|} \rightarrow \mathbf{L}_e & \llbracket \Theta \vdash \Phi \rrbracket &: \mathbf{CPO}_{pe}^{|\Theta|} \rightarrow \mathbf{CPO}_{pe} \\
\llbracket \Theta \vdash \cdot \rrbracket &:= K_I & \llbracket \Theta \vdash \cdot \rrbracket &:= K_1 \\
\llbracket \Theta \vdash \Gamma, x : A \rrbracket &:= \otimes_e \circ \langle \llbracket \Theta \vdash \Gamma \rrbracket, \llbracket \Theta \vdash A \rrbracket \rangle & \llbracket \Theta \vdash \Phi, x : P \rrbracket &:= \times_{pe} \circ \langle \llbracket \Theta \vdash \Phi \rrbracket, \llbracket \Theta \vdash P \rrbracket \rangle, \\
&& \text{where } K_X \text{ is the constant } X \text{ functor.}
\end{aligned}$$

Fig. 7. Interpretation of term contexts (left); non-linear interpretation of non-linear term contexts (right).

Theorem 6.2.1. *For any non-linear term context $\Theta \vdash \Phi$, there exists a natural isomorphism*

$$\alpha^{\Theta \vdash \Phi} : \llbracket \Theta \vdash \Phi \rrbracket \circ F_{pe}^{\times|\Theta|} \Rightarrow F_{pe} \circ \llbracket \Theta \vdash \Phi \rrbracket : \mathbf{CPO}_{pe}^{|\Theta|} \rightarrow \mathbf{L}_e$$

which is given by:

$$\begin{aligned}
\alpha^{\Theta \vdash \cdot} &:= k_u & (\text{The constant } u \text{ natural transformation}) \\
\alpha^{\Theta \vdash \Phi, x : P} &:= m^{pe} \langle \llbracket \Theta \vdash \Phi \rrbracket, \llbracket \Theta \vdash P \rrbracket \rangle \circ \otimes_e \langle \alpha^{\Theta \vdash \Phi}, \alpha^{\Theta \vdash P} \rangle \quad (\text{see Theorem 5.3.8}).
\end{aligned}$$

PROOF. Straightforward verification. $\square$

6.3 Interpretation of LNL-FPC Terms

We introduce some notation for brevity which we use throughout the rest of the section. Given a type context Θ , we shall use $\vec{Z}$ to range over $\text{Ob}(\mathbf{CPO}_{pe}^{|\Theta|}) = \text{Ob}(\mathbf{CPO}^{|\Theta|})$. We write $\vec{FZ}$ for $F_{pe}^{\times|\Theta|} \vec{Z} = F^{\times|\Theta|} \vec{Z}$. We use Υ to range over both types and term contexts and we use Ψ to range over non-linear types and non-linear term contexts. Then, let

$$\llbracket \Upsilon \rrbracket_{\vec{Z}}^{\Theta} := \llbracket \Theta \vdash \Upsilon \rrbracket (\vec{FZ}) \in \text{Ob}(\mathbf{L}), \quad \llbracket \Psi \rrbracket_{\vec{Z}}^{\Theta} := \llbracket \Theta \vdash \Psi \rrbracket (\vec{Z}) \in \text{Ob}(\mathbf{CPO}).$$

With this notation in place, from the previous two subsections we have isomorphisms (in $\mathbf{L}$)

$$\text{unfold}_{\vec{FZ}}^{\Theta \vdash \mu X.A} : \llbracket \mu X.A \rrbracket_{\vec{Z}}^{\Theta} \cong \llbracket A[\mu X.A/X] \rrbracket_{\vec{Z}}^{\Theta} : \text{fold}_{\vec{FZ}}^{\Theta \vdash \mu X.A}, \quad \alpha_{\vec{Z}}^{\Theta \vdash \Psi} : \llbracket \Psi \rrbracket_{\vec{Z}}^{\Theta} \cong F(\llbracket \Psi \rrbracket_{\vec{Z}}^{\Theta}).$$

Remark 6.3.1. The isomorphisms above are natural in $\mathbf{CPO}_{pe}^{|\Theta|}$, which we saw is crucial for defining the type interpretations. However, just like in FPC, this naturality is irrelevant for the term interpretations – we only need the fact that each component is an isomorphism in $\mathbf{L}$. Thus, there is no danger in working within the categories $\mathbf{CPO}$ and $\mathbf{L}$, instead of $\mathbf{CPO}_{pe}$ and $\mathbf{L}_e$.

Next, we show how to interpret the substructural rules of Intuitionistic Linear Logic in a $\mathbf{CPO}$ -LNL model. The isomorphism α plays a fundamental part.

