On Energy Conservation in Data Centers

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We formulate and study an optimization problem that arises in the energy management of data centers and, more generally, multiprocessor environments. Data centers host a large number of heterogeneous servers. Each server has an active state and several standby/sleep states with individual power consumption rates. The demand for computing capacity varies over time. Idle servers may be transitioned to low-power modes so as to rightsize the pool of active servers. The goal is to find a state transition schedule for the servers that minimizes the total energy consumed. On a small scale, the same problem arises in multicore architectures with heterogeneous processors on a chip. One has to determine active and idle periods for the cores so as to guarantee a certain service and minimize the consumed energy.

For this power/capacity management problem, we develop two main results. We use the terminology of the data center setting. First, we investigate the scenario that each server has two states: an active state and a sleep state. We show that an optimal solution, minimizing energy consumption, can be computed in polynomial time by a combinatorial algorithm. The algorithm resorts to a single-commodity minimum-cost flow computation. Second, we study the general scenario that each server has an active state and multiple standby/sleep states. We devise a τ -approximation algorithm that relies on a two-commodity minimum-cost flow computation. Here, τ is the number of different server types. A data center has a large collection of machines but only a relatively small number of different server architectures. Moreover, in the optimization, one can assign servers with comparable energy consumption to the same class. Technically, both of our algorithms involve nontrivial flow modification procedures. In particular, given a fractional two-commodity flow, our algorithm executes advanced rounding and flow packing routines.

CCS Concepts: • **Theory of computation** → **Design and analysis of algorithms; Approximation algorithms analysis; Scheduling algorithms; Discrete optimization**; • **Hardware** → *Power estimation and optimization; Enterprise level and data centers power issues*;

Additional Key Words and Phrases: Heterogeneous machines, efficient algorithms, approximation algorithms, minimum-cost flow

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1 INTRODUCTION

We define and investigate an optimization problem with the objective of energy conservation in multiprocessor environments. We focus on two particularly timely settings.

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Data centers. Energy management is a key issue in data center operations [8]. Electricity costs are a dominant and rapidly growing expense in such centers; about 30% to 50% of their budget is invested into energy. Data centers use about 1.5% of the total electricity worldwide [14]. This corresponds to the energy consumption of more than 90 million households [10]. Surprisingly, the servers of a data center are only utilized 20% to 40% of the time on average [2, 5]. When idle and in active mode, they consume about half of their peak power. Hence, a fruitful approach for energy conservation and capacity management is to transition idle servers into standby and sleep states. Servers have a number of low-power states [7]. However, state transitions, particularly power-up operations, incur energy consumption (cost). Therefore, dynamically matching the varying demand for computing capacity with the number of active servers is a challenging problem.

Multicore architectures. Multicore processors are architectures with multiple and often heterogeneous processing units on a single die. Originally, heterogeneous platforms contained several processor types, i.e., CPUs and GPUs. Modern platforms are also equipped with identical CPUs that have different microarchitectures leading to various levels of energy consumption [6]. To exploit such platforms, effective power management strategies are needed. The optimization problem is identical to that described in the last paragraph, except we have a small number of processing units here. We remark that in such environments, the processors might have several active states as well.

In Section 2, we formally define an optimization problem, Dynamic Power Management (DPM), that captures the preceding scenarios. In short, there are m heterogeneous servers (processors). Each server has several states with associated power consumption rates. State transitions consume energy. The planning horizon contains times $t_1 < t_2 < \dots < t_n$ at which the demand changes. During interval $[t_k, t_{k+1})$, at least d_k servers must be active and available for utilization, $1 \leq k \leq n - 1$. The goal is to find a state transition schedule for the servers minimizing the total energy consumption.

1.1 Previous Work

Augustine et al. [3] and Irani et al. [15] study power-down strategies for a single device that is equipped with an active state and several low-power states. The goal is to minimize the energy consumed in an idle period. Our problem DPM is a generalization with multiple, parallel devices and time-dependent demand. The two articles [3, 15] develop online algorithms that achieve optimal competitive ratios. DPM for a single device with two states is equivalent to the ski-rental problem, a famous rent-or-buy problem [16, 17, 19, 22]. No generalization with several required resources has been examined. Azar et al. [4] study a capital investment problem where machines for manufacturing a product may be purchased over time. The machines differ in the capital and production costs.

Khuller et al. [18] and Li and Khuller [20] introduce machine activation problems that are also motivated by energy conservation in data centers. In Khuller et al. [18], the authors assume that there is an activation cost budget, and jobs have to be scheduled on the selected, activated machines so as to minimize the makespan. They present algorithms that simultaneously approximate the budget and the makespan. The second work [20] considers a generalization where the activation cost of a machine is a nondecreasing function of the load. Demaine et al. [9] investigate the problem of scheduling unit-size jobs on a set of identical parallel processors having one sleep state. Each job can be executed in any of a set of given time intervals. For the objective of minimizing power consumption, they develop an approximation algorithm whose performance factor depends on the energy needed to transition between the active state and the sleep state.

In the more applied computer science literature, power management strategies and the value of sleep states have been studied extensively. The works mostly focus on experimental evaluations.

Analytic results are presented in other articles as well [11–13, 21, 23]. Ghandi et al. [12] model a server farm with setup costs as an $M/M/m$ queuing system. Lin et al. [21] study a dynamic rightsizing of data centers with homogeneous servers having one sleep state. The operating cost of a server is a convex function of the workload.

1.2 Our Contribution

We present an algorithmic study of an important capacity management problem in data centers. Our problem DPM dynamically rightsizes the pool of servers with the objective to minimize the energy consumed. Compared to previous work, the new, essential aspects are that we consider (a) a time horizon with varying demand for computing capacity and (b) power-heterogeneous servers. In fact, with homogeneous servers, the problem is easy to solve. In DPM, the demand for computing capacity is specified by the number of servers needed at any time. In data centers, it is common practice that a number of required servers is determined as a function of the current total workload, ignoring specific jobs. DPM focuses on energy conservation instead of individual job placement.

We investigate DPM as an offline problem, i.e., the varying computing demands are known in advance. From an algorithmic point of view, it is important to explore the tractability and approximability of the problem. The offline setting is also relevant in practice. Data centers usually analyze past workload traces to identify long-term patterns. The findings are used to specify demands in future time windows.

In Section 3, we study DPM in the scenario that each server has two states: one active state and one sleep state. This is a basic setting that, in a first step, abstracts away the full spectrum of low-power modes. Most of the more applied literature on power management strategies assumes the existence of a single sleep state. We show that DPM can be solved in polynomial time by a combinatorial algorithm. We devise an algorithm that resorts to a single-commodity minimum-cost flow computation. In the corresponding network, there is a component for each server. Such a component contains an upper path and a lower path, representing the server's active state and sleep state, respectively. Unfortunately, an arbitrary minimum-cost flow does not correspond to a feasible schedule. Our algorithm modifies flow so that an optimal schedule can be derived.

In Section 4, we investigate DPM in the general scenario that each server has multiple sleep states. We extend our approach based on flow computations. We develop a second algorithm that works with a more complex network in which each component has several lower paths, representing the various low-power states of a server. Furthermore, we need a second commodity to ensure that computing demands are met. With only a single commodity, flow units could switch between lower paths at no cost, and infeasible schedules would result. Given a fractional two-commodity minimum-cost flow, our algorithm executes advanced flow rounding and packing procedures. First, by repeatedly traversing components, the algorithm modifies flow so it becomes integral on the upper paths. Then flow on the lower paths is packed. The final integral flow allows the constructing of a schedule for DPM. Our algorithm achieves an approximation factor of τ , where τ is the number of server types in the problem instance. The servers can be partitioned into τ classes such that, within each class, the servers are identical. Of course, the servers of a class are independent and not synchronized. In practice, a data center has a large collection of machines but a relatively small number of different server architectures. Furthermore, in the optimization, machines with comparable energy consumption characteristics can be assigned to the same server class.

We note that our algorithms can handle the problem extension that the power consumption rates are time dependent. This can model, e.g., scenarios in which servers are temporarily unavailable due to maintenance or because they are reserved for other tasks.

2 PRELIMINARIES

2.1 The Problem DPM

A problem instance $\mathcal{I} = (\mathcal{S}, \mathcal{D})$ is specified by a set of servers and varying computing demands over a time horizon. Let $\mathcal{S} = \{S_1, \dots, S_m\}$ be a set of *heterogeneous servers*. Each server S_i , $1 \leq i \leq m$, has an active state, as well as one or several standby/sleep states. The states of S_i are denoted by $s_{i,0}, \dots, s_{i,\sigma_i}$. Here, $s_{i,0}$ is the active state and $s_{i,1}, \dots, s_{i,\sigma_i}$ are the low-power states. The modes have individual power consumption rates. Let $r_{i,j}$ be the power consumption rate of $s_{i,j}$, i.e., $r_{i,j}$ energy units are consumed per time unit while S_i resides in $s_{i,j}$. The states are numbered in order of decreasing rates such that $r_{i,0} > \dots > r_{i,\sigma_i} \geq 0$. A server can transition between its states. Let $\Delta_{i,j,j'}$ be the non-negative energy needed to move S_i from state $s_{i,j}$ to state $s_{i,j'}$, for any pair $1 \leq j, j' \leq \sigma_i$. The transition energies satisfy the triangle inequality, i.e., the energy to move directly from $s_{i,j}$ to $s_{i,j'}$ is upper bounded by that of visiting an intermediate state $s_{i,k}$. Formally, $\Delta_{i,j,j'} \leq \Delta_{i,j,k} + \Delta_{i,k,j'}$, for any j, j', k .

Over a time horizon, the computing demands are given by a *demand profile* $\mathcal{D} = (T, D)$. Tuple $T = (t_1, \dots, t_n)$ contains the points in time when the computing demands change. There holds $t_1 < t_2 < \dots < t_n$ so that the time horizon is $[t_1, t_n)$. Tuple $D = (d_1, \dots, d_{n-1})$ specifies the demands. More precisely, $d_k \in \mathbb{N}_0$ servers are required for computing during interval $[t_k, t_{k+1})$, for any $1 \leq k \leq n-1$. Thus, at least d_k servers must reside in the active state during $[t_k, t_{k+1})$. We have $d_k \leq m$, for any $1 \leq k \leq n-1$, so that the requirements can be met.

Given $\mathcal{I} = (\mathcal{S}, \mathcal{D})$, a *schedule* Σ specifies, for each S_i and any $t \in [t_1, t_n)$, in which state server S_i resides at time t . Schedule Σ is *feasible* if during any interval $[t_k, t_{k+1})$ at least d_k servers are in the active state, $1 \leq k \leq n-1$. The energy $E(\Sigma)$ incurred by Σ is the total energy consumed by all m servers. Whenever server S_i , $1 \leq i \leq m$, resides in state $s_{i,j}$, it consumes energy at a rate of $r_{i,j}$. Whenever the server transitions from state $s_{i,j}$ to state $s_{i,j'}$, the incurred energy is $\Delta_{i,j,j'}$. The goal is to find an *optimal schedule*, i.e., a feasible schedule Σ that minimizes $E(\Sigma)$. We assume that initially, immediately before t_1 , and at time t_n , all servers reside in the deepest sleep state, i.e., S_i is in s_{i,σ_i} , $1 \leq i \leq m$. Our algorithms and results can be adapted easily to the setting where each server initially/finally takes arbitrary desired states.

2.2 Properties of Optimal Schedules

Given a problem instance $\mathcal{I}$, we characterize optimal schedules. Proposition 2.1 implies that there exists an optimal schedule in which a server never changes state while being in low-power mode. Of course, the low-power states may vary for the various intervals in which a server is not active. Proposition 2.2 states that there exists an optimal schedule executing state transitions only when the computing demands change. A server *powers up* if it transitions from a low-power state to the active state. A server *powers down* if it moves from the active state to a low-power state. The proofs of Propositions 2.1 and 2.2 are given in the appendix.

PROPOSITION 2.1. *There exists an optimal schedule with the following property. Suppose that S_i powers down at time t and next powers up at time t' . Then between t and t' , S_i resides in a single state $s_{i,j}$, where $j > 0$. At time t , S_i transitions directly from $s_{i,0}$ to $s_{i,j}$. At time t' , it moves directly from $s_{i,j}$ to $s_{i,0}$.*

PROPOSITION 2.2. *There exists an optimal schedule that satisfies the property of Proposition 2.1 and performs state transitions only at the times of T .*

We finally argue that without loss of generality, the power-down energies $\Delta_{i,0,j}$ are equal to 0, $1 \leq i \leq m$ and $1 \leq j \leq \sigma_i$. We will always focus on optimal schedules with the property given in Proposition 2.1. At times t_1 and t_n , every server is in its deepest sleep state. The first time server

S_i moves to the active state, the least energy is consumed if it transitions directly from s_{i,σ_i} to $s_{i,0}$. The last time S_i powers down, the best option is to move directly from $s_{i,0}$ to s_{i,σ_i} . Hence, every server S_i performs the same number of transitions from $s_{i,0}$ to $s_{i,j}$ as from $s_{i,j}$ to $s_{i,0}$, for any $1 \leq j \leq \sigma_i$. For any server S_i , only energies $\Delta_{i,0,j}$ and $\Delta_{i,j,0}$, $1 \leq j \leq \sigma_i$, are relevant. Therefore, if $\Delta_{i,0,j} > 0$, we can add this energy to $\Delta_{i,j,0}$, i.e., $\Delta'_{i,j,0} := \Delta_{i,0,j} + \Delta_{i,j,0}$ and $\Delta'_{i,0,j} := 0$.

