Compositional Construction of Bounded Error Over-Approximations of Acyclic Interconnected Continuous Dynamical Systems

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ABSTRACT

We consider the problem of bounded time safety verification of interconnections of input-output continuous dynamical systems. We present a compositional framework for computing bounded error approximations of the complete system from those of the components. The main crux of our approach consists of capturing the input-output *signal* behaviors of a component using an *abstraction* predicate that represents the input-output *sample* behaviors corresponding to the signal behaviors. We define a semantics for the abstraction predicate that captures an over-approximation of the input-output signal behaviors of a component. Next, we define how to compose abstraction predicates of components to obtain an abstraction predicate for the composed system. We instantiate our compositional abstraction construction framework for linear dynamical systems by providing concrete methods for constructing the input-output abstraction predicates for the individual systems.

CCS CONCEPTS

• General and reference → Verification; • Theory of computation → Logic and verification; Abstraction.

KEYWORDS

Continuous Systems, Approximation, Composition, Safety

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1 INTRODUCTION

Safety is an important property required of software that controls physical systems as in autonomous vehicles, process control systems and medical devices. Hybrid systems [1, 18] are a popular formalism to model the interactions between discrete (software) and continuous (physical) systems. Safety verification consists of

checking if all the executions of the system remain within a safety zone. This is typically accomplished by computing the reach set of the system that consists of all states reached by the system executions; and then checking that the reach set does not intersect with the unsafe zone. The challenging part of safety verification of hybrid systems lies in computing the reachable set of the continuous dynamics capturing the physical system behaviors. In this paper, we focus on safety verification of continuous dynamical systems.

The reach set computation problem consists of computing a representation of the reach set such that the problem of checking the emptiness of its intersection with the unsafe zone is decidable. In general, computing the exact reach set is challenging, and is feasible only for certain simple classes of dynamics such as constant rate and rectangular/polyhedral inclusion dynamics [17]. For more complex dynamical systems such as linear dynamical systems, the solutions evolve exponentially with time and hence, require exponential functions to represent the reachable set; checking the safety requires solving constraints involving exponential functions for which there are no known decision procedures [32]. Hence, the research over the past decade has focused on computing over-approximations of the reach set each of which provides a certain trade-off between computational complexity and precision of over-approximation. Broadly, the techniques can be categorized into two classes, namely, flowpipe computation techniques and abstraction techniques. Flowpipe computation [7, 10, 13, 14, 19, 25] consists of computing the solutions of the dynamical system at certain sample time points and using these values to compute an envelope for the reach set. Iteratively precise over-approximation can be obtained by reducing the sampling interval. On the other hand, abstraction techniques construct a simpler abstract system such that the reach set in the abstract system corresponds to an over-approximation of the actual reach set. These techniques, in general, do not provide bounds on the precision of over-approximation, however, they are often accompanied by refinement loops that aim to construct iteratively qualitatively better abstractions [2–4, 11, 12, 26, 27].

The main focus of this paper is to develop a compositional framework for the construction of over-approximations of the system behaviors that have bounded error with respect to the original behaviors. The main motivation for this work is the compositional design of systems, wherein we build systems by composing certain components. Assuming that each of the components comes with a correctness proof (in our case an input-output behavior guarantee), we need to be able to exploit these to provide a correctness proof of the composed system, instead of finding proof from scratch. In addition, if a component changes, then we could reuse the proofs

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for the rest of the components and recompute only the proof of the component that changed, and finally, compose them all. On the flip side, the compositional construction might have its one time overhead which might make it slightly more computationally expensive. In this paper, we focus on over-approximating the input-output behaviors rather than the reach set, since, even if the composed system is a closed system (system with no inputs), the individual components might have inputs and outputs through which they interact with other components. It is straightforward to obtain an over-approximation of the reach set of a closed system from an over-approximation of the input-output behavior.