Definition 6.3.2. We define morphisms, called discarding ($\diamond$), copying (Δ) and promotion ($\square$):

$$\begin{aligned}
\diamond_{\vec{Z}}^{\Theta \vdash \Psi} &:= \llbracket \Psi \rrbracket_{\vec{Z}}^{\Theta} \xrightarrow{\alpha} F(\llbracket \Psi \rrbracket_{\vec{Z}}^{\Theta}) \xrightarrow{F1} F1 \xrightarrow{u^{-1}} I; \\
\Delta_{\vec{Z}}^{\Theta \vdash \Psi} &:= \llbracket \Psi \rrbracket_{\vec{Z}}^{\Theta} \xrightarrow{\alpha} F(\llbracket \Psi \rrbracket_{\vec{Z}}^{\Theta}) \xrightarrow{F\langle \text{id}, \text{id} \rangle} F(\llbracket \Psi \rrbracket_{\vec{Z}}^{\Theta} \times \llbracket \Psi \rrbracket_{\vec{Z}}^{\Theta}) \xrightarrow{m^{-1}} F(\llbracket \Psi \rrbracket_{\vec{Z}}^{\Theta} \otimes \llbracket \Psi \rrbracket_{\vec{Z}}^{\Theta}) \xrightarrow{\alpha^{-1} \otimes \alpha^{-1}} \llbracket \Psi \rrbracket_{\vec{Z}}^{\Theta} \otimes \llbracket \Psi \rrbracket_{\vec{Z}}^{\Theta}; \\
\square_{\vec{Z}}^{\Theta \vdash \Psi} &:= \llbracket \Psi \rrbracket_{\vec{Z}}^{\Theta} \xrightarrow{\alpha} F(\llbracket \Psi \rrbracket_{\vec{Z}}^{\Theta}) \xrightarrow{F\eta} !F(\llbracket \Psi \rrbracket_{\vec{Z}}^{\Theta}) \xrightarrow{! \alpha^{-1}} !\llbracket \Psi \rrbracket_{\vec{Z}}^{\Theta},
\end{aligned}$$

where, for brevity, we have written α instead of $\alpha_{\vec{Z}}^{\Theta \vdash \Psi}$.

Proposition 6.3.3. *The triple $(\llbracket \Psi \rrbracket_{\vec{Z}}^{\Theta}, \Delta_{\vec{Z}}^{\Theta \vdash \Psi}, \diamond_{\vec{Z}}^{\Theta \vdash \Psi})$ forms a cocommutative comonoid in $\mathbf{L}$.*

PROOF. By the axioms of symmetric monoidal adjunctions and the cartesian structure of $\mathbf{CPO}$. $\square$