3 SERVERS WITH TWO STATES

We consider the variant of DPM in which each server S_i has exactly two states: an active state $s_{i,0}$ and a sleep state $s_{i,1}$, $1 \leq i \leq m$.

THEOREM 3.1. *Let $\mathcal{I}$ be an instance of DPM in which each server has exactly two states. An optimal schedule for $\mathcal{I}$ can be computed in polynomial time by a combinatorial algorithm that uses a minimum-cost flow computation.*

In the remainder of this section, we prove Theorem 3.1. We first argue that we may assume without loss of generality that the power consumption rates in the sleep states are equal to 0. More specifically, for any problem instance $\mathcal{I}$, an optimal schedule can be derived from an optimal solution to a modified instance $\mathcal{I}'$ in which the power consumption rates in the sleep states are indeed 0. Formally, given $\mathcal{I} = (\mathcal{S}, \mathcal{D})$, define an instance $\mathcal{I}' = (\mathcal{S}', \mathcal{D})$. Set $\mathcal{S}'$ consists of servers $S'_1, \dots, S'_m$, where each server S'_i has again an active state and a sleep state. For any S'_i , let $r'_{i,0} = r_{i,0} - r_{i,1}$ and $r'_{i,1} = 0$, i.e., the rates are reduced by $r_{i,1}$. All other problem parameters of $\mathcal{I}'$, namely the state transition energies and the demand profile, are identical to those of $\mathcal{I}$. The next proposition states that an optimal schedule for $\mathcal{I}$ translates into an optimal schedule for $\mathcal{I}'$ and vice versa. Only the consumed energy differs by $\sum_{i=1}^m r_{i,1}(t_n - t_1)$.

PROPOSITION 3.2. *Any schedule Σ for $\mathcal{I}$ that is executed for $\mathcal{I}'$ consumes an energy of $E(\Sigma) - \sum_{i=1}^m r_{i,1}(t_n - t_1)$. Any schedule Σ' for $\mathcal{I}'$ that is executed for $\mathcal{I}$ consumes an energy of $E(\Sigma') + \sum_{i=1}^m r_{i,1}(t_n - t_1)$.*

PROOF. Considering that the demand profiles of $\mathcal{I}$ and $\mathcal{I}'$ are identical, any schedule for $\mathcal{I}$ can be executed for $\mathcal{I}'$ and vice versa. In $\mathcal{I}'$, for any server S'_i with $1 \leq i \leq m$, the power consumption rates in the active and sleep states are reduced by $r_{i,1}$, compared to those of S_i . In addition, considering that at any time a server either resides in the active state or the sleep state, the proposition follows. $\square$

In the following, let $\mathcal{I} = (\mathcal{S}, \mathcal{D})$ be a problem instance in which the power consumption rates in the servers' sleep states are 0. To simplify notation, let $r_i := r_{i,0}$ be the power consumption rate of S_i in the active state, $1 \leq i \leq m$. Moreover, let $\Delta_i := \Delta_{i,1,0}$ be the energy needed to transition S_i from the sleep state to the active state. We develop an algorithm $\mathcal{A}_1$ that computes an optimal schedule. Based on Proposition 2.2, we focus on schedules that perform state transitions only at the times of T . Given $\mathcal{I} = (\mathcal{S}, \mathcal{D})$, $\mathcal{A}_1$ constructs a network $\mathcal{N}(\mathcal{I})$. Any feasible schedule Σ for $\mathcal{I}$ translates into a feasible flow of cost $E(\Sigma)$ in $\mathcal{N}(\mathcal{I})$. Any feasible flow of cost C in $\mathcal{N}(\mathcal{I})$ can be converted so that it corresponds to a feasible schedule consuming energy C . The conversion requires some work but can be performed in a polynomial number of steps.

3.1 Construction of the Network

Consider any input $\mathcal{I} = (\mathcal{S}, \mathcal{D})$.

Network components. Network $\mathcal{N}(\mathcal{I})$ contains a *component* C_i , for each server S_i , $1 \leq i \leq m$. Such a component C_i , which is depicted in Figure 1, consists of an *upper path* and a *lower path*. The upper path represents the active state of S_i ; the lower path models the server's sleep state. The

Fig. 1. The component C_i for server S_i . Fig. 2. The network $\mathcal{N}(I)$.

computing demands change at the times $t_1 < \dots < t_n$ in T . For any t_k , $1 \leq k \leq n$, there is a vertex $u_{i,k}$ on the upper path. Vertices $u_{i,k}$ and $u_{i,k+1}$ are connected by a directed edge $(u_{i,k}, u_{i,k+1})$ of cost $r_i(t_{k+1} - t_k)$, $1 \leq k \leq n-1$. This cost is equal to the energy consumed if S_i is in the active state during $[t_k, t_{k+1})$. Similarly, for any t_k , $1 \leq k \leq n$, there is a vertex $l_{i,k}$ on the lower path. To ensure that at least d_k servers are in the active state during $[t_k, t_{k+1})$, if $k < n$, we need two auxiliary vertices $l_{i,k}^a$ and $l_{i,k}^b$. These vertices are again connected by directed edges. There is an edge $(l_{i,k}, l_{i,k}^a)$, followed by two edges $(l_{i,k}^a, l_{i,k}^b)$ and $(l_{i,k}^b, l_{i,k+1})$, for any k with $1 \leq k \leq n-1$. The cost of each of these edges is 0 because the energy consumption in the sleep state is 0.

The lower and the upper path are connected by additional edges that model state transitions. Recall that all servers are in the sleep state at times t_1 and t_n . For any k with $1 \leq k \leq n-1$, there is a directed edge $(l_{i,k}, u_{i,k})$ of cost Δ_i , representing a power-up operation of S_i at time t_k . For any k with $1 < k \leq n$, there is a directed edge $(u_{i,k}, l_{i,k})$ of cost 0, modeling a power-down operation of S_i at time t_k . The capacity of each edge of C_i is equal to 1.

The entire network. In network $\mathcal{N}(I)$, the components $C_1, \dots, C_m$ are aligned in parallel and connected to a source a_0 and a sink b_0 . The general structure of $\mathcal{N}(I)$ is depicted in Figure 2. There is a directed edge from a_0 to $l_{i,1}$ in C_i , for any $1 \leq i \leq m$. Furthermore, there is a directed edge from $l_{i,n}$ to b_0 , for any $1 \leq i \leq m$. Each of these edges has a cost of 0 and a capacity of 1. Vertex a_0 has a supply of m , and b_0 has a demand of m . Hence, m units of flow must be shipped through $C_1, \dots, C_m$. Considering that all edges have a capacity of 1, one unit of flow must be routed through each C_i , $1 \leq i \leq m$. Whenever the unit traverses the upper path, S_i is in the active state. Whenever the unit traverses the lower path, S_i is in the sleep state.

To ensure that at least d_k servers are in the active state during $[t_k, t_{k+1})$, $1 \leq k \leq n-1$, we introduce additional sources and sinks. Network $\mathcal{N}(\mathcal{I})$ has a source a_k and a sink b_k with supply/demand d_k , for any $1 \leq k \leq n-1$. There is a directed edge from a_k to $l_{i,k}^a$ on the lower path of each C_i , $1 \leq i \leq m$. Furthermore, there is a directed edge from each $l_{i,k}^b$ to b_k , $1 \leq i \leq m$. The cost and capacity of each of these edges is equal to 0 and 1, respectively. Considering that d_k flow units have to be shipped from a_k to b_k , there must exist at least d_k components C_i in which the flow unit from a_0 to b_0 traverses the upper path from $u_{i,k}$ to $u_{i,k+1}$. Hence, the corresponding servers are in the active state during $[t_k, t_{k+1})$. The encoding length of $\mathcal{N}(\mathcal{I})$ is polynomial in that of $\mathcal{I}$.

LEMMA 3.3. *Any feasible schedule Σ in which state transitions are performed only at the times of T corresponds to a feasible flow of cost $E(\Sigma)$.*

PROOF. Given Σ , we construct a flow as follows. One flow unit is transferred from a_0 to b_0 via component C_i , for any $1 \leq i \leq m$. Within C_i , the flow unit mimics the actions of S_i . Whenever S_i powers up at a time t_k , $1 \leq k \leq n-1$, the flow unit traverses edge $(l_{i,k}, u_{i,k})$. If S_i resides in the active state in $[t_k, t_{k+1})$, the flow unit traverses (u_k, u_{k+1}) ; otherwise, it traverses the path from l_k to l_{k+1} . Whenever the server powers down at time t_k , $1 < k \leq n$, the flow unit takes edge $(u_{i,k}, l_{i,k})$. The cost incurred by the flow unit in C_i is equal to the energy consumed by S_i . Consider any k with $1 \leq k \leq n-1$. Considering that Σ is feasible, at least d_k servers are in the active state in $[t_k, t_{k+1})$. Hence, there are at least d_k components C_i in which the flow unit traverses the edge (u_k, u_{k+1}) on the upper path. We may choose any d_k of these components. For each such C_i , one unit of flow is shipped from a_k to b_k via the auxiliary vertices $l_{i,k}^a$ and $l_{i,k}^b$. Thus, d_k flow units are transferred from a_k to b_k and the overall flow is feasible. The total cost of the flow is equal to $E(\Sigma)$ because the shipment of flow from any a_k to b_k , $1 \leq k \leq n-1$, does not incur cost. $\square$

3.2 Analysis of Flows

We analyze feasible flows in $\mathcal{N}(\mathcal{I})$. The goal is to show that any feasible flow f can be converted into one that corresponds to a feasible schedule Σ for $\mathcal{I}$; the energy consumed by Σ will be equal to the cost of f . The conversion is not immediate. A feasible flow might not be well behaved, i.e., flow shipped out of a source a_k is not necessarily routed to b_k , $0 \leq k \leq n-1$. It may happen that flow leaving a_k is routed to a sink $b_{k'}$, where $k' > k$, or to b_0 .

In $\mathcal{N}(\mathcal{I})$, all edge capacities and supplies/demands are integer values. Hence, in $\mathcal{N}(\mathcal{I})$, there exists a minimum-cost flow that is integral. A flow f is called *integral* if the flow $f(e)$ along any edge e takes an integer value. Moreover, there exist polynomial time combinatorial algorithms that compute an integral minimum-cost flow, given a network with integer edge capacities and supplies/demands; see Ahuja et al. [1].

We will always work with a flow f in $\mathcal{N}(\mathcal{I})$ that is integral. Such a flow translates into a state transition schedule for the servers if, for each C_i and each k , one flow unit traverses either the upper path from $u_{i,k}$ to $u_{i,k+1}$ or the lower path from $l_{i,k}$ to $l_{i,k+1}$. Formally, we call an integral flow *consistent* in $[t_k, t_{k+1})$, where $1 \leq k \leq n-1$, if $f(u_{i,k}, u_{i,k+1}) + f(l_{i,k}, l_{i,k+1}^a) = 1$ holds for all $i = 1, \dots, m$. In this definition, we only consider flow from $l_{i,k}$ to $l_{i,k+1}^a$. This will be sufficient for our purposes. An integral flow is called *consistent* if it is consistent in all intervals $[t_k, t_{k+1})$, $1 \leq k \leq n-1$. In the following, we will prove that any feasible integral flow can be converted into one that is consistent. The next lemma identifies some useful properties.

LEMMA 3.4. *Let f be any feasible integral flow in $\mathcal{N}(\mathcal{I})$.*

- (a) *Then f is consistent in $[t_1, t_2)$.*
- (b) *Suppose that f is consistent in $[t_1, t_2), \dots, [t_k, t_{k+1})$ but not in $[t_{k+1}, t_{k+2})$. Then there exist C_i and C_j such that $f(u_{i,k+1}, u_{i,k+2}) + f(l_{i,k+1}, l_{i,k+1}^a) = 2$ and $f(u_{j,k+1}, u_{j,k+2}) + f(l_{j,k+1}, l_{j,k+1}^a) = 0$.*

Fig. 3. A flow that is not consistent in $[t_{k+1}, t_{k+2})$.