There are several challenges to the design of a compositional over-approximation computation framework. Consider, for instance, a system that consists of two components $\dot{x}(t) = f(x(t), y(t))$ and $\dot{y}(t) = g(y(t), z(t))$ where the dot represent the derivative with respect to time. Note that the behaviors of this system are continuous trajectories $x, y, z : [0, T] \rightarrow \mathbb{R}$. The first challenge is an effective representation of the input-output continuous behaviors. We choose to capture the behaviors of a component $\mathcal{S}$ using a predicate ϕ whose free variables capture the values of the input and output trajectories at certain sample times. However, ϕ only captures discrete behaviors rather than continuous behaviors. Hence, we define a semantics for ϕ which captures a set of continuous behaviors. More precisely, given positive real numbers θ, ϵ and r , we define $\llbracket \phi \rrbracket^{\theta, \epsilon, r}$ to be a set of input-output trajectories (u, y) such that the rate of change of input trajectory u is bounded by θ , the sampled values of u and that of some signal y' which is within ϵ of y satisfy the formula ϕ when sampled at intervals of size r . The motivation for the semantics is as follows. Every input-output sequence $(\bar{u}, \bar{y})$ that satisfies ϕ is intended to capture a set of trajectories. There are uncountably many trajectories that have $\bar{u}$ as a sampled sequence; however, not all of the corresponding output trajectories will have $\bar{y}$ as the sampled trace. However, if we restrict the input trajectories to have bounded deviation, specified using rate θ , from a piecewise linear trajectory corresponding to $\bar{u}$, then the output trajectories will also be bounded by some ϵ with respect to the output trajectory of the piecewise continuous input signal. There are many choices to capture input-output deviations, this is one of the choices for which the compositional construction works.

Our main results are the following. First we define a predicate $\phi_{\mathcal{S}}^r$ (Definition 8) that captures a set of input-output sequences with sample interval r depending on $\mathcal{S}$ such that $\llbracket \phi \rrbracket^{\theta, \epsilon, r}$ is a bounded error over-approximation of $\mathcal{S}$. This is formally stated in Theorem 1 using certain functions that capture the input-output deviations of the system $\mathcal{S}$. Next, we provide a definition of composition of predicates $\phi_{\mathcal{S}_1}^r$ and $\phi_{\mathcal{S}_2}^r$ (Definition 12) such that the composed predicate is a bounded error over-approximation of the input-output behaviors of the composition of $\mathcal{S}_1$ and $\mathcal{S}_2$ (Theorem 2). Finally, we instantiate the framework for the class of linear dynamical systems, that is, we provide a specific construction for $\phi_{\mathcal{S}}^r$ which satisfies Definition 8. Also, we provide an algorithm that takes as input an error bound γ and computes a γ bounded input-output over-approximation of a composed system by inductively computing the same for its components. We have analyzed two case studies based on these results to demonstrate the feasibility of the approach.

Related work. Compositional modeling of hybrid systems has been explored in the realm of hybrid input-output automata [22]. Compositional analysis based on theorem proving has been studied in [24] for hybrid systems. Compositional analysis based on assume guarantee reasoning [9, 21] as well as methods for learning assumptions [5, 8, 23] have been explored. In addition, there is some recent work on assume guarantee reasoning for linear hybrid systems based on hybridization [6]. Also, simulation based compositional analysis [15, 16] of interconnected systems and compositional abstraction [28–31] of hybrid systems have been studied. However, in general, these methods do not construct bounded error approximations, which is the objective of the work in this paper. Although compositional bounded error approximation [20] has been studied for hybrid systems, it is not applicable for interconnected dynamical systems.

2 MOTIVATION

Complex systems consist of multiple components that interact with each other. Each component can be interpreted as a system that transforms certain input signals into output signals; we refer to these as input-output systems (IOSs). The components are interconnected through input/output variables as shown in Figure 1.

There are two input-output systems $\mathcal{S}_1$ and $\mathcal{S}_2$; $\mathcal{S}_1$ has 3 input variables u_1, u_2 and u_3 and 3 output variables y_1, y_2 and y_3 , and $\mathcal{S}_2$ has 4 input variables u_1, y_2, y_3 and u_4 and 2 output variables y_4 and y_5 . The interconnection is implicitly specified using common variable names. Hence, in this composition, the two outputs y_2 and y_3 of $\mathcal{S}_1$ are connected to inputs y_2 and y_3 of $\mathcal{S}_2$, respectively. The composed system has as inputs u_1, u_2, u_3 and u_4 , namely, the inputs of the first system, and the inputs of the second system which are not connected to any outputs of the first system. The outputs of the composed system are y_4, y_5 and y_1 , which correspond to the outputs of the second system and the outputs of the first system that are not connected to the inputs of the second system.