$$\begin{aligned}
\llbracket \Phi, x : A \vdash x : A \rrbracket &:= \llbracket \Phi \rrbracket \otimes \llbracket A \rrbracket \xrightarrow{\circ \otimes \text{id}} I \otimes \llbracket A \rrbracket \xrightarrow{\cong} \llbracket A \rrbracket \\
\llbracket \Gamma \vdash \text{left}_{A,B} m : A + B \rrbracket &:= \llbracket \Gamma \rrbracket \xrightarrow{\llbracket m \rrbracket} \llbracket A \rrbracket \xrightarrow{\text{left}} \llbracket A \rrbracket + \llbracket B \rrbracket = \llbracket A + B \rrbracket \\
\llbracket \Gamma \vdash \text{right}_{A,B} m : A + B \rrbracket &:= \llbracket \Gamma \rrbracket \xrightarrow{\llbracket m \rrbracket} \llbracket B \rrbracket \xrightarrow{\text{right}} \llbracket A \rrbracket + \llbracket B \rrbracket = \llbracket A + B \rrbracket \\
\llbracket \Phi, \Gamma, \Sigma \vdash \text{case } m \text{ of } \{\text{left } x \rightarrow n \mid \text{right } y \rightarrow p\} : C \rrbracket &:= \llbracket \Phi \rrbracket \otimes \llbracket \Gamma \rrbracket \otimes \llbracket \Sigma \rrbracket \xrightarrow{\Delta \otimes \text{id}} \\
&\llbracket \Phi \rrbracket \otimes \llbracket \Phi \rrbracket \otimes \llbracket \Gamma \rrbracket \otimes \llbracket \Sigma \rrbracket \xrightarrow{\cong} \llbracket \Phi \rrbracket \otimes \llbracket \Sigma \rrbracket \otimes \llbracket \Phi \rrbracket \otimes \llbracket \Gamma \rrbracket \xrightarrow{\text{id} \otimes \llbracket m \rrbracket} \llbracket \Phi \rrbracket \otimes \llbracket \Sigma \rrbracket \otimes \llbracket A + B \rrbracket \xrightarrow{\cong} \\
&(\llbracket \Phi \rrbracket \otimes \llbracket \Sigma \rrbracket \otimes \llbracket A \rrbracket) + (\llbracket \Phi \rrbracket \otimes \llbracket \Sigma \rrbracket \otimes \llbracket B \rrbracket) \xrightarrow{[\llbracket n \rrbracket, \llbracket p \rrbracket]} \llbracket C \rrbracket \\
\llbracket \Phi, \Gamma, \Sigma \vdash \langle m, n \rangle : A \otimes B \rrbracket &:= \llbracket \Phi \rrbracket \otimes \llbracket \Gamma \rrbracket \otimes \llbracket \Sigma \rrbracket \xrightarrow{\Delta \otimes \text{id}} \llbracket \Phi \rrbracket \otimes \llbracket \Phi \rrbracket \otimes \llbracket \Gamma \rrbracket \otimes \llbracket \Sigma \rrbracket \xrightarrow{\cong} \\
&\llbracket \Phi \rrbracket \otimes \llbracket \Gamma \rrbracket \otimes \llbracket \Phi \rrbracket \otimes \llbracket \Sigma \rrbracket \xrightarrow{\llbracket m \rrbracket \otimes \llbracket n \rrbracket} \llbracket A \rrbracket \otimes \llbracket B \rrbracket = \llbracket A \otimes B \rrbracket \\
\llbracket \Phi, \Gamma, \Sigma \vdash \text{let } \langle x, y \rangle = m \text{ in } n : C \rrbracket &:= \llbracket \Phi \rrbracket \otimes \llbracket \Gamma \rrbracket \otimes \llbracket \Sigma \rrbracket \xrightarrow{\Delta \otimes \text{id}} \llbracket \Phi \rrbracket \otimes \llbracket \Phi \rrbracket \otimes \llbracket \Gamma \rrbracket \otimes \llbracket \Sigma \rrbracket \xrightarrow{\cong} \\
&\llbracket \Phi \rrbracket \otimes \llbracket \Gamma \rrbracket \otimes \llbracket \Phi \rrbracket \otimes \llbracket \Sigma \rrbracket \xrightarrow{\llbracket m \rrbracket \otimes \text{id}} \llbracket A \otimes B \rrbracket \otimes \llbracket \Phi \rrbracket \otimes \llbracket \Sigma \rrbracket \xrightarrow{\cong} \llbracket \Phi \rrbracket \otimes \llbracket \Sigma \rrbracket \otimes \llbracket A \rrbracket \otimes \llbracket B \rrbracket \xrightarrow{\llbracket n \rrbracket} \llbracket C \rrbracket \\
\llbracket \Gamma \vdash \lambda x^A. m : A \multimap B \rrbracket &:= \llbracket \Gamma \rrbracket \xrightarrow{\text{curry}(\llbracket m \rrbracket)} (\llbracket A \rrbracket \multimap \llbracket B \rrbracket) = \llbracket A \multimap B \rrbracket \\
\llbracket \Phi, \Gamma, \Sigma \vdash mn : B \rrbracket &:= \llbracket \Phi \rrbracket \otimes \llbracket \Gamma \rrbracket \otimes \llbracket \Sigma \rrbracket \xrightarrow{\Delta \otimes \text{id}} \llbracket \Phi \rrbracket \otimes \llbracket \Phi \rrbracket \otimes \llbracket \Gamma \rrbracket \otimes \llbracket \Sigma \rrbracket \xrightarrow{\cong} \\
&\llbracket \Phi \rrbracket \otimes \llbracket \Gamma \rrbracket \otimes \llbracket \Phi \rrbracket \otimes \llbracket \Sigma \rrbracket \xrightarrow{\llbracket m \rrbracket \otimes \llbracket n \rrbracket} (\llbracket A \rrbracket \multimap \llbracket B \rrbracket) \otimes \llbracket A \rrbracket \xrightarrow{\text{eval}} \llbracket B \rrbracket \\
\llbracket \Phi \vdash \text{lift } m : !A \rrbracket &:= \llbracket \Phi \rrbracket \xrightarrow{\square} !\llbracket \Phi \rrbracket \xrightarrow{!\llbracket m \rrbracket} !\llbracket A \rrbracket = \llbracket !A \rrbracket \\
\llbracket \Gamma \vdash \text{force } m : A \rrbracket &:= \llbracket \Gamma \rrbracket \xrightarrow{\llbracket m \rrbracket} !\llbracket A \rrbracket \xrightarrow{\epsilon} \llbracket A \rrbracket \\
\llbracket \Gamma \vdash \text{fold}_{\mu X.A} m : \mu X.A \rrbracket &:= \llbracket \Gamma \rrbracket \xrightarrow{\llbracket m \rrbracket} \llbracket A[\mu X.A/X] \rrbracket \xrightarrow{\text{fold}} \llbracket \mu X.A \rrbracket \\
\llbracket \Gamma \vdash \text{unfold } m : A[\mu X.A/X] \rrbracket &:= \llbracket \Gamma \rrbracket \xrightarrow{\llbracket m \rrbracket} \llbracket \mu X.A \rrbracket \xrightarrow{\text{unfold}} \llbracket A[\mu X.A/X] \rrbracket
\end{aligned}$$

where, for brevity, we write $\llbracket \Upsilon \rrbracket$ for $\llbracket \Upsilon \rrbracket_{\vec{Z}}^{\Theta}$ for all types or term contexts Υ and where we omit the subscripts and type superscripts of some morphisms (which can be easily inferred).

Fig. 8. Interpretation of LNL-FPC terms.

We can now define the interpretation of LNL-FPC terms. A term $\Theta; \Gamma \vdash m : A$ is interpreted as a family morphisms of $\mathbf{L}$ indexed by $\vec{Z}$. That is

$$\llbracket \Theta; \Gamma \vdash m : A \rrbracket := \left\{ \llbracket \Gamma \vdash m : A \rrbracket_{\vec{Z}}^{\Theta} : \llbracket \Gamma \rrbracket_{\vec{Z}}^{\Theta} \rightarrow \llbracket A \rrbracket_{\vec{Z}}^{\Theta} \mid \vec{Z} \in \text{Ob}(\mathbf{CPO}^{|\Theta|}) \right\},$$

where $\llbracket \Gamma \vdash m : A \rrbracket_{\vec{Z}}^{\Theta}$ is defined by induction on the derivation of $\Theta; \Gamma \vdash m : A$ in Figure 8 (notice the remark). Just like in FPC, the interpretation of terms is invariant with respect to both Θ and $\vec{Z}$, so omitting the subscript and superscript annotations (as we have done) should not lead to confusion, but on the other hand it improves readability. For brevity, we also informally write $\llbracket m \rrbracket$ instead of $\llbracket \Gamma \vdash m : A \rrbracket_{\vec{Z}}^{\Theta}$, which can be inferred from context.