$l_{j,k+1}^a = 0$, cf. Figure 3. In C_i , flow f satisfies $f(u_{i,k}, u_{i,k+1}) = 1$. One flow unit is shipped from source a_k to $l_{i,k}^a$ and is further routed to $l_{i,k+1}^a$. In C_j , flow f satisfies $f(l_{j,k}, l_{j,k}^a) = 1$. This unit is routed via $l_{j,k}^b$ to sink b_k .

PROOF. For the proof of part (a), we observe that a supply of m is shipped from a_0 to the components $C_1, \dots, C_m$. Considering that all edge capacities are 1, exactly one unit of flow enters $l_{i,1}$ in C_i , $1 \leq i \leq m$. The unit can traverse the edge to $l_{i,1}^a$. Alternatively, it can move to the upper path of C_i and traverse $(u_{i,1}, u_{i,2})$. In addition, considering that further flow can be injected into C_i only from a source a_k reaching vertex $l_{i,k}^a$, $1 \leq k \leq n-1$, flow f is consistent in $[t_1, t_2)$.

We next prove part (b). Flow f is consistent in $[t_1, t_2), \dots, [t_k, t_{k+1})$. Considering the last of these intervals and all $C_1, \dots, C_m$, we obtain $\sum_{i=1}^m (f(u_{i,k}, u_{i,k+1}) + f(l_{i,k}, l_{i,k+1}^a)) = m$. Source a_k injects a total of d_k flow units into $C_1, \dots, C_m$, but the same amount is also absorbed by sink b_k . Hence, exactly m flow units traverse the set of edges $(u_{i,k+1}, u_{i,k+2})$ and $(l_{i,k+1}, l_{i,k+1}^a)$, $1 \leq i \leq m$. Formally, $\sum_{i=1}^m (f(u_{i,k+1}, u_{i,k+2}) + f(l_{i,k+1}, l_{i,k+1}^a)) = m$. Considering that f is not consistent in $[t_{k+1}, t_{k+2})$, then there must exist C_i and C_j such that $f(u_{i,k+1}, u_{i,k+2}) + f(l_{i,k+1}, l_{i,k+1}^a) = 2$ and $f(u_{j,k+1}, u_{j,k+2}) + f(l_{j,k+1}, l_{j,k+1}^a) = 0$; see Figure 3. In C_i , a total of two flow units leave $u_{i,k+1}$ and $l_{i,k+1}$ along the upper and lower paths, respectively. The second flow unit must be injected from a_k via $l_{i,k}^a$. The unit is further routed to $l_{i,k}^b$ and $l_{i,k+1}$. This also implies that $f(u_{i,k}, u_{i,k+1}) = 1$ because f is consistent in $[t_k, t_{k+1})$. In C_j , the flow unit present along $(u_{j,k}, u_{j,k+1})$ or $(l_{j,k}, l_{j,k}^a)$ can only leave the component via $l_{j,k}^b$ by traversing the edge to the sink b_k . This implies that $f(l_{j,k}, l_{j,k}^a) = 1$. $\square$

3.3 Making a Flow Consistent

Let f be a feasible integral flow in $\mathcal{N}(\mathcal{I})$. We describe how algorithm $\mathcal{A}_1$ modifies f so that the resulting flow is consistent. By Lemma 3.4(a), f is consistent in $[t_1, t_2)$. Suppose that f is consistent in $[t_1, t_2), \dots, [t_k, t_{k+1})$ but not in $[t_{k+1}, t_{k+2})$. $\mathcal{A}_1$ modifies the flow so that it becomes consistent in $[t_1, t_2), \dots, [t_{k+1}, t_{k+2})$. The modifications are performed sequentially for all further intervals.

Modifying flow. By assumption, f is consistent in $[t_1, t_2), \dots, [t_k, t_{k+1})$ but not in $[t_{k+1}, t_{k+2})$. Hence, there must exist components C_i and C_j with the properties specified in Lemma 3.4(b); see Figure 3. In C_i , a total of two flow units leave $u_{i,k+1}$ and $l_{i,k+1}$ along the upper and lower paths, respectively. On the upper path, one flow unit traverses the edge from $u_{i,k}$ and $u_{i,k+1}$. On the lower path, one unit is injected from a_k . This unit reaches $l_{i,k}^a$ and continues via $l_{i,k}^b$ to $l_{i,k+1}$. In C_j , no flow leaves $u_{j,k+1}$ or $l_{j,k+1}$. A flow unit is shipped from $l_{j,k}$ to $l_{j,k}^a$, and this unit is routed to sink b_k via $l_{j,k}^b$.

As long as there exist components C_i and C_j as specified earlier, $\mathcal{A}_1$ works as follows. It determines the smallest integer k' , with $k' > k$, such that a flow unit is routed from C_i to sink $b_{k'}$, i.e., $f(l_{i,k'}, b_{k'}) = 1$. Such an integer must exist, because otherwise a total of two flow units must

Fig. 4. No edge of $P_j(k, k')$ ships flow. Fig. 5. Some edge of $P_j(k, k')$ ships flow.

reach the end of C_i at $u_{i,n}$ and $l_{i,n}$. These two flow units cannot feasibly be routed to b_0 along the unit-capacity edge $(l_{i,n}, b_0)$. Let $P_i(k, k')$ be the path from $l_{i,k}^b$ to $l_{i,k'}^b$ that uses only edges of the lower path of C_i . All edges of $P_i(k, k')$ carry one unit of flow. Similarly, let $P_j(k, k')$ be the path from $l_{j,k}^b$ to $l_{j,k'}^b$ that uses only edges of the lower path of C_j . In the flow modification, there are two cases depending on whether or not $P_j(k, k')$ carries any flow.

Flow modification, type 1. Suppose that $P_j(k, k')$ does not ship any flow; see Figure 4. Loosely speaking, $\mathcal{A}_1$ replaces flow along $P_i(k, k')$ by flow on $P_j(k, k')$. Formally, the modified flow is as follows. In C_i , the flow unit entering $l_{i,k}^b$ is routed to b_k , i.e., $f'(l_{i,k}^b, b_k) = 1$. In C_j , algorithm $\mathcal{A}_1$ removes the flow unit leaving $l_{j,k}^b$, i.e., $f'(l_{j,k}^b, b_k) = 0$. For all edges e of $P_i(k, k')$, the algorithm sets $f'(e) = 0$. For all edges e of $P_j(k, k')$, it sets $f'(e) = 1$. Finally, it removes the flow unit leaving $l_{i,k'}^b$, i.e., $f'(l_{i,k'}^b, b_{k'}) = 0$, and routes one unit from $l_{j,k'}^b$ to $b_{k'}$, i.e., $f'(l_{j,k'}^b, b_{k'}) = 1$. For all other edges not considered here, the flow remains unchanged. Obviously, after these modifications, the amount of flow routed into b_k and $b_{k'}$ has not changed. The flow conservation law is observed at all vertices of $P_i(k, k')$ and $P_j(k, k')$. Hence, the new flow is feasible. Furthermore, the cost of the flow has not changed because the flow update only affects edges of cost 0. Note that $f'(l_{i,k+1}, l_{i,k+1}^a) = 0$ and $f'(l_{j,k+1}, l_{j,k+1}^a) = 1$. Hence, restricted to C_i and C_j , the new flow is consistent in $[t_{k+1}, t_{k+2})$.

Flow modification, type 2. Assume that some edge of $P_j(k, k')$ carries flow; see Figure 5. Then this flow must enter C_j from some source among $a_{k+1}, \dots, a_{k'}$. $\mathcal{A}_1$ determines the smallest integer k^* , with $k^* > k$, such that $f(a_{k^*}, l_{j,k^*}^a) = 1$. Then $k^* \leq k'$. In component C_i , the corresponding edge (a_{k^*}, l_{i,k^*}^a) does not ship flow because all edges of $P_i(k, k')$ carry one unit of flow and no further unit can be injected from a_{k^*} . Let $P_i(k, k^*)$ be the path from $l_{i,k}^b$ to l_{i,k^*}^a that uses only edges of the lower path of C_i . Analogously, let $P_j(k, k^*)$ be the path from $l_{j,k}^b$ to l_{j,k^*}^a that uses only edges of the lower path of C_j . $\mathcal{A}_1$ replaces flow on $P_i(k, k^*)$ by flow on $P_j(k, k^*)$. More specifically, the flow unit routed into $l_{i,k}$ is shipped to b_k , i.e., $f'(l_{i,k}^b, b_k) = 1$. The flow unit on edge $(l_{j,k}^b, b_k)$ is removed. For all edges e of $P_i(k, k^*)$, $\mathcal{A}_1$ sets $f'(e) = 0$. For all edges e of $P_j(k, k^*)$, it sets $f'(e) = 1$. Finally, it sets $f'(a_{k^*}, l_{i,k^*}^a) = 1$ and $f'(a_{k^*}, l_{j,k^*}^a) = 0$. The new flow is feasible, and during the modification

the cost has not changed. Restricted to C_i and C_j , the new flow is consistent in $[t_{k+1}, t_{k+2})$ because $f'(l_{i,k+1}, l_{i,k+1}^a) = 0$ and $f'(l_{j,k+1}, l_{j,k+1}^a) = 1$.

We remark that the preceding flow modifications do not change flow in components other than C_i and C_j . By repeating the flow update operations for other pairs of network components violating consistency, $\mathcal{A}_1$ obtains a flow that is consistent in $[t_1, t_2), \dots, [t_{k+1}, t_{k+2})$. The total number of steps to perform the modifications is polynomial in $\mathcal{N}(\mathcal{I})$. The next lemma summarizes the result.

LEMMA 3.5. *Let f be a feasible integral flow of cost C in $\mathcal{N}(\mathcal{I})$. Then f can be transformed into a feasible integral flow that is consistent and has cost C . The transformation takes a polynomial number of steps.*

3.4 Establishing the Theorem

We show an important property of feasible consistent flows in $\mathcal{N}(\mathcal{I})$.

LEMMA 3.6. *In any feasible integral flow f that is consistent, all flow leaving a_k is routed to b_k , $1 \leq k \leq n-1$.*

PROOF. Fix a k with $1 \leq k \leq n-1$. Consider a unit of flow that leaves a_k and enters a component C_i via $l_{i,k}^a$. The unit is routed to $l_{i,k}^b$. Considering that f is consistent in $[t_k, t_{k+1})$, it satisfies $f(u_{i,k}, u_{i,k+1}) + f(l_{i,k}, l_{i,k}^a) = 1$. In addition, considering that all edge capacities are 1, no further flow can enter $l_{i,k}^a$. Hence, $f(u_{i,k}, u_{i,k+1}) = 1$. First assume that $k < n-1$. In this case, the flow unit on $(u_{i,k}, u_{i,k+1})$ will be routed on either $(u_{i,k+1}, u_{i,k+2})$ or $(l_{i,k+1}, l_{i,k+1}^a)$. As f is consistent in $[t_{k+1}, t_{k+2})$, the flow unit entering $l_{i,k}^b$ must be shipped to b_k and cannot be sent to $l_{i,k+1}$. If $k = n-1$, then the flow unit on $(u_{i,k}, u_{i,k+1})$ is routed to $l_{i,n}$ and b_0 . Considering that all edge capacities are 1, no further flow can be shipped on $(l_{i,n}, b_0)$. Again, the flow unit entering $l_{i,k}^b$ must be routed to b_k and cannot be sent to $l_{i,k+1}$. $\square$

We finish the proof of Theorem 3.1. Given problem instance $\mathcal{I}$, $\mathcal{A}_1$ constructs $\mathcal{N}(\mathcal{I})$ and computes an integral minimum-cost flow f^* using a combinatorial algorithm. Executing the flow modifications described earlier, the algorithm obtains an integral minimum-cost flow f that is consistent. Lemma 3.6 implies that in f , all flow units leaving a_0 are transferred to b_0 . By the edge capacity constraints, one unit of flow is transferred through each C_i , $1 \leq i \leq m$. $\mathcal{A}_1$ derives a schedule Σ for $\mathcal{I}$ by keeping track of these flow units in $C_1, \dots, C_m$. Consider component C_i , $1 \leq i \leq m$. While the flow unit traverses the upper path, server S_i is in the active state. While the flow unit traverses the lower path, S_i is in the sleep state. If the flow traverses an edge $(l_{i,k}, u_{i,k})$, S_i powers up at time t_k . If the flow traverses $(u_{i,k}, l_{i,k})$, the server powers down at time t_k . The energy consumed by S_i is exactly equal to the cost incurred by the flow unit traversing C_i . Hence, the energy consumed by Σ is equal to the cost of f , and this is equal to the cost of f^* .