Figure 1: Composition of input-output systems

The composed system has two special subcases, namely, serial and parallel composition wherein serial, the set of output variables of the first system is the same as the input variables of the second system in serial, and in parallel, the input variables for the two systems are the same, however, the output variables of the first system and the input variables of the second system are disjoint. A widely prevalent class of composed systems consist of plant and control that are connected through a feedback loop. However, here we focus on systems that are built compositionally from smaller components, and assume that the interconnections do not have loops.

3 PRELIMINARIES

Numbers and Functions. Let $\mathbb{R}$ be the set of real numbers and $[n]$ be the set $\{1, 2, \dots, n\}$. We use $\text{dom}(x)$ for the domain of the

variable x . Given function $h : [0, T] \rightarrow A$, $r \geq 0$ such that $T = rn$, we define a set of points of the function h at ir , $i = 0, 1, \dots, n$ as a tuple $[h]_r = (h(0), h(r), h(2r), \dots, h(\frac{T}{r}))$. Given tuple $g = (g_0, g_1, \dots, g_n)$, where $g_i \in A$, $T \geq 0$ such that $T = rn$, we define a piece-wise constant function $\langle g \rangle_r : [0, T] \rightarrow A$ such that $\langle g \rangle_r(t) = g_i$, $ir \leq t < (i+1)r$, for $i = 0, 1, \dots, n-1$, $\langle g \rangle_r(T) = g_n$. For $x \in \mathbb{R}^n$, we use $\|x\| = \max\{|x_1|, \dots, |x_n|\}$.

Concatenation. Given two tuples $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_m)$, we use $x \frown y$ to denote the concatenation of x and y which is defined as follows: $x \frown y = (x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m)$. It can be extended for sets.

Projection. Given tuple $x = (x_1, x_2, \dots, x_n)$, and an index set $I \subseteq [n]$, we use $x|_I$ to denote the projection of the tuple x with respect to the index set I , for example, $x|_I = (x_{i_1}, x_{i_2}, \dots, x_{i_p})$, where $1 \leq i_1 < i_2 < \dots < i_p \leq n$ and $I = \{i_1, \dots, i_p\}$. It can be extended for sets.

4 SYSTEM DESCRIPTION

In this section, we provide the formal definition of the interconnected systems that we study. In this paper, we consider compositions that do not lead to feedback loops.

4.1 Input-output systems

We formally define an input-output system that models a continuous time system using a set of pairs of input-output trajectories.

DEFINITION 1. Given a set $\mathcal{A}$, a trajectory over $\mathcal{A}$ is a function $\tau : [0, T] \rightarrow \mathcal{A}$ for some $T \geq 0$. We represent $\text{Traj}(\mathcal{A})$ to denote the set of trajectories over the set $\mathcal{A}$.

DEFINITION 2. Given a set $\mathcal{A}$ and $r \geq 0$, a piece-wise constant trajectory with sample interval r is a function $\tau : [0, T] \rightarrow \mathcal{A}$, for some $T \geq 0$ such that $T = rn$ for some natural number n and for every $i \in [n]$, there exists c_i such that $\tau(t) = c_i$ for all $(i-1)r \leq t < ir$. Let $\text{Traj}_r(\mathcal{A})$ represent the set of all piece-wise constant trajectories over the set $\mathcal{A}$ with sample interval r .

DEFINITION 3. An input-output system (IOS) is a tuple $S = (I, O, \mathcal{U}, \mathcal{Y}, \mathcal{T})$, where

- (a) $I = (u_1, u_2, \dots, u_m)$ is a tuple of input variables;
- (b) $O = (y_1, y_2, \dots, y_n)$ is a tuple of output variables;
- (c) $\mathcal{U} = U_1 \times U_2 \times \dots \times U_m$ is the input domain;
- (d) $\mathcal{Y} = Y_1 \times Y_2 \times \dots \times Y_n$ is the output domain;
- (e) $\mathcal{T} \subseteq \text{Traj}(\mathcal{U}) \times \text{Traj}(\mathcal{Y})$ is a set of input-output trajectories such that for all $(u, y) \in \mathcal{T}$, $\text{dom}(u) = \text{dom}(y)$.