The careful reader might have noticed that the above notation is not completely precise, because terms do not have unique derivations. Technically, the definition should be read as providing an interpretation for a derivation D of $\Theta; \Gamma \vdash m : A$. However, the next theorem justifies our notation.

Theorem 6.3.4. *Let D_1 and D_2 be two derivations of a judgement $\Theta; \Gamma \vdash m : A$. Then $\llbracket D_1 \rrbracket_{\vec{Z}}^{\Theta} = \llbracket D_2 \rrbracket_{\vec{Z}}^{\Theta}$.*

PROOF. As previously explained, two derivations may only differ in whether some non-linear variables are propagated up into both premises or just one. Variables which are propagated up but not used are always discarded. The proof follows by induction using Proposition 6.3.3. $\square$

In order to prove our semantics sound, we have to first formulate a substitution lemma for terms. This lemma, in turn, requires that the interpretation of *non-linear values* behave well with respect to the substructural morphisms of Definition 6.3.2. To show this, we reason as follows.

Definition 6.3.5. Given a non-linear term context $\Theta \vdash \Phi$ and a non-linear type $\Theta \vdash P$, a morphism $f : \llbracket \Phi \rrbracket_{\vec{Z}}^{\Theta} \rightarrow \llbracket P \rrbracket_{\vec{Z}}^{\Theta}$ in **L** is called *non-linear*, if there exists a morphism $f' : \langle \Phi \rangle_{\vec{Z}}^{\Theta} \rightarrow \langle P \rangle_{\vec{Z}}^{\Theta}$ in **CPO**,

$$\text{such that } f = \left(\llbracket \Phi \rrbracket_{\vec{Z}}^{\Theta} \xrightarrow{\alpha_{\vec{Z}}^{\Theta, \Phi}} F \langle \Phi \rangle_{\vec{Z}}^{\Theta} \xrightarrow{Ff'} F \langle P \rangle_{\vec{Z}}^{\Theta} \xrightarrow{(\alpha_{\vec{Z}}^{\Theta, P})^{-1}} \llbracket P \rrbracket_{\vec{Z}}^{\Theta} \right).$$

Proposition 6.3.6. *If $f : \llbracket \Phi \rrbracket_{\vec{Z}}^{\Theta} \rightarrow \llbracket P \rrbracket_{\vec{Z}}^{\Theta}$ is non-linear, then:*

$$\begin{aligned} \diamond_{\vec{Z}}^{\Theta \vdash P} \circ f &= \diamond_{\vec{Z}}^{\Theta \vdash \Phi}; \\ \Delta_{\vec{Z}}^{\Theta \vdash P} \circ f &= (f \otimes f) \circ \Delta_{\vec{Z}}^{\Theta \vdash \Phi}; \\ \square_{\vec{Z}}^{\Theta \vdash P} \circ f &= !f \circ \square_{\vec{Z}}^{\Theta \vdash \Phi}. \end{aligned}$$

In particular, non-linear morphisms are comonoid homomorphisms (with respect to Proposition 6.3.3).

Recall that by Lemma 2.2.3, non-linear values necessarily have a non-linear term context.

Proposition 6.3.7. *Let $\Theta; \Phi \vdash v : P$ be a non-linear value. Then $\llbracket \Phi \vdash v : P \rrbracket_{\vec{Z}}^{\Theta}$ is non-linear.*

PROOF. We define a non-linear interpretation

$$\langle \Phi \vdash v : P \rangle_{\vec{Z}}^{\Theta} : \langle \Phi \rangle_{\vec{Z}}^{\Theta} \rightarrow \langle P \rangle_{\vec{Z}}^{\Theta}$$

in **CPO** by induction on the derivation of $\Theta; \Phi \vdash v : P$ as follows:

$$\begin{aligned} \langle \Phi, x : P \vdash x : P \rangle &:= \langle \Phi \rangle \times \langle P \rangle \xrightarrow{\pi_2} \langle P \rangle \\ \langle \Phi \vdash \text{left}_{P,R} v : P + R \rangle &:= \langle \Phi \rangle \xrightarrow{\langle v \rangle} \langle P \rangle \xrightarrow{\text{inl}} \langle P \rangle \amalg \langle R \rangle = \langle P + R \rangle \\ \langle \Phi \vdash \text{right}_{P,R} v : P + R \rangle &:= \langle \Phi \rangle \xrightarrow{\langle v \rangle} \langle R \rangle \xrightarrow{\text{inr}} \langle P \rangle \amalg \langle R \rangle = \langle P + R \rangle \\ \langle \Phi \vdash \langle v, w \rangle : P \otimes R \rangle &:= \langle \Phi \rangle \xrightarrow{\langle \langle v \rangle, \langle w \rangle \rangle} \langle P \rangle \times \langle R \rangle = \langle P \otimes R \rangle \\ \langle \Phi \vdash \text{lift } m : !A \rangle &:= \langle \Phi \rangle \xrightarrow{\eta} GF \langle \Phi \rangle \xrightarrow{G\alpha^{-1}} G \llbracket \Phi \rrbracket \xrightarrow{G[m]} G \llbracket A \rrbracket = \langle !A \rangle \\ \langle \Phi \vdash \text{fold}_{\mu X.P} v : \mu X.P \rangle &:= \langle \Phi \rangle \xrightarrow{\langle v \rangle} \langle P[\mu X.P/X] \rangle \xrightarrow{\text{fold}} \langle \mu X.P \rangle, \end{aligned}$$

where π_2 is the second projection in **CPO**; inl and inr are the coproduct injections in **CPO**; $\langle f, g \rangle$ is the unique map induced by the product in **CPO** and fold is defined in Definition 6.1.6. The definition is invariant with respect to Θ and $\vec{Z}$, so again, we omit them from the subscript and superscript to improve readability.