It remains to verify that Σ is feasible. By Lemma 3.6, in f all flow units leaving a_k are shipped to b_k , $1 \leq k \leq n-1$. Consider any fixed k , $1 \leq k \leq n-1$. There must exist d_k components C_i such that a flow unit is routed from a_k to $l_{i,k}^a$ and further on to $l_{i,k}^b$ and b_k . Considering that f is consistent in $[t_k, t_{k+1})$, it satisfies $f(u_{i,k}, u_{i,k+1}) = 1$. If $f(l_{i,k}, l_{i,k}^a) = 1$, then two units of flow would leave $l_{i,k}^a$, violating the capacity of the outgoing edge. Thus, in $[t_k, t_{k+1})$, at least d_k servers are in the active state. Optimality of Σ follows from Proposition 2.2 and Lemma 3.3.

4 SERVERS WITH MULTIPLE STATES

We develop an approximation algorithm for DPM in the general setting that each server may have an arbitrary number of states. Let $\mathcal{I} = (\mathcal{S}, \mathcal{D})$ be an input with τ server types, i.e., each server of $\mathcal{S}$ belongs to one of τ classes, where $\tau \in \mathbb{N}$. Formally, $\mathcal{S}$ is partitioned into $\mathcal{S}_1, \dots, \mathcal{S}_\tau$. Within

Fig. 6. The component C_i for server type i . Unlabeled edges connecting two vertices have cost 0.

each server type/class $\mathcal{S}_i$, $1 \leq i \leq \tau$, all servers are identical. Every server of $\mathcal{S}_i$ has $\sigma_i + 1$ states $s_{i,0}, \dots, s_{i,\sigma_i}$ with power consumption rates $r_{i,0} > \dots > r_{i,\sigma_i}$. Here, $s_{i,0}$ is again the active state; the other states are low-power modes. The energy needed to transition from $s_{i,j}$ to $s_{i,0}$ is denoted by $\Delta_{i,j}$, $1 \leq j \leq \sigma_i$. The state transition energy from the active state to any lower-power state is 0. The servers of $\mathcal{S}_i$ are independent and not synchronized. Over the time horizon, each server may reside in individual states and perform state transitions independent of the other servers. Let m_i be the number of servers in $\mathcal{S}_i$. Then $\sum_{i=1}^{\tau} m_i = m$.

THEOREM 4.1. *Let $\mathcal{I}$ be an instance of DPM with τ server types. A schedule whose energy consumption is at most τ times the minimum one for $\mathcal{I}$ can be computed in polynomial time based on a minimum-cost two-commodity flow computation.*

In the remainder of this section, we develop an algorithm $\mathcal{A}_2$ that, given $\mathcal{I} = (\mathcal{S}, \mathcal{D})$, constructs a feasible schedule attaining a τ -approximation to the optimal energy consumption. By Proposition 2.2, we restrict ourselves to schedules with the following two properties. (1) While a server is in low-power mode, it uses a single state. (2) State transitions are performed only at the times of T .

Algorithm $\mathcal{A}_2$ constructs a network $\mathcal{N}(\mathcal{I})$. Compared to the construction in Section 3, the main differences are as follows. Each network component will represent a class of servers so that the encoding length of $\mathcal{N}(\mathcal{I})$ is polynomial in that of $\mathcal{I}$. A component has a collection of lower paths corresponding to the various low-power modes of the servers. We need a second commodity to ensure that computing demands are met. This will allow us to reduce the number of auxiliary vertices on the lower paths. We will explain the need of the second commodity later, before Lemma 4.2.

Any schedule Σ as specified in Proposition 2.2 corresponds to a feasible flow of cost $E(\Sigma)$ in $\mathcal{N}(\mathcal{I})$. Given $\mathcal{N}(\mathcal{I})$, $\mathcal{A}_2$ computes a minimum-cost flow f^* . Considering that the network involves two commodities, f^* is not integral but fractional in general. In a sequence of rounding and packing operations, $\mathcal{A}_2$ transforms f^* into an integral flow that guides the construction of a feasible schedule for $\mathcal{I}$. The cost of the integral flow and the constructed schedule will be at most τ times that of f^* .

4.1 Construction of the Network

We describe $\mathcal{N}(\mathcal{I})$, given $\mathcal{I} = (\mathcal{S}, \mathcal{D})$.

Network components with multiple paths. For each server type i , the network contains a component C_i , $1 \leq i \leq \tau$. The component represents all m_i servers of $\mathcal{S}_i$. Exactly m_i flow units will be routed through C_i , modeling the individual states and actions of the servers. Component C_i consists of an *upper path* and σ_i *lower paths*. The general structure is depicted in Figure 6. We search for an optimal schedule in which state transitions are performed only at times $t_1 < \dots < t_n$ in T , cf. Proposition 2.2. In C_i , the upper path corresponds to the active state of the servers of $\mathcal{S}_i$. For any t_k , $1 \leq k \leq n$, there is a vertex $u_{i,k}$ on the upper path. Vertices $u_{i,k}$ and $u_{i,k+1}$ are connected by a directed edge $(u_{i,k}, u_{i,k+1})$ of cost $r_{i,0}(t_{k+1} - t_k)$, $1 \leq k \leq n - 1$. The cost is equal to energy

Fig. 7. The network $\mathcal{N}(\mathcal{I})$. To maintain the readability of the figure, for a_k and b_k with $1 \leq k \leq n-1$, only the links into C_i are shown as solid lines. For the other components, the links are shown as dashed lines.

consumed by one server of $\mathcal{S}_i$ if it resides in the active state during $[t_k, t_{k+1})$. The capacity of $(u_{i,k}, u_{i,k+1})$, and in fact of all edges of C_i , is equal to m_i , reflecting that C_i represents m_i servers in $\mathcal{S}_i$.

The σ_i lower paths correspond to the σ_i low-power states. Consider any j with $1 \leq j \leq \sigma_i$. On lower path j , there is a vertex $l_{i,j,k}$ and an auxiliary vertex $l_{i,j,k}^a$, for any $1 \leq k \leq n-1$. Moreover, there is a final vertex $l_{i,j,n}$. The auxiliary vertices will help to ensure that a total of at least d_k flow units traverse the edges $(u_{i,k}, u_{i,k+1})$ on the upper paths, considering all of the components C_i , $1 \leq i \leq \tau$. We do not need a second auxiliary vertex $l_{i,j,k}^b$ because we work with two commodities. On lower path j , the vertices are connected as follows. For any k , where $1 \leq k \leq n-1$, there is a directed edge $(l_{i,j,k}, l_{i,j,k}^a)$ of cost $r_{i,j}(t_{k+1} - t_k)$, representing the energy consumed by a server if it is in state $s_{i,j}$ during $[t_k, t_{k+1})$. Furthermore, there is an edge $(l_{i,j,k}^a, l_{i,j,k+1})$ of cost 0.

The upper path is connected to the lower paths by additional edges that model state transitions. We assume that at times t_1 and t_n , all servers of $\mathcal{S}_i$ are in the deepest low-power state s_{i,σ_i} . Thus, there is a directed edge $(l_{i,\sigma_i,1}, u_{i,1})$ of cost Δ_{i,σ_i} modeling possible power-up operations of servers at time t_1 . Furthermore, there is a directed edge $(u_{i,n}, l_{i,\sigma_i,n})$ of cost 0 representing power-down operations at time t_n . For any $1 < k < n$ and any $1 \leq j \leq \sigma_i$, there is a directed edge $(u_{i,k}, l_{i,j,k})$ of cost 0 and a directed edge $(l_{i,j,k}, u_{i,k})$ of cost $\Delta_{i,j}$. As we consider schedules specified in Proposition 2.1, there are no state transitions among low-power states; thus, there are no edges between the lower paths. (We remark that on lower path j , $1 \leq j < \sigma_i$, we could remove the first and the last vertex, but it is not important.) Note again that the capacity of each edge of C_i is m_i .

The network with two commodities. In $\mathcal{N}(\mathcal{I})$, components $C_1, \dots, C_\tau$ are aligned in parallel and connected to vertices a_0 and b_0 , cf. Figure 7. These two vertices inject and absorb flow of commodity 1. Specifically, a_0 has a supply of m and b_0 has a demand of m of commodity 1. The connections are as follows. At times t_1 and t_n , the servers are in the deepest low-power mode. Hence, for any $1 \leq i \leq \tau$, there exist directed edges $(a_0, l_{i,\sigma_i,1})$ and $(l_{i,\sigma_i,n}, b_0)$. Each of these edges has a cost of 0 and a capacity of m_i so that m_i flow units can be routed from a_0 to b_0 via C_i .

Network $\mathcal{N}(\mathcal{I})$ contains further sources and sinks that inject and absorb flow of commodity 2. This second commodity will ensure that the computing demands are met. Consider any k with $1 \leq k \leq n-1$. There is a source a_k and a sink b_k with a supply/demand of

$$D_k = \sum_{i=1}^{\tau} m_i(\sigma_i - 1) + d_k \quad (1)$$

Fig. 8. Feasible flow and state changes with a single commodity.

of commodity 2. Vertices a_k and b_k are connected to all lower paths in the components. For any i and j with $1 \leq i \leq \tau$ and $1 \leq j \leq \sigma_i$, there is a directed edge $(a_k, l_{i,j,k}^a)$ into the auxiliary vertex on lower path j in C_i . Moreover, there is a directed edge $(l_{i,j,k+1}, b_k)$ from the following vertex on the lower path into b_k . Each of these edges has a cost of 0 and a capacity of m_i . Lemma 4.2 states that in any feasible flow, for any $1 \leq k \leq n-1$, at least d_k units will be shipped along the edges $(u_{i,k}, u_{i,k+1})$ on the upper paths of the components $C_1, \dots, C_\tau$.

So far, we have specified the total capacity of any edge in $\mathcal{N}(\mathcal{I})$. It remains to specify edge capacity constraints for the two commodities. Consider any $1 \leq i \leq \tau$. For any edge of C_i , the capacity of commodity 1 is m_i . The same holds true for the edges $(a_0, l_{i,\sigma_i,1})$ and $(l_{i,\sigma_i,n}, b_0)$. On all edges leaving a_k or entering b_k , $1 \leq k \leq n-1$, the capacity of commodity 1 is equal to 0. Hence, flow of commodity 1 must not traverse these edges. In the network components, commodity 2 may only traverse the edges from the auxiliary vertices to the subsequent vertices on the lower paths. Hence, commodity 2 has a capacity constraint of m_i on each of the edges $(a_k, l_{i,j,k}^a)$, $(l_{i,j,k}^a, l_{i,j,k+1})$ and $(l_{i,j,k+1}, b_k)$, where $1 \leq i \leq \tau$, $1 \leq j \leq \sigma_i$ and $1 \leq k \leq n-1$. For all other edges in $\mathcal{N}(\mathcal{I})$, commodity 2 has a capacity of 0. Hence, all flow from a_k , $1 \leq k \leq n-1$, must be shipped to b_k via edges $(l_{i,j,k}^a, l_{i,j,k+1})$. The encoding length of $\mathcal{N}(\mathcal{I})$ is polynomial in that of $\mathcal{I}$ because the m_i identical servers of $\mathcal{S}_i$ are modeled by a single component C_i , $1 \leq i \leq \tau$.

We explain why in $\mathcal{N}(\mathcal{I})$ two commodities are indeed essential. Consider the network as described earlier but with a single commodity only. In this case, a feasible flow can change among low-power states at no cost and a minimum-cost flow does not correspond to a feasible schedule with the same energy consumption. More specifically, consider the situation depicted in Figure 8, where we focus on a network component C_i in the time interval $[t_k, t_{k+1})$. Suppose for simplicity that C_i represents a single server. This server resides in the deepest sleep state prior to t_k and also during $[t_k, t_{k+1})$. Now, a flow unit is injected from a_k into the path representing the highest sleep state $s_{i,1}$. This flow unit continues on to the active state, whereas the flow unit along the lowest path is shipped to b_k . Thus, at time t_{k+1} , the server powers up to the active state but at a reduced cost smaller than Δ_{i,σ_i} . In general, with a single commodity, state changes, particularly power-up operations, can be performed without accurate payments.

We proceed with the analysis of the network $\mathcal{N}(\mathcal{I})$, incorporating two commodities, as specified earlier. Lemma 4.2, as mentioned, identifies an important property of feasible flow.