In the above definitions U_i corresponds to the input domain of the variable u_i , and Y_i to that of y_i . We will often refer to an input-output system as just system.

Next, we illustrate how to model a linear system as an input-output system. Consider a linear system,

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y = Cx(t) \quad (1)$$

where $x(t) \in X \subseteq \mathbb{R}^n$, $u(t) \in U \subseteq \mathbb{R}^m$, and $y(t) \in Y \subseteq \mathbb{R}^p$, where X, U and Y are compact sets; $X_0 \subseteq X$ is an initial convex compact polyhedral set. Here $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $C \in \mathbb{R}^{p \times n}$ are state, input, and output matrices, respectively. The input-output system corresponding to the System (1) up to a time bound T is given by $S = (I, O, \mathcal{U}, \mathcal{Y}, \mathcal{T})$, where

- (a) $I = (u_1, u_2, \dots, u_m)$; $O = (y_1, y_2, \dots, y_p)$;
- (b) $\mathcal{U} = U$; $\mathcal{Y} = Y$;
- (c) $\mathcal{T} = \{(u, y) \in \text{Traj}(\mathcal{U}) \times \text{Traj}(\mathcal{Y}) \mid \text{dom}(u) = \text{dom}(y) \text{ and } \exists x_0 \in X_0, y(t) = Cx(t), \text{ where } x(t) = e^{At}x_0 + \int_0^t e^{A(t-t')}Bu(t')dt'\}$.

ASSUMPTION 1. We assume that for every input trajectory $u \in \text{Traj}(\mathcal{U})$, there is an output trajectory $y \in \text{Traj}(\mathcal{Y})$ such that $(u, y) \in \mathcal{T}$. It is not restrictive because most continuous dynamical systems have this property.

4.2 Composition of input-output systems

In this section, we define the composition of input-output systems. We use $S_1 \triangleright S_2$ to denote the composition of IOSs S_1 and S_2 . First, we define the conditions under which two input-output systems can be composed; this will eliminate the possibility of feedback loops in the system.

DEFINITION 4. An input-output system $S_1 = (I_1, O_1, \mathcal{U}_1, \mathcal{Y}_1, \mathcal{T}_1)$ can be composed with another input-output system $S_2 = (I_2, O_2, \mathcal{U}_2, \mathcal{Y}_2, \mathcal{T}_2)$ if following conditions hold:

- (a) $\llbracket I_1 \rrbracket \cap \llbracket O_2 \rrbracket = \emptyset$; (b) $\llbracket O_1 \rrbracket \cap \llbracket O_2 \rrbracket = \emptyset$.

Next, we construct the composition of two input-output systems.

DEFINITION 5. Given two input-output systems $S_i = (I_i, O_i, \mathcal{U}_i, \mathcal{Y}_i, \mathcal{T}_i)$, $i = 1, 2$, which can be composed, the composed system is $S_1 \triangleright S_2 = (I, O, \mathcal{U}, \mathcal{Y}, \mathcal{T})$, where

- (a) $I = I_1 \frown I_2 \downarrow_{\text{diff}(I_2, O_1)}$; $O = O_1 \downarrow_{\text{diff}(O_1, I_2)} \frown O_2$;
- (b) $\mathcal{U} = \mathcal{U}_1 \frown \mathcal{U}_2 \downarrow_{\text{diff}(I_2, O_1)}$; $\mathcal{Y} = \mathcal{Y}_1 \downarrow_{\text{diff}(O_1, I_2)} \frown \mathcal{Y}_2$;
- (c) $\mathcal{T} = \{(u, y) \mid \exists (u_1, y_1) \in \mathcal{T}_1, (u_2, y_2) \in \mathcal{T}_2, u = u_1 \frown u_2 \downarrow_{\text{diff}(I_2, O_1)}, y = y_1 \downarrow_{\text{diff}(O_1, I_2)} \frown y_2\}$.