To complete the proof, one has to show

$$\llbracket \Phi \vdash v : P \rrbracket_{\bar{Z}}^{\Theta} = \left(\llbracket \Phi \rrbracket_{\bar{Z}}^{\Theta} \xrightarrow{\alpha_{\bar{Z}}^{\Theta \vdash \Phi}} F(\llbracket \Phi \rrbracket_{\bar{Z}}^{\Theta}) \xrightarrow{F(\llbracket \Phi \vdash v : P \rrbracket_{\bar{Z}}^{\Theta})} F(\llbracket P \rrbracket_{\bar{Z}}^{\Theta}) \xrightarrow{(\alpha_{\bar{Z}}^{\Theta \vdash P})^{-1}} \llbracket P \rrbracket_{\bar{Z}}^{\Theta} \right),$$

which follows by simple verification using the available categorical data. The fold case is the most complicated (in general), but because of the preparatory work we have done, it may be established simply by using Theorem 6.1.7. $\square$

Next, as usual, we also have to formulate a substitution lemma for terms. Before we do so, recall Lemma 2.2.2.

Lemma 6.3.8 (Substitution). *Let $\Theta; \Phi, \Gamma, x : A \vdash m : B$ be a term and $\Theta; \Phi, \Sigma \vdash v : A$ a value with $\Gamma \cap \Sigma = \emptyset$. Then*

$$\llbracket m[v/x] \rrbracket = \left(\llbracket \Phi \rrbracket \otimes \llbracket \Gamma \rrbracket \otimes \llbracket \Sigma \rrbracket \xrightarrow{\cong \circ (\Delta \otimes id)} \llbracket \Phi \rrbracket \otimes \llbracket \Gamma \rrbracket \otimes \llbracket \Phi \rrbracket \otimes \llbracket \Sigma \rrbracket \xrightarrow{id \otimes \llbracket v \rrbracket} \llbracket \Phi \rrbracket \otimes \llbracket \Gamma \rrbracket \otimes \llbracket A \rrbracket \xrightarrow{\llbracket m \rrbracket} \llbracket B \rrbracket \right),$$

where, again, we have omitted the obvious subscript and superscript annotations, for brevity.

PROOF. First, one has to formulate and prove suitable permutation and contraction lemmas. Then, the current lemma is proved by induction using Lemma 2.2.2 and Propositions 6.3.6 - 6.3.7. $\square$

With this in place, we may now show our semantics is sound.

Theorem 6.3.9 (Soundness). *If $\Theta; \Gamma \vdash m : A$ and $m \Downarrow v$, then $\llbracket \Gamma \vdash m : A \rrbracket_{\bar{Z}}^{\Theta} = \llbracket \Gamma \vdash v : A \rrbracket_{\bar{Z}}^{\Theta}$.*

PROOF. Straightforward induction. $\square$

6.4 Notation for closed types

In the special case when $\Theta = \cdot$, the notation for terms and term contexts can be simplified, as we showed in §2.2. The same is also true for the denotational semantics and we will use this simplified notation for the proof of adequacy in the next section. We denote with $*$ the unique object of the terminal category $\mathbf{1}$.

Let $\llbracket \Upsilon \rrbracket := \llbracket \cdot \vdash \Upsilon \rrbracket(*) \in \text{Ob}(\mathbf{L})$. Let $\llbracket \Psi \rrbracket := \llbracket \cdot \vdash \Psi \rrbracket(*) \in \text{Ob}(\mathbf{CPO})$. For $\chi \in \{\alpha, \diamond, \Delta, \square\}$, we write $\chi^{\Psi} := \chi_*^{+ \Psi}$. Then, we have isomorphisms $\alpha^{\Psi} : \llbracket \Psi \rrbracket \cong F(\llbracket \Psi \rrbracket)$, and substructural morphisms $\diamond^{\Psi} : \llbracket \Psi \rrbracket \rightarrow I$, $\Delta^{\Psi} : \llbracket \Psi \rrbracket \rightarrow \llbracket \Psi \rrbracket \otimes \llbracket \Psi \rrbracket$ and $\square^{\Psi} : \llbracket \Psi \rrbracket \rightarrow !\llbracket \Psi \rrbracket$. Writing $\text{fold}^{\mu X.A}$ for $\text{fold}_*^{+ \mu X.A}$ and $\text{unfold}^{\mu X.A}$ for its inverse, we get an isomorphism $\text{unfold}^{\mu X.A} : \llbracket \mu X.A \rrbracket \cong \llbracket A[\mu X.A/X] \rrbracket : \text{fold}^{\mu X.A}$. For the interpretation of terms, we shall also write $\llbracket \Gamma \vdash m : A \rrbracket := \llbracket \Gamma \vdash m : A \rrbracket_* : \llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket$ which is a morphism in $\mathbf{L}$ and its definition can be simply read off from Figure 8, disregarding the remark at the bottom.