LEMMA 4.2. *In $\mathcal{N}(\mathcal{I})$, there exists a feasible flow. Any feasible flow f satisfies $\sum_{i=1}^{\tau} f(u_{i,k}, u_{i,k+1}) \geq d_k$, for $1 \leq k \leq n-1$.*

PROOF. A feasible flow can easily be constructed as follows. For any i , $1 \leq i \leq \tau$, exactly m_i flow units of commodity 1 are shipped from a_0 to b_0 along the upper path of C_i . When entering C_i at $l_{i,\sigma_i,1}$ the flow units move directly to the upper path. When reaching the end of C_i at $u_{i,0,n}$, the flow is routed to $l_{i,\sigma_i,n}$. Consider any $1 \leq k \leq n-1$. No flow of commodity 1 is routed along the edges $(l_{i,j,k}^a, l_{i,j,k+1})$ on the lower paths, where $1 \leq i \leq \tau$ and $1 \leq j \leq \sigma_i$. Hence, the total residual

capacity of these edges is $\sum_{i=1}^{\tau} m_i \sigma_i = \sum_{i=1}^{\tau} m_i (\sigma_i - 1) + m \geq D_k$; see (1) for the definition of D_k . We conclude that the supply of commodity 2 at a_k can be shipped in an arbitrary way to b_k along the edges of the lower paths. We only have to observe that the capacity of $(l_{i,j,k}^a, l_{i,j,k+1})$ is m_i , which is equal to the capacity of the edges from a_k to $l_{i,j,k}^a$ and from $l_{i,j,k+1}$ to b_k .

Let f be any feasible flow and consider any k with $1 \leq k \leq n-1$. On the upper paths of the network components, there is a collection of edges $(u_{i,k}, u_{i,k+1})$, $1 \leq i \leq \tau$. On the lower paths, there is a collection of edges $(l_{i,j,k}^a, l_{i,j,k+1})$, where $1 \leq i \leq \tau$ and $1 \leq j \leq \sigma_i$. Together, these edges must route $\sum_{i=1}^{\tau} m_i$ units of commodity 1 and $D_k = \sum_{i=1}^{\tau} m_i (\sigma_i - 1) + d_k$ units of commodity 2. Thus, the total flow along all of these edges is $\sum_{i=1}^{\tau} m_i \sigma_i + d_k$. The total capacity of the edges $(l_{i,j,k}^a, l_{i,j,k+1})$, where $1 \leq i \leq \tau$ and $1 \leq j \leq \sigma_i$, is $\sum_{i=1}^{\tau} m_i \sigma_i$. Hence, at least d_k units of flow must be shipped along the edges of the upper paths. $\square$

The following lemma states that every optimal schedule with the properties of Proposition 2.2 translates to a feasible flow with the same energy/cost.

LEMMA 4.3. *Let Σ be an optimal schedule as specified in Proposition 2.2. Then Σ corresponds to a feasible flow of cost $E(\Sigma)$ in $\mathcal{N}(\mathcal{I})$.*

PROOF. Given Σ , we construct a feasible flow. Consider any server type i with set $\mathcal{S}_i$, $1 \leq i \leq \tau$. Exactly m_i units of commodity 1 are shipped from a_0 to b_0 via component C_i . For each server $S \in \mathcal{S}_i$, one flow unit is routed, modeling the states and actions of S . If S transitions from low-power state $s_{i,j}$, $1 \leq j \leq \sigma_i$, to the active state at time t_k , the flow unit traverses $(l_{i,j,k}, u_{i,k})$. Conversely, if S moves from the active state to low-power state $s_{i,j}$ at time t_k , the flow unit traverses $(u_{i,k}, l_{i,j,k})$. If S resides in the active state during $[t_k, t_{k+1})$, the flow unit traverses $(u_{i,k}, u_{i,k+1})$ on the upper path. If S is in state $s_{i,j}$ during $[t_k, t_{k+1})$, the flow unit traverses lower path j from $l_{i,j,k}$ to $l_{i,j,k+1}$, where $1 \leq j \leq \sigma_i$. All edge capacities are observed because all edges of C_i , as well as edges $(a_0, l_{i,\sigma_i,1})$ and $(l_{i,\sigma_i,n}, b_0)$, have a capacity of m_i , for commodity 1 and globally.

It remains to route the flow of commodity 2. Consider any k , $1 \leq k \leq n-1$. Considering that Σ is feasible, at least d_k servers of $\mathcal{S}$ reside in the active state during $[t_k, t_{k+1})$. Thus, a total of at least d_k flow units traverse the edges $(u_{i,k}, u_{i,k+1})$ on the upper paths of the components C_i , $1 \leq i \leq \tau$. This implies that the total residual capacity of the edges $(l_{i,j,k}^a, l_{i,j,k+1})$, where $1 \leq i \leq \tau$ and $1 \leq j \leq \sigma_i$, is at least $\sum_{i=1}^{\tau} m_i \sigma_i - (\sum_{i=1}^{\tau} m_i - d_k)$. The first term corresponds to the total capacity of the edges $(l_{i,j,k}^a, l_{i,j,k+1})$, where $1 \leq i \leq \tau$ and $1 \leq j \leq \sigma_i$. The second term is an upper bound on the flow of commodity 1 shipped along these edges. Hence, the residual capacity is at least D_k , cf. (1). The capacity of $(a_k, l_{i,j,k}^a)$ and $(l_{i,j,k+1}, b_k)$ is not smaller than the residual capacity of $(l_{i,j,k}^a, l_{i,j,k+1})$, for any $1 \leq i \leq \tau$ and $1 \leq j \leq \sigma_i$. Thus, the supply of D_k of commodity 2 can feasibly be routed to b_k . The cost of the flow of commodity 1 is exactly $E(\Sigma)$. The cost incurred by commodity 2 is 0 because this flow only traverses edges of cost 0. $\square$

4.2 Outline of the Algorithm and Flow Properties

Given $\mathcal{N}(\mathcal{I})$, algorithm $\mathcal{A}_2$ computes a feasible minimum-cost flow f^* . By Lemma 4.3, an optimal schedule with the properties of Proposition 2.2 corresponds to a feasible flow with the same cost. Thus, the cost of f^* , denoted by $\text{cost}(f^*)$, is a lower bound on the energy consumed by an optimal schedule for $\mathcal{I}$. Considering that f^* involves two commodities, it is fractional in general. In particular, it may be fractional on the upper paths of the components. On the corresponding edges the flow has to be raised, for sufficiently many components, so that a feasible schedule for $\mathcal{I}$ can be derived later. $\mathcal{A}_2$ modifies f^* in three main steps. The resulting flow is integral. (1) First, f^* is scaled by a factor of τ . (2) The scaled flow is modified so that it becomes integral on the upper

paths of the components. Specifically, on edge $(u_{i,k}, u_{i,k+1})$, exactly

$$d_{i,k} = \min\{m_i, \lfloor \tau f^*(u_{i,k}, u_{i,k+1}) \rfloor\} \quad (2)$$

units of flow are routed, where $1 \leq i \leq \tau$ and $1 \leq k \leq n-1$. In Lemma 4.4, we show that $\sum_{i=1}^{\tau} d_{i,k} \geq d_k$, for any $1 \leq k \leq n-1$. This property will later admit the construction of a feasible schedule in which the computing demands of $\mathcal{I}$ are met. (3) Given the flow of Step 2, $\mathcal{A}_2$ packs fractional flows on the lower paths of the components $C_1, \dots, C_{\tau}$. Using the integral flow obtained in Step 3, $\mathcal{A}_2$ constructs a feasible schedule for $\mathcal{I}$ whose energy consumption is upper bounded by the cost of that flow. Once f^* has been scaled in Step 1, the subsequent flow modifications of Steps 2 and 3 never increase cost. Thus, the energy consumed by the schedule is at most $\tau \text{cost}(f^*)$.

LEMMA 4.4. *For $k = 1, \dots, n-1$, we have $\sum_{i=1}^{\tau} d_{i,k} \geq d_k$.*

PROOF. Fix any k , $1 \leq k \leq n-1$. For any i with $1 \leq i \leq \tau$, let $f^*(u_{i,k}, u_{i,k+1}) = m_{i,k} + \delta_{i,k}$, where $m_{i,k} \in \mathbb{N}_0$ and $0 \leq \delta_{i,k} < 1$. Set $\delta_k = \sum_{i=1}^{\tau} \delta_{i,k}$. We first argue that there exist at least $\lfloor \delta_k \rfloor$ values $i \in \{1, \dots, \tau\}$ such that $\delta_{i,k} \geq 1/\tau$. If $\lfloor \delta_k \rfloor = 0$, there is nothing to show. So suppose that $\lfloor \delta_k \rfloor \geq 1$. If there were less than $\lfloor \delta_k \rfloor$ values i such that $\delta_{i,k} \geq 1/\tau$, then there would be at most $\lfloor \delta_k \rfloor - 1$ values i with $1/\tau \leq \delta_{i,k} < 1$. For the remaining values i , there holds $\delta_{i,k} < 1/\tau$. Hence, $\sum_{i=1}^{\tau} \delta_{i,k} < \lfloor \delta_k \rfloor - 1 + (\tau - \lfloor \delta_k \rfloor + 1)/\tau = \lfloor \delta_k \rfloor + (1 - \lfloor \delta_k \rfloor)/\tau \leq \delta_k$, which is a contradiction to the definition of δ_k . The last inequality uses the fact that $\lfloor \delta_k \rfloor \geq 1$.

For any i such that $\delta_{i,k} \geq 1/\tau$, there holds $\lfloor \tau f^*(u_{i,k}, u_{i,k+1}) \rfloor \geq \lfloor \tau(m_{i,k} + 1/\tau) \rfloor \geq m_{i,k} + 1$. Moreover, $m_{i,k} + 1 \leq m_i$ because $\delta_{i,k} > 0$, and thus $m_{i,k} < f^*(u_{i,k}, u_{i,k+1}) < m_i$. For any i such that $\delta_{i,k} < 1/\tau$, there holds $\lfloor \tau f^*(u_{i,k}, u_{i,k+1}) \rfloor \geq m_{i,k}$. In addition, $m_{i,k} \leq m_i$. As shown in the last paragraph, there exist at least $\lfloor \delta_k \rfloor$ values $i \in \{1, \dots, \tau\}$ such that $\delta_{i,k} \geq 1/\tau$. Considering the definition of $d_{i,k}$, see (2), we obtain $\sum_{i=1}^{\tau} d_{i,k} \geq \sum_{i=1}^{\tau} m_{i,k} + \lfloor \delta_k \rfloor = \lfloor \sum_{i=1}^{\tau} m_{i,k} + \delta_k \rfloor = \lfloor \sum_{i=1}^{\tau} (m_{i,k} + \delta_{i,k}) \rfloor = \lfloor \sum_{i=1}^{\tau} f^*(u_{i,k}, u_{i,k+1}) \rfloor \geq d_k$. The last inequality follows from Lemma 4.2 and the fact that d_k is an integer value. $\square$

Given Lemma 4.4, a natural idea for finding an integral solution is to use a flow computation: Determine a single-commodity minimum-cost flow that ships m_i flow units through component C_i and, importantly, exactly $d_{i,k}$ units along edge $(u_{i,k}, u_{i,k+1})$, where $1 \leq i \leq \tau$ and $1 \leq k \leq n-1$. However, one has to prove that the cost of such a flow is upper bounded by $\tau \text{cost}(f^*)$. Such a proof involves arguments and flow modifications contained in Steps 1 and 2 of $\mathcal{A}_2$. Therefore, we describe them explicitly as algorithmic steps. Step 3 could indeed be replaced by a min-cost flow computation. However, we instead devise a faster $O(n^2 \sum_{i=1}^{\tau} \sigma_i)$ time routine for constructing an integral flow along the lower paths of the components.

In the following, when describing flow modifications, we will always focus on one particular network component. All flow updates will be performed independently for the components. Hence, in the corresponding exposition, we consider an arbitrary but fixed component $C = C_i$, $1 \leq i \leq \tau$. This allows us to simplify notation and omit the index i . On the upper path, the vertices are $u_1, \dots, u_n$. Component C has $\sigma = \sigma_i$ lower paths. On the lower path j , $1 \leq j \leq \sigma$, the vertices are $l_{j,k}$ and $l_{j,k}^a$, for $k = 1, \dots, n-1$, followed by the final vertex $l_{j,n}$. Let $m_c = m_i$ be the number of servers in class $\mathcal{S}_i$ represented by $C = C_i$.

Nested structure of flows. We show that in each network component C flow f^* has a crucial property, i.e., it exhibits a nested structure. Let $P_j(k, k')$ be the path from u_k to $u_{k'}$ along lower path j , where $1 \leq j \leq \sigma$ and $1 < k < k' < n$. More specifically, the path consists of $(u_k, l_{j,k})$, followed by the path from $l_{j,k}$ to $l_{j,k'}$ on lower path j , followed by $(l_{j,k'}, u_{k'})$. For $k = 1$ and $1 < k' < n$, we define $P_{\sigma}(1, k')$ as the path consisting of the edges from $l_{\sigma,1}$ to $l_{\sigma,k'}$ on lower path σ , followed by the edge $(l_{\sigma,k'}, u_{k'})$. For $k' = n$ and $1 < k < n$, path $P_{\sigma}(k, n)$ consists of edge $(u_k, l_{\sigma,k})$, followed by

Fig. 9. Paths that are *not* nested. Only the relevant vertices and edges are shown.

the edges from $l_{\sigma,k}$ to $l_{\sigma,n}$ on lower path σ . Finally, $P(k, k')$ is the path connecting u_k and $u_{k'}$ on the upper path of the component, for any $1 \leq k < k' \leq n$. In the sequel, unless otherwise stated, flow always refers to commodity 1. Consider any path P . We say that P *routes flow* if, for any edge of P , the flow is strictly positive.