REMARK 1. The composition $S_1 \triangleright S_2$ of two input-output systems S_1 and S_2 is a serial composition if $I_2 = O_1$ and $\mathcal{U}_2 = \mathcal{Y}_1$, and it is a parallel composition if $I_1 = I_2$, $\mathcal{U}_1 = \mathcal{U}_2$, $\llbracket I_2 \rrbracket \cap \llbracket O_1 \rrbracket = \emptyset$ and $\llbracket I_1 \rrbracket \cap \llbracket O_2 \rrbracket = \emptyset$.

4.3 Bounded Error Approximation

Our objective is to compute bounded error over-approximations for input-output behaviors of systems. This computation captures more input-output behaviors than the input-output behaviors of the systems, which are defined by the notion of γ perturbation of the behaviors. We will restrict ourselves to input trajectories whose rate is bounded by some assigned value θ .

DEFINITION 6. Given system S , $\theta \geq 0$, and $\gamma \geq 0$, we define $\llbracket S \rrbracket^{(\theta, \gamma)}$ to be the set of all input-output signals (u, y) such that $\|\dot{u}\| \leq \theta$ and y is within γ of y' , where $(u, y') \in \mathcal{T}$. In other words,

$$\llbracket S \rrbracket^{(\theta, \gamma)} = \{(u, y) \mid \|\dot{u}\| \leq \theta, \exists (u, y') \in \mathcal{T}, \|y - y'\| \leq \gamma\}.$$

When $\gamma = 0$, we use $\llbracket S \rrbracket^\theta$ instead of $\llbracket S \rrbracket^{(\theta, 0)}$.

Hence, a γ -bounded over-approximation is any set of input-output behaviors that contains $\llbracket S \rrbracket^\theta$ but is contained in $\llbracket S \rrbracket^{(\theta, \gamma)}$.

5 COMPOSITIONAL FRAMEWORK FOR ABSTRACTION CONSTRUCTION

In this section, we present our compositional framework to construct an abstraction of system executions over a bounded time by

using the abstractions of the components. The crux of our construction relies on an abstraction predicate that captures the sampled input-output behaviors of a system. In Section 5.1, we formalize the notion of an abstraction predicate, and show how it can be used to capture bounded error over-approximations of input-output behaviors of component systems. Next, in Section 5.2, we present how to compose abstraction predicates of components to obtain abstraction predicates for the composed system.

5.1 Abstraction of Components

The broad goal of this section is to define a bounded error over-approximation of input-output behaviors of a system $\mathcal{S}$, where the rate of change of input is bounded by some value θ . In other words, given an error bound γ , we intend to compute a set of input-output behaviors $\mathcal{S}'$ that contains $\llbracket \mathcal{S} \rrbracket^\theta$, but is contained in $\llbracket \mathcal{S} \rrbracket^{(\theta, \gamma)}$. Our broad approach towards defining $\mathcal{S}'$ is to define a predicate $\phi_{\mathcal{S}}^r$ that captures the input-output sample behaviors of $\mathcal{S}$, and define a semantics $\llbracket \phi_{\mathcal{S}}^r \rrbracket^{(\theta, \epsilon, r)}$ that provides the required $\mathcal{S}'$. In particular, $\llbracket \phi_{\mathcal{S}}^r \rrbracket^{(\theta, \epsilon, r)}$ is a γ bounded over-approximation of $\mathcal{S}$ only for certain values of ϵ and r for a fixed value of θ and γ . In order to obtain the values of ϵ and r , we use certain properties of the dynamics $\mathcal{S}$ which are captured using auxiliary functions $C_{\mathcal{S}}$ and $\mathcal{B}_{\mathcal{S}}$. Hence, we formally define these notions below.