7 COMPUTATIONAL ADEQUACY

In this section we show that computational adequacy holds at non-linear types for a class of CPO-LNL models that satisfy some additional conditions (Theorem 7.0.10). In the process, we also provide sufficient conditions for program termination at *any* type (Theorem 7.0.9). We begin by specifying the additional properties of CPO-LNL models which we use in the proof of adequacy.

Definition 7.0.1. A *computationally adequate* CPO-LNL model is a CPO-LNL model where:

- (1) $\text{id}_I \neq \perp_{I,I}$ (or equivalently $I \not\cong 0$).
- (2) For any two pairs of parallel morphisms $f_1, g_1 : I \rightarrow A$ and $f_2, g_2 : I \rightarrow B$, such that $f_1 \neq \perp \neq g_1$ and $f_2 \neq \perp \neq g_2 : f_1 \otimes f_2 \leq g_1 \otimes g_2 \Rightarrow f_1 \leq g_1$ and $f_2 \leq g_2$.

For instance, the CPO-LNL model of Theorem 5.4.1 is computationally adequate.

Remark 7.0.2. Condition (1) is clearly necessary for adequacy. If $I \cong 0$, then $L \simeq 1$, and since L is completely degenerate, then clearly adequacy cannot hold. Condition (2) is a strong assumption which requires the tensor product of L to behave like the smash product of domain theory. This is a sufficient condition, but we do not know if it is necessary. However, it appears to be unavoidable when using the standard proof strategy based on formal approximation relations, like we do. In principle, there could be a different proof strategy which does not make this assumption, but we do not know of such a strategy which works in our setting.

Assumption 7.0.3. Throughout the rest of the section we consider an arbitrary, but fixed, computationally adequate CPO-LNL model.

We begin by showing that non-linear values correspond to non-zero morphisms.

Lemma 7.0.4. *Let $\cdot \vdash v : P$ be a non-linear value. Then $\llbracket v \rrbracket \neq \perp$.*

PROOF. Assume for contradiction $\llbracket v \rrbracket = \perp$. Therefore $\diamond^P \circ \llbracket v \rrbracket = \perp_{I,I}$. Since $\cdot \vdash v : P$ is a non-linear value, it may be discarded so that $\diamond^P \circ \llbracket v \rrbracket = \diamond' = \text{id}_I$ (Propositions 6.3.6-6.3.7). But then $\perp_{I,I} = \text{id}_I$, which contradicts Definition 7.0.1. $\square$

Notation 7.0.5. Given morphisms $f_1 : I \rightarrow A_1, \dots, f_n : I \rightarrow A_n$, we write $\langle\langle f_1, \dots, f_n \rangle\rangle$ for the morphism

$$\langle\langle f_1, \dots, f_n \rangle\rangle := \left(I \xrightarrow{\cong} I \otimes \dots \otimes I \xrightarrow{f_1 \otimes \dots \otimes f_n} A_1 \otimes \dots \otimes A_n \right).$$

For example, $\langle\langle f_1, f_2 \rangle\rangle = (f_1 \otimes f_2) \circ \lambda_I^{-1}$.

Let $\text{Values}(A) := \{v \mid v \text{ is a value and } \cdot \vdash v : A\}$ and $\text{Programs}(A) := \{p \mid \cdot \vdash p : A\}$ for any closed type $\cdot \vdash A$. We prove adequacy using the standard method based on *formal approximation relations*, a notion first devised in [Plotkin 1985].

Lemma 7.0.6. *For each closed type $\cdot \vdash A$, there exist formal approximation relations:*

$$\begin{aligned} \sqsubseteq_A &\subseteq (\mathbf{L}(I, \llbracket A \rrbracket) - \{\perp\}) \times \text{Values}(A) \\ \sqsubseteq_A &\subseteq \mathbf{L}(I, \llbracket A \rrbracket) \times \text{Programs}(A) \end{aligned}$$

with the properties:

- (A1.1) $f \sqsubseteq_{A+B} \text{left } v \text{ iff } \exists f'. f = \text{left} \circ f' \text{ and } f' \sqsubseteq_A v$;
- (A1.2) $f \sqsubseteq_{A+B} \text{right } v \text{ iff } \exists f'. f = \text{right} \circ f' \text{ and } f' \sqsubseteq_B v$;
- (A2) $f \sqsubseteq_{A \otimes B} \langle v, w \rangle \text{ iff } \exists f', f''. f = \langle f', f'' \rangle \text{ and } f' \sqsubseteq_A v \text{ and } f'' \sqsubseteq_B w$;
- (A3) $f \sqsubseteq_{A \rightarrow B} \lambda x^A. p \text{ iff } \forall f' \in \mathbf{L}(I, \llbracket A \rrbracket), \forall v \in \text{Values}(A). f' \sqsubseteq_A v \Rightarrow \text{eval} \circ \langle f, f' \rangle \sqsubseteq_B p[v/x]$;
- (A4) $f \sqsubseteq_{!A} \text{lft } p \text{ iff } \exists g \in \mathbf{CPO}(1, G[\llbracket A \rrbracket]). f = I \xrightarrow{u} F(1) \xrightarrow{F(g)} !\llbracket A \rrbracket \text{ and } \epsilon \circ f \sqsubseteq_A p$;
- (A5) $f \sqsubseteq_{\mu X. A} \text{fold } v \text{ iff } \text{unfold}^{\mu X. A} \circ f \sqsubseteq_{A[\mu X. A/X]} v$;
- (B) $f \sqsubseteq_A p \text{ iff } f \neq \perp \Rightarrow \exists v \in \text{Values}(A). p \Downarrow v \text{ and } f \sqsubseteq_A v$.