The following property of a flow will be important. A flow f in component C is *nested* if it satisfies the following condition. Let $P_i(k_1, k_2)$ and $P_j(k_3, k_4)$ be two paths such that both route flow and $i < j$. Then one of the relations (a–c) holds: (a) $k_2 < k_3$, (b) $k_4 < k_1$, or (c) $k_3 \leq k_1 < k_2 \leq k_4$. Intuitively, the endpoints of the two paths do not alternate. Both endpoints of $P_i(k_1, k_2)$ occur either before, after, or in between those of $P_j(k_3, k_4)$. Figure 9 show two forbidden structures of paths that are not nested.

LEMMA 4.5. *In each component C , flow f^* is nested.*

PROOF. Let $P_i(k_1, k_2)$ and $P_j(k_3, k_4)$, with $i < j$, be two paths that route flow. We assume that none of the relations (a–c) holds and derive a contradiction to the optimality of f^* . Determine a $\delta > 0$ such that δ units of flow are routed on all edges of $P_i(k_1, k_2)$ and $P_j(k_3, k_4)$.

First assume that $k_2 = k_3$. Considering that $i < j \leq \sigma$, there holds $1 < k_1$. We modify f^* in the following way. On $P_i(k_1, k_2)$, we delete δ units of flow and instead route this flow on $P_j(k_1, k_2)$ using lower path j . We then delete δ units of flow entering/leaving u_{k_2} . More specifically, we delete δ units of flow on (l_{j,k_2}, u_{k_2}) and on (u_{k_2}, l_{j,k_2}) . The flow conservation law is observed at vertices l_{j,k_2} and u_{k_2} . The flow of commodity 2 can be adapted easily: For any k with $k_1 \leq k < k_2$, up to δ units of flow that cannot be shipped anymore from a_k to b_k using lower path j can now be routed via lower path i . The new flow has a strictly smaller cost. The cost saving is $\delta(r_i - r_j)(t_{k_2} - t_{k_1}) + \delta\Delta_i > 0$. The last inequality holds because $r_i > r_j$. Here, r_i and r_j are the cost coefficients on lower path i and j . More precisely, for any $1 \leq k \leq n - 1$, the cost of edges $(l_{i,k}, l_{i,k}^a)$ and $(l_{j,k}, l_{j,k}^a)$ is $r_i(t_{k+1} - t_k)$ and $r_j(t_{k+1} - t_k)$, respectively. Moreover, Δ_i is the cost of an edge connecting lower path i to the upper path. In the same way, we can handle the case $k_4 = k_1$. Then $k_2 < n$ holds.

Next, assume that $k_1 < k_3 < k_2 \leq k_4$. Again, $1 < k_1$. We remove δ units of flow from both $P_i(k_1, k_2)$ and $P_j(k_3, k_4)$. Instead, we route δ units of flow on $P_i(k_3, k_2)$ and $P_j(k_1, k_4)$. Both of these paths exist. If $k_4 < n$, this is obvious. If $k_4 = n$, then $k_2 < k_4$ because $i < j \leq \sigma$. Path $P_i(k_3, k_2)$ exists because $1 < k_3 < k_2 < n$, and $P_j(k_1, k_4)$ exists because $1 < k_1$. Again, we can easily update commodity 2. The flow modifications save a cost of $\delta(r_i - r_j)(t_{k_3} - t_{k_1}) > 0$. Analogously, we can argue if $k_3 \leq k_1 < k_4 < k_2$.

It remains to analyze the case $k_1 < k_3 < k_4 < k_2$. There holds $1 < k_1$ and $k_2 < n$. We remove δ units of flow from $P_i(k_1, k_2)$ and $P_j(k_3, k_4)$. Instead, we ship δ units of flow on $P_j(k_1, k_2)$

and $P_i(k_3, k_4)$. Flow of commodity 2 is modified accordingly. The new flow saves a cost of $\delta(r_i - r_j)(t_{k_3} - t_{k_1} + t_{k_2} - t_{k_4}) > 0$. $\square$

Loop freeness. Given f^* , $\mathcal{A}_2$ slightly modifies it so that it becomes *loop free* in each C . A flow is loop free in C if there exists no vertex u_k such that edges $(l_{i,k}, u_k)$ and $(u_k, l_{j,k})$ both route flow, where $1 \leq i, j \leq \sigma$. Suppose that there exists such a vertex u_k . Considering that f^* is nested, $i = j$ must hold. Otherwise, there would exist paths $P_i(k_1, k)$ and $P_j(k, k_2)$ with $k_1 < k < k_2$ that route flow. As $i = j$, $\mathcal{A}_2$ can simply remove $\min\{f^*(l_{i,k}, u_k), f^*(u_k, l_{i,k})\}$ units of flow from both $(l_{i,k}, u_k)$ and $(u_k, l_{i,k})$. The flow conservation law is observed at $l_{i,k}$ and u_k . The cost of the flow can only decrease. By performing these updates, one obtains a loop-free flow f^* that is nested.

4.3 Constructing an Integral Flow

We describe the three main modification steps.

Step 1: Flow scaling. Let f^* be the minimum-cost, loop-free flow. Algorithm $\mathcal{A}_2$ multiplies f^* by a factor of τ on all edges of the network. At the same time, it multiplies all edge capacities and supplies/demands by τ . Then it deletes the flow of commodity 2 and the supplies/demands at a_k and b_k , $1 \leq k \leq n - 1$. The resulting flow f^1 of commodity 1 is feasible. Additionally, in each component, it is nested and loop free. There holds $\text{cost}(f^1) \leq \tau \text{cost}(f^*)$.

In Steps 2 and 3, flow f^1 is modified. As indicated earlier, the modification are executed independently for the components. Therefore, in the description of Steps 2 and 3, we concentrate on one component C that ships τm_c units of flow. The flow modifications never increase the cost. At all times, the flow remains nested and loop free.

Step 2: Rounding flow on the upper path. Given f^1 , $\mathcal{A}_2$ rounds it so the flow becomes integral on the upper path of C . On (u_k, u_{k+1}) , the flow will be $\min\{m_c, \lfloor \tau f^*(u_k, u_{k+1}) \rfloor\}$. Recall that m_c is the number of servers in the class represented by C . We first describe how to reduce f^1 so that on any edge (u_k, u_{k+1}) the flow is $\lfloor \tau f^*(u_k, u_{k+1}) \rfloor$. $\mathcal{A}_2$ makes four passes over C . First, in Step 2.1, it rounds valleys with a nonintegral amount of flow. Then, in Steps 2.2 and 2.3, it modifies flow on edge sequences with increasing and decreasing flow, respectively. Finally, in Step 2.4, it takes care of flow peaks. Given this flow, we then describe how to further reduce it so that the flow on any edge of the upper path does not exceed m_c . At any time, for a current flow f , we say that the flow *increases at u_k* if $k = 1$ and $f(u_1, u_2) > 0$ or if $1 < k < n$ and $f(u_{k-1}, u_k) < f(u_k, u_{k+1})$. Similarly, the flow *decreases at u_k* if $1 < k < n$ and $f(u_{k-1}, u_k) > f(u_k, u_{k+1})$ or if $k = n$ and $f(u_{n-1}, u_n) > 0$. Initially in Step 2, let $f = f^1$.

Step 2.1: Valleys. A valley is a path $P(k, k')$, $1 < k < k' < n$, on the upper path of C such that the flow decreases at u_k , increases at $u_{k'}$ and is constant for all edges of $P(k, k')$. Formally, the last condition indicates that $f(e) = f(u_k, u_{k+1})$, for all edges e of $P(k, k')$. $\mathcal{A}_2$ scans C . Whenever it encounters a valley $P(k, k')$ with a nonintegral amount of flow, it invokes the following procedure that reduces the flow to $\lfloor f(u_k, u_{k+1}) \rfloor$.

Flow update procedure. For the given valley $P(k, k')$, the flow decreases at u_k . The procedure determines the smallest integer j such that flow is routed from u_k to $l_{j,k}$ and shipped on lower path j .

CLAIM 1. $P_j(k, k')$ routes flow.

PROOF. Suppose that δ_j units of flow are routed from u_k to $l_{j,k}$. The flow along $P(k, k')$ is constant. Considering that the flow is loop free in C , no flow is routed from a vertex $l_{j,k''}$ to $u_{k''}$, where $k < k'' < k'$. Thus, all edges between $l_{j,k}$ and $l_{j,k'}$ on path $P_j(k, k')$ carry δ_j units of flow. This implies that δ_j units of flow reach $l_{j,k'}$ via lower path j . If none of these flow units are shipped to

$u_{k'}$, then the flow of C would not be nested: A path $P_j(k, k_3)$ with $k_3 > k'$ would route flow. Furthermore, a path $P_i(k_4, k')$ with $i \neq j$ and $k_4 \leq k$ would carry flow into $u_{k'}$. This holds true because the flow increases at $u_{k'}$ and the flow is loop free in C . If $i < j$, then by the choice of j there holds $k_4 < k$ and the flow of C is not nested. If $i > j$, then the flow is not nested either, even if $k_4 = k$. $\square$

Hence, as $P_j(k, k')$ routes flow, in the unscaled minimum-cost flow f^* path $P_j(k, k')$ also ships flow. Routing the flow on the upper path would have been a feasible option as well. This implies that the total edge cost of $P_j(k, k')$ is upper bounded by that of $P(k, k')$. The procedure updates the flow as follows. It remove $\delta = f(u_k, u_{k+1}) - \lfloor f(u_k, u_{k+1}) \rfloor$ units of flow from $P(k, k')$ and instead routes them along $P_j(k, k')$. This does not increase the cost. The resulting flow in C remains nested. Modifying all valleys takes $O(n\sigma)$ time.

Step 2.2: Flow increases. In a second pass over C algorithm $\mathcal{A}_2$ identifies vertices u_k at which the flow increases. If $f(u_k, u_{k+1})$ is not integral, it is reduced to $\lfloor f(u_k, u_{k+1}) \rfloor$. Starting at u_1 or at a vertex representing the end of a valley, $\mathcal{A}_2$ performs a sequence of vertex inspections and possible flow updates. The sequence ends at a vertex at which the flow decreases. The algorithm then searches for the end of the next valley and continues.

Formally, let $u_{k'}$ be a vertex such that $k' = 1$ or $u_{k'}$ is the last vertex of a valley. When located at $u_{k'}$, $\mathcal{A}_2$ determines the smallest k'' with $k'' > k'$ such that the flow decreases at $u_{k''}$. The algorithm inspects the vertices u_k , $k' \leq k < k'' - 1$, in order of increasing index. If $f(u_k, u_{k+1})$ is not integral, the procedure described in the next paragraph is invoked, which reduces the flow to $\lfloor f(u_k, u_{k+1}) \rfloor$. When the procedure is executed at u_k , we have $u_k = u_1$ or the flow $f(u_{k-1}, u_k)$ on the preceding edge is integral. The latter condition holds true because if $u_k = u_{k'}$ is the last vertex of a valley, then the flow along the incoming edge has been made integral in Step 2.1. $\mathcal{A}_2$ considers vertices in order of increasing index, starting at $u_{k'}$. When u_k , $k' < k < k'' - 1$, is inspected, the flow on the edges between $u_{k'}$ and u_k is already integral. We note that if $f(u_k, u_{k+1})$ is reduced by a call of the procedure, then u_{k+1} becomes a vertex where the flow increases, provided that this has not been the case before. We also remark that flow on $(u_{k''-1}, u_{k''})$ is not modified. If $f(u_{k''-1}, u_{k''})$ is not integral, then $(u_{k''-1}, u_{k''})$ is a peak when $\mathcal{A}_2$ reaches $u_{k''-1}$, i.e., the flow increases at $u_{k''-1}$ and decreases at $u_{k''}$. Peaks will be handled in Step 2.4. When located at $u_{k''}$, $\mathcal{A}_2$ moves to the end of the next valley, if it exists.