We use formula $\varphi(\bar{\mathbf{u}}, \bar{\mathbf{y}})$ to denote a first-order logic predicate with free variables $\bar{\mathbf{u}}$ and $\bar{\mathbf{y}}$, each of which represents a sequence of free variables. We will assume that the sequences are of the same length. Our results in this section are not affected by the specific vocabulary of the predicates, however, it will become important when we want a decision procedure for safety verification. We assume that there is a standard semantics for the predicate $\varphi(\bar{\mathbf{u}}, \bar{\mathbf{y}})$ which consists of pair of sequences of real values for the variables $\bar{\mathbf{u}}$ and $\bar{\mathbf{y}}$ for which the predicate is evaluated as true. We use $\llbracket \varphi \rrbracket_{pw}$ to denote the standard semantics. Next, we associate another semantics with $\varphi(\bar{\mathbf{u}}, \bar{\mathbf{y}})$, which captures a set of input-output signals corresponding to the input-output sequences captured by the predicate in the standard semantics. We will use bold letters such as $\bar{\mathbf{u}}$ and $\bar{\mathbf{y}}$ to denote variable names, and regular letters such as $\bar{u}$ and $\bar{y}$ to denote valuations for the variables.

DEFINITION 7. Given $\theta \geq 0, r > 0, \epsilon > 0$, and predicate $\varphi(\bar{\mathbf{u}}, \bar{\mathbf{y}})$, $\llbracket \varphi(\bar{\mathbf{u}}, \bar{\mathbf{y}}) \rrbracket^{(\theta, \epsilon, r)}$ be the set of all input-output signals (u, y) such that $\|\bar{u}\| \leq \theta$ and there exists $(\bar{u}, \bar{y}) \in \llbracket \varphi(\bar{\mathbf{u}}, \bar{\mathbf{y}}) \rrbracket_{pw}$ which satisfy the following conditions: $[u]_r = \bar{u}$, $\|y - \langle \bar{y} \rangle_r\| \leq \epsilon$. In other words, $\llbracket \varphi(\bar{\mathbf{u}}, \bar{\mathbf{y}}) \rrbracket^{(\theta, \epsilon, r)} = \{(u, y) \mid \exists (\bar{u}, \bar{y}) \in \llbracket \varphi(\bar{\mathbf{u}}, \bar{\mathbf{y}}) \rrbracket_{pw}, \|\bar{u}\| \leq \theta, [u]_r = \bar{u}, \|y - \langle \bar{y} \rangle_r\| \leq \epsilon\}$.

Next, we define an abstraction predicate that captures all input-output sample points of the input-output trajectories corresponding to a system $\mathcal{S}$ with sample time r .

DEFINITION 8. Given a system $\mathcal{S} = (I, O, \mathcal{U}, \mathcal{Y}, \mathcal{T})$ and $r > 0$, a predicate $\phi_{\mathcal{S}}^r(\bar{\mathbf{u}}, \bar{\mathbf{y}})$ is an abstraction predicate if it captures all sequences $(\bar{u}, \bar{y})$ such that there exists an output signal y satisfying $(\langle \bar{u} \rangle_r, y) \in \mathcal{T}$ and $[y]_r = \bar{y}$. In other words,

$$\llbracket \phi_{\mathcal{S}}^r \rrbracket_{pw} = \{(\bar{u}, \bar{y}) \mid \exists y \text{ such that } (\langle \bar{u} \rangle_r, y) \in \mathcal{T}, [y]_r = \bar{y}\}.$$

Note that Definition 8 does not provide a specific predicate, but specifies the set defined by the predicate in the standard semantics.

Our main objective is to find ϵ and r such that $\llbracket \phi_{\mathcal{S}}^r \rrbracket^{(\theta, \epsilon, r)}$ is a γ bounded over-approximation of $\mathcal{S}$ with respect to input signals whose rate of change is bounded by θ . We define certain auxiliary functions for $\mathcal{S}$, which will help in establishing the values of ϵ and r , given θ and γ . First, we define an input-output value function that captures the maximum distance between two values of an output trajectory. These values are points of the output trajectory at two time points within a sample interval corresponds to a constant input signal.

DEFINITION 9. Given a system $\mathcal{S}$, we call $C_{\mathcal{S}} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$, a constant input-output value function if for any $r \geq 0$, for all $t \geq 0$, and for all $t_1, t_2 \in [t, t + T]$ with $\|t_1 - t_2\| \leq r$, for all $u \in \text{Traj}(\mathcal{U})$ such that $\forall t \in [t_1, t_2], u(t) = u(t_1)$, and for all y such that $(u, y) \in \mathcal{T}$, we have $\|y(t_1) - y(t_2)\| \leq C_{\mathcal{S}}(r)$.