PROOF. Because of (A5), the relations are defined by recursion and proving their existence requires considerable effort. Our proof strategy is very similar to the one in [Fiore 1994, Chapter 9]. $\square$

Lemma 7.0.7. *If $f \sqsubseteq_P v$, where P is a non-linear type, then f is a non-linear morphism.*

PROOF. By induction on $\cdot \vdash v : P$, essentially the same as Proposition 6.3.7. $\square$

Proposition 7.0.8. *Let $\Gamma \vdash m : A$, where $\Gamma = x_1 : A_1, \dots, x_n : A_n$. Assume further we are given v_i and f_i , such that $f_i \trianglelefteq_{A_i} v_i$. Then $\llbracket m \rrbracket \circ \langle\langle f_1, \dots, f_n \rangle\rangle \sqsubseteq_A m[\vec{v} / \vec{x}]$.*

PROOF. By induction on $\Gamma \vdash m : A$, using Lemma 7.0.7 where necessary. $\square$

The next theorem establishes sufficient conditions for termination at any type.

Theorem 7.0.9 (Termination). *Let $\cdot \vdash p : A$ be a well-typed program. If $\llbracket p \rrbracket \neq \perp$, then $p \Downarrow$.*

PROOF. This is a special case of the previous proposition when $\Gamma = \cdot$. We get $\llbracket \cdot \vdash p : A \rrbracket \sqsubseteq_A p$, and thus $p \Downarrow$ by definition of $\sqsubseteq_A$. $\square$

Even though the above theorem is true at all types, the other direction for showing adequacy is true at non-linear types, but not at arbitrary types, in general. Consider, for instance, a non-terminating program $\cdot \vdash p : A$. Theorem 7.0.9 implies $\llbracket p \rrbracket = \perp$. But then $\llbracket \lambda x^I. p \rrbracket = \perp$ and clearly $\lambda x^I. p \Downarrow \lambda x^I. p$. Therefore, there can be no hope of proving computational adequacy at arbitrary linear types in any CPO-LNL model. For most linear/non-linear type systems, adequacy is proven only at non-linear types (see §8 for more discussion on this) and we do so as well.

Theorem 7.0.10 (Adequacy). *Let $\cdot \vdash p : P$ be a well-typed program, with P non-linear. Then*

$$p \Downarrow \text{ iff } \llbracket p \rrbracket \neq \perp.$$

PROOF. ($\Rightarrow$) By soundness and Lemma 7.0.4; ($\Leftarrow$) by Theorem 7.0.9. $\square$

8 RELATED WORK

LNL-FPC can be seen as an extension with recursive types of the circuit-free fragment of Proto-Quipper-M [Rios and Selinger 2017], which corresponds to the CLNL calculus in [Lindenhovius et al. 2018]. The present paper adds a syntactic, operational and denotational treatment of recursive types to these type systems.

Our work is also closely related to the LNL calculus [Benton 1995] (also known as the adjoint calculus [Benton and Wadler 1996]). The LNL type assignment system has two kinds of term judgements – a linear one and a non-linear one, which are interpreted in a monoidal category and a cartesian category, respectively. However, LNL-FPC differs syntactically, because it has only one kind of term judgement, which we believe results in a more convenient syntax for programming.

Eppendahl [Eppendahl 2003] extends LNL with recursive types only for the linear judgements. In that work, recursive types are treated linearly and are interpreted in the linear category only. In LNL-FPC, we also show how to add non-linear recursive types. We are able to treat them non-linearly, because we present new techniques for solving recursive domain equations within CPO (the cartesian category) which we use for their interpretation. Moreover, in LNL-FPC, both term-level recursion and type-level recursion are handled elegantly and agnostically with respect to the linearity of the type/term in question, which we think is a point in favor of our syntax. In contrast, term recursion in LNL and the adjoint calculus seems to be less elegant [Benton and Wadler 1996, pp. 11]. In an LNL-style syntax, it is also natural to add type recursion over the non-linear types, including non-linear function space, but it is an open problem how to interpret this in the cartesian category. In LNL-FPC, we do not expose any syntax for the internal-hom of the cartesian category and we have shown how to interpret the remaining non-linear recursive types (in both categories, see Figure 4). Of course, we may decompose the non-linear function space as $!(P \multimap R)$, which we can obviously interpret.

Some mixed linear/non-linear type systems, such as the adjoint calculus [Benton and Wadler 1996], also support additive conjunction. We did not include additive conjunction in our type system, because we do not think it is interesting from a programming perspective. However, it

looks like it could be included without causing problems – on the denotational side we merely have to assume that our linear category has CPO-products, which is true in all of our concrete models.