Flow update procedure. The procedure that updates flow at a vertex u_k where the flow increases and $f(u_k, u_{k+1})$ is not integral. Let again $\delta = f(u_k, u_{k+1}) - \lfloor f(u_k, u_{k+1}) \rfloor$. Either $k = 1$ or the flow on (u_{k-1}, u_k) is integral. Hence, at least δ units of flow are shipped from lower paths into u_k . While $\delta > 0$, the procedure executes the following steps. Let j be the largest integer such that the flow from $l_{j,k}$ to u_k is positive. Figure 10(a) depicts the general flow configuration. Let $\delta_j = f(l_{j,k}, u_k)$ and $\delta' = \min\{\delta, \delta_j\}$. The procedure reduces flow on $(l_{j,k}, u_k)$ and (u_k, u_{k+1}) by δ units. Instead, it ships δ units of flow from $l_{j,k}$ to u_{k+1} along lower path j , i.e., via $l_{j,k}^a$ and $l_{j,k+1}$. Then δ is reduced by δ' . The flow update decreases the cost of the flow by $(r_0 - r_j)(t_{k+1} - t_k) > 0$. Here, r_0 and r_j are the cost coefficients along the upper path and lower path j , respectively. More precisely, edge (u_k, u_{k+1}) has a cost of $r_0(t_{k+1} - t_k)$ and $(l_{j,k}, l_{j,k}^a)$ has a cost of $r_j(t_{k+1} - t_k)$. The modified flow remains nested: By the choice of j , there exists no path $P_{j'}(k_1, k)$ with $j' > j$ and $k_1 < k$ that routes flow. Moreover, since the flow does not decrease at u_{k+1} and is loop free, there is no path $P_{j'}(k+1, k_2)$, $1 \leq j' \leq \sigma$ and $k+1 < k_2$, that routes flow. The running time of one execution of the procedure is $O(\sigma)$. The running time of the entire pass over C is $O(n\sigma)$.

Step 2.3: Flow decreases. The flow modifications are symmetric to those described in Step 2.2. Algorithm $\mathcal{A}_2$ makes another pass over C , this time from right to left starting at u_n . It searches for vertices u_k at which the flow decreases. If the flow $f(u_{k-1}, u_k)$ on the incoming edge is not

Fig. 10. Flow configurations. (a) A flow increase. (b) A peak.

integral, then it is reduced to $\lfloor f(u_{k-1}, u_k) \rfloor$. Let $u_{k''}$ be a vertex such that $u_{k''} = n$ or $u_{k''}$ is the first vertex of a valley. When located at $u_{k''}$, $\mathcal{A}_2$ determines the largest k' with $k' < k''$ such that $u_{k'}$ is a vertex where the flow increases. Then it considers the vertices u_k , $k' + 1 < k \leq k''$, in order of decreasing index. If $f(u_{k-1}, u_k)$ is not integral, $\mathcal{A}_2$ invokes the procedure described in the next paragraph. When the procedure is executed for u_k , we have $k = n$ or the flow $f(u_k, u_{k+1})$ on the outgoing edge is integral. The last condition holds because vertices are inspected in order of decreasing index starting at $u_{k''}$ such that $k'' = n$ or $u_{k''}$ is the first vertex of a valley, for which the flow along the outgoing edge is integral. When $\mathcal{A}_2$ reaches u_k , the flow on the edges between u_k and $u_{k''}$ is integral. Moreover, when the procedure has been executed at u_k , the flow decreases at u_{k-1} . When reaching $u_{k'+1}$, $\mathcal{A}_2$ does not adjust flow on $(u_{k'}, u_{k'+1})$, as this edge is a peak. Instead, it searches for the first vertex of the preceding valley and continues.

Flow update procedure. Given a vertex u_k where the flow decreases, the procedure works as follows. Let $\delta = f(u_{k-1}, u_k) - \lfloor f(u_{k-1}, u_k) \rfloor$. While $\delta > 0$, the procedure determines the largest j such that flow is routed from u_k to $l_{j,k}$. Let $\delta_j = f(u_k, l_{j,k})$ and $\delta' = \min\{\delta, \delta_j\}$. The procedure reduces the flow on (u_{k-1}, u_k) and $(u_k, l_{j,k})$ by δ units. Instead, the flow is shipped from u_{k-1} to $l_{j,k}$ via $l_{j,k-1}$ and $l_{j,k-1}^a$. Then δ is reduced by δ' . The modified flow has a strictly smaller cost; the savings is $(r_0 - r_j)(t_k - t_{k-1})$. It is still nested because there is no path $P_{j'}(k, k_1)$, $j' > j$, that routes flow. Moreover, there is no path $P_{j'}(k_2, k - 1)$, with $k_2 < k - 1$, that routes flow because the flow on the upper path does not increase at u_{k-1} . The entire scan over C takes $O(n\sigma)$ time.

Step 2.4: Peaks. A *peak* is an edge (u_k, u_{k+1}) such that the flow increases at u_k and decreases at u_{k+1} . After $\mathcal{A}_2$ has executed Steps 2.1 through 2.3, the only edges on the upper path with a nonintegral amount of flow are peaks: Edges considered but not handled in Step 2.2 end in vertices $u_{k''}$ at which the flow decreases. Such a vertex $u_{k''}$ is followed by vertices at which the flow does not increase. The vertex sequence ends at the beginning of a valley or at u_n . The edges of the corresponding path are handled in Step 2.3. If $f(u_{k''-1}, u_{k''})$ is not integral, then the flow at $u_{k''-1}$ increases because the flow along $f(u_{k''-2}, u_{k''-1})$ is integral.

Algorithm $\mathcal{A}_2$ traverses C . For each peak (u_k, u_{k+1}) with a nonintegral amount of flow, it invokes the following routine.

Flow update procedure. As always, $\delta = f(u_k, u_{k+1}) - \lfloor f(u_k, u_{k+1}) \rfloor$. Let j_1 be the largest integer such that $f(l_{j_1,k}, u_k) > 0$, i.e., flow is routed from lower path j_1 to u_k . Furthermore, let j_2 be the largest integer such that $f(u_{k+1}, l_{j_2,k+1}) > 0$. There are two basic cases. We describe the flow update and then show that the new flow remains nested.

If $j_1 = j_2$, then let $\delta_1 = f(l_{j_1,k}, u_k)$ and $\delta_2 = f(u_{k+1}, l_{j_2,k+1})$. Furthermore, let $\delta' = \min\{\delta, \delta_1, \delta_2\}$. The procedure removes δ' units of flow from the path connecting $l_{j_1,k}$ and $l_{j_1,k+1}$ along the upper

path. Specifically, it removes δ' units of flow from the edges $(l_{j_1,k}, u_k)$, (u_k, u_{k+1}) and $(u_{k+1}, l_{j_1,k+1})$. Instead, it sends δ' units of flow from $l_{j_1,k}$ to $l_{j_1,k+1}$ via $l_{j_1,k}^a$ on lower path j_1 . The reduction in the cost of the flow is $\delta(r_0 - r_{j_1})(t_{k+1} - t_k) + \delta\Delta_{j_1} > 0$. Here, Δ_j is the cost of $(l_{j,k}, u_k)$, for any $1 \leq j \leq \sigma$ and $1 \leq k < n - 1$.

If $j_1 \neq j_2$, then let k_1 be the smallest integer such that $P_{j_1}(k_1, k)$ routes flow. Let δ_1 be the largest value such that every edge of $P_{j_1}(k_1, k)$ routes at least δ_1 units of flow. Similarly, let k_2 be the largest integer such that $P_{j_2}(k + 1, k_2)$ routes flow. Let δ_2 be the largest value such that every edge of $P_{j_2}(k + 1, k_2)$ routes at least δ_2 units of flow. Figure 10(b) shows the case that $j_1 < j_2$. Let $\delta' = \min\{\delta, \delta_1, \delta_2\}$. The procedure removes δ' units of flow from $P_{j_1}(k_1, k)$, edge (u_k, u_{k+1}) , and $P_{j_2}(k + 1, k_2)$. If $j_1 < j_2$, it instead sends these δ' units on path $P_{j_2}(k_1, k_2)$. If $j_1 > j_2$, it routes the δ' units along $P_{j_1}(k_1, k_2)$. Thus, in any case, the deeper low-power state is used. Note that the flow conservation law is maintained, particularly at vertices u_{k_1} and u_{k_2} . In case $j_1 < j_2$, the reduction in the cost of the flow is $\delta'((r_{j_1} - r_{j_2})(t_k - t_{k_1}) + \Delta_{j_1} + (r_0 - r_{j_2})(t_{k+1} - t_k))$. If $j_1 > j_2$, the reduction in the cost of the flow is $\delta'((r_0 - r_{j_1})(t_{k+1} - t_k) + (r_{j_2} - r_{j_1})(t_{k_2} - t_{k+1}) + \Delta_{j_1})$.

In any case, δ is reduced by δ' . The next claim ensures that the new flow is nested. One call of the procedure takes $O(n^2\sigma)$ time such that the rounding of all peaks can be accomplished in $O(n^3\sigma)$ time.

CLAIM 2. *The modified flow is nested.*

PROOF. Consider the flow before modification. First, assume that $j_1 = j_2$. By the choice of j_1 and j_2 , there exist no paths $P_j(k_1, k)$ or $P_j(k + 1, k_2)$, where $j > j_1 = j_2$ and $k_1 < k < k + 1 < k_2$, that route flow. Thus, the flow shipment along lower path j_1 preserves the nested structure of the flow.

Next, assume $j_1 < j_2$. We show that there exists no path $P_j(k_3, k_4)$ with $j_1 \leq j < j_2$ and $k_3 < k_1 < k_4$ that routes flow. This ensures that the routing of flow on $P_{j_2}(k_1, k_2)$ preserves the nested structure of the flow. First, consider $j = j_1$. If there were a path $P_{j_1}(k_3, k_4)$ with $k_3 < k_1 < k_4$ that routes flow, then $P_{j_1}(k_3, k)$ would also route flow, which contradicts the choice of k_1 . Next, consider any j with $j_1 < j < j_2$. We argue that edge (l_{j,k_1}, l_{j,k_1}^a) cannot route flow. Flow along this edge cannot be shipped to vertex $u_{k'}$ with $k_1 < k' < k$ because this would imply the existence of a path $P_j(k_3, k')$ with $k_3 \leq k_1$ that routes flow. This in turn would contradict the fact that the flow is nested. By the choice of j_1 , the flow cannot be routed to u_k . Hence, the flow must reach $l_{j,k+1}$. If $k + 1 = n$, then the flow cannot reach $l_{\sigma,n}$ at the end of the component C . Thus, the flow in C would not be feasible. Assume that $k + 1 < n$. The flow arriving at $l_{j,k+1}$ is not sent to u_{k+1} , because the flow is loop free. Hence, there must exist a path $P_j(k_3, k_4)$, where $k_3 \leq k_1 < k + 1 < k_4$, that routes flow. Considering that $P_{j_2}(k + 1, k_2)$ routes flow, the flow in C would not be nested. It follows that no edge (l_{j,k_1}, l_{j,k_1}^a) , $j_1 < j < j_2$, routes flow. Consequently, there exists no path $P_j(k_3, k_4)$ with $j_1 < j < j_2$ and $k_3 < k_1 < k_4$ that routes flow.

The analysis of $j_1 > j_2$ is similar. As in the previous paragraph, one can show that there exists no path $P_j(k_3, k_4)$ with $j_1 > j \geq j_2$ and $k_3 < k_2 < k_4$ that routes flow. Hence, the routing of flow on $P_{j_1}(k_1, k_2)$ observes the nested structure of the flow. For the case $j_1 > j > j_2$, one can easily argue that there exists no edge $(l_{j,k_2-1}, l_{j,k_2-1}^a)$ that carries flow. $\square$

Step 2.5: Flow reduction to m_c . It remains to reduce the flow to m_c on edges (u_k, u_{k+1}) where the flow after Steps 2.1 through 2.4 is higher. This can be done using the procedures that handle flow increases and peaks. Again, $\mathcal{A}_2$ scans the vertices u_k in order of increasing index k . Suppose that on the upper path the edges between u_1 and u_k already ship a flow of at most m_c . Assume that $f(u_k, u_{k+1}) > m_c$. If the flow does not decrease at u_{k+1} , then $\mathcal{A}_2$ executes the procedure that handles flow increases. The value δ is set to $\delta = f(u_k, u_{k+1}) - m_c$. If the flow decreases at u_{k+1} , then $\mathcal{A}_2$ first invokes the procedure that handles peaks. If $k + 1 = n$, then the reduction value is $\delta =$

$f(u_k, u_{k+1}) - m_c$. Otherwise, it is $\delta = f(u_k, u_{k+1}) - \max\{m_c, f(u_{k+1}, u_{k+2})\}$. If $f(u_{k+1}, u_{k+2}) > m_c$, then the algorithm then executes the procedure for handling flow increases with $\delta = f(u_k, u_{k+1}) - m_c$, where $f(u_k, u_{k+1})$ is the flow value after peak reduction. The running time is $O(n^3\sigma)$.

Step 3: Packing flow on the lower paths. Given the flow f^2 constructed in Step 2, $\mathcal{A}_2$ packs flow on the lower paths of the considered component C so that the final flow becomes integral. During the modification, the flow on the upper path of C does not change. Moreover, the cost of the flow will not increase.