Next, we want to capture the output behaviors when input signals are not constant. We define bounded input-output value function that expresses the maximum variation between two output trajectories when their corresponding input signals are within a certain value and the rate of one of the input signals is bounded by some assigned value.

DEFINITION 10. Given a system $\mathcal{S}$, we name $\mathcal{B}_{\mathcal{S}} : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$, a bounded input-output value function if for any $\theta \geq 0, \delta \geq 0$, for any two trajectories $(u, y), (u', y') \in \mathcal{T}$ such that $\|\bar{u}\| \leq \theta$ and $\|u - u'\| \leq \delta$, we have $\|y - y'\| \leq \mathcal{B}_{\mathcal{S}}(\theta, \delta)$, where $y(0) = y'(0)$.

Next, we state our main result, that is, given an error bound γ , we can compute an ϵ and an r , such that $\llbracket \phi_{\mathcal{S}}^r \rrbracket^{(\theta, \epsilon, r)}$ is a γ bounded over-approximation of $\mathcal{S}$.

THEOREM 1. Given system $\mathcal{S}$, $\gamma > 0$ and $\theta > 0$, constant input-output value function $C_{\mathcal{S}}$, bounded input-output value function $\mathcal{B}_{\mathcal{S}}$, let $r > 0, \epsilon > 0$ be such that $\epsilon = C_{\mathcal{S}}(r) + \mathcal{B}_{\mathcal{S}}(\theta, \theta r)$ and $2\epsilon \leq \gamma$. Then we have $\llbracket \mathcal{S} \rrbracket^\theta \subseteq \llbracket \phi_{\mathcal{S}}^r \rrbracket^{(\theta, \epsilon, r)} \subseteq \llbracket \mathcal{S} \rrbracket^{(\theta, \gamma)}$.

Next, we define γ -bounded over-approximation for the composition of input-output systems.

5.2 Compositional Abstraction

The broad goal of this section is to define the abstraction predicate for a composed system in terms of its components. Recall that we constructed the abstraction predicate and the corresponding parameters ϵ and r in the semantics depending on the error bound γ and rate of change of input θ . Note that when given a composed system $\mathcal{S}_1 \triangleright \mathcal{S}_2$ and a bound on the rate of change of input θ , the rate of change of input to $\mathcal{S}_1$ is still bounded by θ . However, the input to $\mathcal{S}_2$ might consist of some outputs of $\mathcal{S}_1$ which do not obey the bound θ . Hence, we need to compute an abstraction predicate of $\mathcal{S}_2$ with respect to a different bound on the rate of change of input. The bound is computed using a function called the output rate value function that expresses an upper bound on the rate of change of output for a given bound on the rate of change of input.

DEFINITION 11. Given a system $\mathcal{S}$, and a value $\theta \geq 0$, we call $\mathcal{D}_{\mathcal{S}} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$, an output rate value function, if for any trajectory $(u, y) \in \mathcal{T}$ such that $\|\bar{u}\| \leq \theta$, we have $\|\bar{y}\| \leq \mathcal{D}_{\mathcal{S}}(\theta)$.

The next proposition capture the relation between input-output trajectories of the composed system and that of its components using the notion of output rate value function.

PROPOSITION 1. *Given systems $\mathcal{S}_1$ and $\mathcal{S}_2$, rate $\theta \geq 0$, and output rate value function $\mathcal{D}_{\mathcal{S}_1}$, let $\mathcal{S} = \mathcal{S}_1 \triangleright \mathcal{S}_2$. Then $\llbracket \mathcal{S} \rrbracket^\theta$ is the set of composition of trajectories $(u, w) \in \mathcal{T}_1$ and $(w, y) \in \mathcal{T}_2$ such that $(u, w) \in \llbracket \mathcal{S}_1 \rrbracket^\theta$ and $(w, y) \in \llbracket \mathcal{S}_2 \rrbracket^{\mathcal{D}_{\mathcal{S}_1}(\theta)}$. Alternatively,*

$$\llbracket \mathcal{S} \rrbracket^\theta = \{(u, y) \mid \exists w, (u, w) \in \llbracket \mathcal{S}_1 \rrbracket^\theta, (w, y) \in \llbracket \mathcal{S}_2 \rrbracket^{\mathcal{D}_{\mathcal{S}_1}(\theta)}\}.$$

Next, we define the main component of our compositional abstraction framework, namely, the construction of the abstraction of the composed system from that of the components.