The Lily language [Bierman et al. 2000] is a polymorphic linear/non-linear lambda calculus where recursive types may be encoded using polymorphism. The non-linear types in Lily are those of the form $!A$, whereas in LNL-FPC the non-linear types are also closed under sums, pairs and formation of recursive types, which is a major focus of our paper and which we believe improves the usability of the language. A version of Lily where the function space $A \multimap B$ is *strict*, rather than linear, was considered in [Rosolini and Simpson 2004] and the language was equipped with a denotational model based on synthetic domain theory. The authors prove two computational adequacy results (one for call-by-name and one for call-by-value evaluation) which hold at all types of the form $!A$. Computational adequacy for Lily at types $!A$ was established in [Mögelberg et al. 2008], again using a model based on synthetic domain theory. These are the only adequacy results, that we know of, for languages with a similar feature set to LNL-FPC. Our adequacy result, however, covers a larger range of types (all non-linear types, rather than only those of the form $!A$) and our categorical model is considerably simpler and easier to understand. LNL-FPC does not support polymorphism, however.

LNL-FPC is also related to Barber’s Dual Intuitionistic Linear Logic (DILL) [Barber 1996], which, like LNL-FPC, also has one kind of term judgement, but it enforces a strict separation between linear and non-linear contexts. As a result, for a lambda abstraction $\lambda x^A. p$ in DILL, the variable x is always treated as a linear variable, whereas LNL-FPC allows it to be treated as non-linear, provided A is, resulting in increased convenience for programming.

Other type systems like λ^q , [Bernardy et al. 2017], $^a\lambda_{ms}$ [Tov and Pucella 2011] and Quill [Morris 2016] support linear or affine types where the substructural operations are implicit, whereas in LNL-FPC only the weakening and contraction rules are implicit. However, none of them have so far been equipped with a categorical semantics (which is the main contribution of this paper).

Fiore and Plotkin model recursive types within FPC using categories which are algebraically compact in an enriched sense [Fiore and Plotkin 1994; Fiore 1994]. Any category which is algebraically compact in such a sense and is also cartesian closed is necessarily degenerate. In a mixed linear/non-linear setting, we also have to solve recursive domain equations within the cartesian closed category, and because of this we cannot model recursive types using these techniques.

Another approach to modeling recursive types involves *bilimit compact categories* [Levy 2004] where type expressions are modelled as functors $T : \mathbf{A}^{\text{op}} \times \mathbf{B}^{\text{op}} \times \mathbf{A} \times \mathbf{B} \rightarrow \mathbf{B}$ for which one can solve recursive domain equations by using domain-theoretic methods [Pitts 1996]. We do not know if these techniques would also work in a mixed linear/non-linear setting like ours, but we think that our presentation is simpler and more compact, because we simply interpret type expressions as functors $T : \mathbf{A}_e \times \mathbf{B}_e \rightarrow \mathbf{B}_e$. In addition, our categorical models can be easily simplified for languages that have only inductive datatypes (recursive types allowing only $\otimes$ and $+$). In such a case, the category CPO from our models can be replaced with a cartesian closed category with finite coproducts and ω -colimits, CPO-enrichment from the model may be removed and our semantics can be adapted quite easily.

9 CONCLUSION AND FUTURE WORK

We introduced a new language called LNL-FPC which can be seen as extending FPC with a mixed linear/non-linear typing discipline (§2). An interesting feature of our type system is that it has implicit weakening and contraction rules – the users of the language do not have to explicitly specify when to copy and discard non-linear variables, while still enjoying linear type checking for the linear variables.

We presented a big-step call-by-value operational semantics for the language and showed that a previously known canonical fixpoint operator on terms [Lindenhovius et al. 2018] may be derived via the isorecursive type structure, thereby recreating an important result from FPC (§3).

In preparation for the denotational semantics of the language, we first had to establish some new coherence properties between (parameterised) initial algebras defined in potentially different categories and related by appropriate mediating functors (§4).

We then described our categorical model, which is a CPO-enriched model of Intuitionistic Linear Logic with suitable ω -colimits (§5). A crucial insight of our approach and of our model, is that non-linear types which correspond to mixed-variance functors can be seen as *covariant* functors both on the linear category of embeddings, but also on the non-linear category of *pre-embeddings*. We also presented an entire class of concrete models which have found applications not only in classical functional programming, but also in programming languages for certain emerging fields (quantum computation and diagrammatic languages).

We proved our denotational semantics sound (§6) and cemented our results by demonstrating a computational adequacy result (§7). In doing so, we provided two kinds of denotational interpretations for non-linear types, terms and contexts – one standard interpretation within the linear category and one within the cartesian one – which we showed are very strongly related to one another via a coherent natural isomorphism. This isomorphism, in turn, allowed us to elegantly interpret the substructural operations of Intuitionistic Linear Logic and in particular we precisely described the canonical comonoid structure of the non-linear (recursive) types. This, in turn, required developing new techniques for solving recursive domain equations within a cartesian category.

As part of future work we would like to investigate what additional categorical structure (if any) is required in order to design and model a type system similar to ours, but where the substructural rules for promotion and dereliction are also implicit. In practical terms, this would result in a programming language that is more convenient for use, while still retaining the benefits of linear type checking. In theoretical terms, the categorical treatment could lead to a better understanding of the underlying design principles.

Another line of future work is to consider a more abstract model of LNL-FPC where the CPO-enrichment of our model is axiomatised away. This is a challenging problem, but if resolved, it would allow us to replace the category CPO (currently an integral part of our model) by some abstract cartesian closed category $\mathbf{C}$ with suitable additional structure to solve the necessary recursive domain equations, not only in $\mathbf{L}$, but also in $\mathbf{C}$. In turn, this could help to discover additional models of LNL-FPC.

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