DEFINITION 12. *Given $\epsilon \geq 0$, predicates φ_1 and φ_2 , we define $\varphi_1 \circ_\epsilon \varphi_2$ to be the predicate $\varphi_1 \circ_\epsilon \varphi_2(\bar{u}, \bar{y})$ which is given by:*

$$\exists \bar{w}, \bar{w}', \varphi_1(\bar{u}, \bar{w}) \wedge \varphi_2(\bar{w}', \bar{y}) \wedge \|\bar{w} - \bar{w}'\| \leq \epsilon.$$

Next, we show that if we have abstraction predicates φ_1 and φ_2 for systems $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively, which satisfy Theorem 1, then we can construct an abstraction predicate φ as in Definition 12 that captures the behaviors of the composed system.

THEOREM 2. *Consider systems $\mathcal{S}_1$ and $\mathcal{S}_2$, predicates φ_1 and φ_2 , and values $\theta, \epsilon_1, \epsilon_2, \gamma_1$ and γ_2 such that $\llbracket \mathcal{S}_1 \rrbracket^\theta \subseteq \llbracket \varphi_1 \rrbracket^{(\theta, \epsilon_1, r)} \subseteq \llbracket \mathcal{S}_1 \rrbracket^{(\theta, \gamma_1)}$ and $\llbracket \mathcal{S}_2 \rrbracket^{\mathcal{D}_{\mathcal{S}_1}(\theta)} \subseteq \llbracket \varphi_2 \rrbracket^{(\mathcal{D}_{\mathcal{S}_1}(\theta), \epsilon_2, r)} \subseteq \llbracket \mathcal{S}_2 \rrbracket^{(\mathcal{D}_{\mathcal{S}_1}(\theta), \gamma_2)}$. Then, we have*

$$\llbracket \mathcal{S}_1 \triangleright \mathcal{S}_2 \rrbracket^\theta \subseteq \llbracket \varphi \rrbracket^{(\theta, \epsilon_2, r)} \subseteq \llbracket \mathcal{S}_1 \triangleright \mathcal{S}_2 \rrbracket^{(\theta, \gamma)}, \text{ where}$$

- (a) $\varphi = \varphi_1 \circ_{\epsilon_1} \varphi_2$;
- (b) $\gamma \geq 3\gamma_2 + \mathcal{B}_{\mathcal{S}_2}(\mathcal{D}_{\mathcal{S}_1}(\theta), \gamma_1 + \mathcal{D}_{\mathcal{S}_1}(\theta)r) + \mathcal{B}_{\mathcal{S}_2}(\mathcal{D}_{\mathcal{S}_1}(\theta), \gamma_1)$.

Finally, given an unsafe region U_R , we construct a predicate Ψ_{unsafe} to express that the system is approximately unsafe as follows: $\exists (\bar{u}, \bar{w}) \in \llbracket \varphi_{\mathcal{S}_1}^{r_1} \circ_{\epsilon_1} \varphi_{\mathcal{S}_2}^{r_2} \rrbracket_{pw}, s \in U_R$, such that $\bigvee_{i=1}^n \|s - \bar{w}_i\| \leq \epsilon_2$. Unsatisfiability of the predicate Ψ_{unsafe} guarantees the safety of the composed system.

6 CONCLUSIONS

In this paper, we presented a compositional framework to compute bounded error approximations of composed systems, and instantiated that framework for the case of linear dynamical systems. Constructing compositional abstractions has computational overheads, however, compositionality provides additional benefits towards construction of abstraction as well as analysis. Firstly, if a particular component changes, we need to only recompute the abstraction for the component. Also, by treating the input-output abstractions as assumptions/guarantees for the components, we might be able to do away with re-analysis by just checking if the new component satisfies these assumptions and guarantees. In the future, we intend to extend the framework for non-linear and hybrid systems.

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