Auxiliary edges. To separate flow that has already been made integral from the original one, we need auxiliary edges. For every edge e in C , except for those on the upper path, we add an auxiliary edge e' . More precisely, for every link $e = (v, w)$ not contained on the upper path, there is the original edge and a new auxiliary edge. Hence, during the flow packing operation, component C is a multigraph.

Initially, the flow f^2 is routed on the upper path and the original edges of the lower paths. In a series of rounds, $\mathcal{A}_2$ removes flow from the original edges, packs it, and adds it to the auxiliary edges. On the auxiliary edges, the flow is always integral. The process ends when there is no flow on the original edges. Then the original edges are removed so that, for each edge, there is only one copy.

We observe that since f^2 is integral on the upper path and loop free, only integral amounts of flow enter/leave the upper path from/to the lower paths. At every vertex u_k where the flow on the upper path increases, an integral amount of flow is routed into u_k from lower paths. At every vertex u_k where the flow decreases, an integral amount of flow is sent from u_k to lower paths. This invariant will be maintained at all times during the flow transformation. Moreover, the flow will always be nested and loop free, considering the combined flow on the original and auxiliary edges. Last but not least, the flow will always be feasible. The flow conservation law is observed. For every edge, the total flow on the original and auxiliary copy is at most τm_c .

Matching pairs. The flow packing procedure uses the notion of a *matching pair*. A matching pair consists of two vertices u_k and $u_{k'}$, $1 < k < k' < n$, with the following properties: (a) Flow is routed from u_k to the lower paths on original edges $(u_k, l_{j,k})$, $1 \leq j \leq \sigma$; (b) flow is routed into $u_{k'}$ from lower paths on original edges $(l_{j,k'}, u_{k'})$, $1 \leq j \leq \sigma$; and (c) there exists no vertex $u_{k''}$ with $k < k'' < k'$ that satisfies (a) or (b). While there exists a matching pair, $\mathcal{A}_2$ executes the following flow packing routine. Let f denote the current flow. Unless stated otherwise, $f(e)$ always refers to the flow on the original copy of e .

Packing procedure. Let u_k and $u_{k'}$ be the given matching pair. Let δ_k be the total amount of flow routed from u_k to lower paths on original edges $(u_k, l_{j,k})$, $1 \leq j \leq \sigma$. Similarly, let $\delta_{k'}$ be the total amount of flow shipped into $u_{k'}$ from lower paths along original edges $(l_{j,k'}, u_{k'})$, $1 \leq j \leq \sigma$. Both δ_k and $\delta_{k'}$ are integral. If $\delta_k \leq \delta_{k'}$, then let J be the set of integers j with $1 \leq j \leq \sigma$ such that $f(u_k, l_{j,k}) > 0$. Define $\delta_j = f(u_k, l_{j,k})$, for any $j \in J$. We have $\sum_{j \in J} \delta_j = \delta_k$. If $\delta_k > \delta_{k'}$, then let J be the set of integers j with $1 \leq j \leq \sigma$ such that $f(l_{j,k'}, u_{k'}) > 0$. Define $\delta_j = f(l_{j,k'}, u_{k'})$, for any $j \in J$. There holds $\sum_{j \in J} \delta_j = \delta_{k'}$. In any case, for $j \in J$, consider the path $P_j(k, k')$ from u_k to $u_{k'}$ along lower path j . We say that $P_j(k, k')$ routes δ units of flow if, for every original edge of $P_j(k, k')$, the amount of flow is at least δ . Claim 3 states that $P_j(k, k')$ routes δ_j units of flow, for every $j \in J$.

The procedure for packing flow works as follows. It determines the integer $j' \in J$ such that the total edge cost of $P_{j'}(k, k')$ is minimal among $P_j(k, k')$ with $j \in J$. Then, for every $j \in J$, it removes δ_j units of flow from the original edges of $P_j(k, k')$. Finally, it routes $\min\{\delta_k, \delta_{k'}\}$ units of flow on the new edges of $P_{j'}(k, k')$. The new flow remains nested because an already existing routing path with positive flow is selected. The cost does not increase because the total edge cost of $P_{j'}(k, k')$ is upper bounded by that of any $P_j(k, k')$, $j \in J$.

Finding a matching pair takes $O(n)$ time. An execution of the preceding procedure requires $O(\sigma)$ time. Every time the procedure is invoked for a matching pair u_k and $u_{k'}$, the flow leaving u_k or entering $u_{k'}$ on original edges drops to 0. Thus, the total time to pack flow using the preceding procedure takes $O(n^2\sigma)$ time. Consider the point in time when there exists no matching pair anymore. At every vertex $u_{k'}$ where the flow increases by injections from original edges, the flow is routed on path $P_\sigma(1, k')$. At every vertex u_k where the flow decreases by shipment along original edges, the flow is routed on path $P_\sigma(k, n)$. The flow on any of these paths is integral because at $u_{k'}$ and u_k , the increase/decrease is integral. Thus, the remaining flow on original edges can be transferred without modification to the auxiliary edges. In summary, Step 3 takes $O(n^2\sigma)$ time.

CLAIM 3. *For every $j \in J$, path $P_j(k, k')$ routes δ_j units of flow.*

PROOF. First, assume that $\delta_k \leq \delta_{k'}$. Suppose that the statement of the claim does not hold. We derive a contradiction. In the following, we always analyze flow on original edges. By the definition of a matching pair, the flow on $(u_k, l_{j,k})$, which then traverses lower path j , is not shipped to any vertex between u_k and $u_{k'}$ on the upper path. Hence, it reaches $l_{j,k'}$. Let $j_1 \in J$ be the smallest integer such that less than $f(u_k, l_{j_1,k})$ units of flow are sent along $f(l_{j_1,k'}, u_{k'})$. Hence, some flow entering $l_{j_1,k'}$ must continue on lower path j_1 . This implies the existence of a path $P_{j_1}(k, k_1)$, with $k' < k_1$, that routes flow. Considering that a total of at least δ_k units of flow are shipped to $u_{k'}$, some flow must be routed from a vertex u_{k_2} with $k_2 < k$, or from $l_{\sigma,1}$ at the beginning of the component. Hence, there must exist a path $P_j(k_2, k')$, $j \neq j_1$, that routes flow. We obtain a contradiction to the fact that the flow is nested because $P_{j_1}(k, k_1)$ ships flow and $k_2 < k < k' < k_1$. The case $\delta_k > \delta_{k'}$ is analogous. $\square$

4.4 Construction of the Schedule

Let f^3 denote the flow obtained in Step 3. We need two lemmas. The following Lemma 4.6 relates f^3 to schedules with τm_i servers of type i , $1 \leq i \leq \tau$.

LEMMA 4.6. *Flow f^3 corresponds to a schedule Σ with τm_i servers of type i , $1 \leq i \leq \tau$. In $[t_k, t_{k+1})$, exactly $d_{i,k} = \min\{m_i, \lfloor \tau f^*(u_{i,k}, u_{i,k+1}) \rfloor\}$ servers of type i are in the active state, $1 \leq i \leq \tau$ and $1 \leq k \leq n-1$. The energy consumed by the servers of type i is equal the cost of f^3 in C_i , $1 \leq i \leq \tau$.*

PROOF. For any component C_i of $\mathcal{N}(\mathcal{I})$, $1 \leq i \leq \tau$, consider the flow f^3 . It ships τm_i flow units through C_i . Along edge $(u_{i,k}, u_{i,k+1})$, exactly $d_{i,k}$ units of flow are routed. Considering that the flow is integral, it can be decomposed into τm_i unit flows/paths through C_i : Start at vertex $l_{i,\sigma_i,1}$. At any vertex v , traverse an arbitrary outgoing edge $e = (v, w)$ with positive flow and remove one unit from it. A path ends at $l_{i,\sigma_i,n}$. Each of these paths corresponds to a schedule for a server of type i . The energy consumed by the server is equal to the cost incurred by the flow unit/path. Considering that $d_{i,k}$ units of flow are sent along $(u_{i,k}, u_{i,k+1})$, $d_{i,k}$ servers are in the active state during $[t_k, t_{k+1})$, $1 \leq k \leq n-1$. $\square$

The next Lemma 4.7 allows us to reduce a schedule with τm_i servers of type i , where $1 \leq i \leq \tau$, to one with only m_i servers of this type. At any time, the lowest numbered servers are in the active state.

LEMMA 4.7. *Let Σ_i be a schedule for τm_i servers of type i in which exactly $d_{i,k}$ servers are in the active state during $[t_k, t_{k+1})$, where $d_{i,k} \leq m_i$ and $1 \leq k \leq n-1$. Then there exists a schedule Σ'_i for m_i servers of type i in which the servers numbered 1 to $d_{i,k}$ are in the active state during $[t_k, t_{k+1})$. The energy consumed by Σ'_i is upper bounded by that of Σ_i .*

PROOF. Given Σ_i , we modify it such that, among the τm_i servers, those numbered 1 to $d_{i,k}$ are active in $[t_k, t_{k+1})$, for $k = 1, \dots, n-1$. The modifications never increase the consumed energy. Let

$[t_k, t_{k+1})$ be the first interval such that at least one server is active. Determine the set I of server numbers such that the respective servers reside in the active state during $[t_k, t_{k+1})$. Construct an arbitrary one-to-one mapping π between $\{1, \dots, d_{i,k}\}$ and I . In Σ_i , replace the server numbered $\pi(h)$ by that numbered h and vice versa, for any $h \in I$. More specifically, in the entire schedule, server h assumes the states and performs the state transitions of server $\pi(h)$ and vice versa, for any $h \in I$. These swaps are just renaming operations that do not affect the energy consumed by the schedule.

Let Σ_i be a schedule for τm_i servers such that during any interval $[t_{k'}, t_{k'+1})$, the servers numbered 1 to $d_{i,k'}$ are in the active state, for $k' = 1, \dots, k-1$. We assume that in low-power mode, a server never changes state. Otherwise, we can change schedule Σ_i accordingly, without increasing its cost; cf. Proposition 2.1. We consider interval $[t_k, t_{k+1})$. If the servers residing in the active state are not those numbered 1 to $d_{i,k}$, then let $h \geq 1$ be the smallest integer such that server h is not active. Let h' be the highest server number such that the respective server is active. Let $s_{i,j}$ be the low-power state of server h at time t_k . We distinguish two cases.

Suppose that server h was active during $[t_{k-1}, t_k)$. If server h' was also active during $[t_{k-1}, t_k)$, then we change the schedule in the following way. At time t_k , server h remains active and h' transitions to low-power state $s_{i,j}$. If server h' was not active during $[t_{k-1}, t_k)$, then let $s_{i,j'}$ be the state of the server during that interval. If $j' \geq j$, server h' simply remains in state $s_{i,j'}$ at time t_k . Otherwise, let t' be the most recent time when server h' transitioned to $s_{i,j'}$. We modify the schedule such that server h' transitions to state $s_{i,j}$ at time t' . Starting at time t_k , we interchange the roles of servers h and h' in the rest of the schedule.

Next, assume that server h was not active during $[t_{k-1}, t_k)$. Then by induction hypothesis, server h' was not active either. Let $s_{i,j'}$ be the state of the server h' during $[t_{k-1}, t_k)$. We change the schedule as follows. At time t_k , server h powers up while server h' remains in low-power mode. If $j > j'$, we additionally have to change low-power states. Again, let t' be the most recent time when server h' transitioned to $s_{i,j'}$. Similarly, let t be the most recent time when server h powered down. By induction hypothesis, $t' \leq t$. If $j > j'$, then at time t' server h' powers down to $s_{i,j}$. At time t , server h transitions to $s_{i,j'}$.

In the schedule obtained after the preceding transformation, the highest numbered $(\tau - 1)m_i$ servers are never active. Hence, we can simply remove them; the consumed energy can only decrease. $\square$

Given the integral flow f^3 , algorithm $\mathcal{A}_2$ constructs a feasible schedule Σ^* for $\mathcal{I}$. For each server type $\mathcal{S}_i$, $1 \leq i \leq \tau$, $\mathcal{A}_2$ builds an optimal schedule Σ_i^* such that $d_{i,k}$ of the m_i servers in $\mathcal{S}_i$ are in the active state during $[t_k, t_{k+1})$, $1 \leq k \leq n-1$. These schedules $\Sigma_1^*, \dots, \Sigma_\tau^*$ are then combined to form Σ^* . More specifically, consider any i with $1 \leq i \leq \tau$. In a first step, given Lemma 4.7, Σ_i^* just specifies that the servers numbered 1 to $d_{i,k}$ are in the active state during $[t_k, t_{k+1})$, for any $1 \leq k \leq n-1$. Then, while a server is not required to be active according to the specification, $\mathcal{A}_2$ selects an optimal state. Suppose that at time t_k the number of required servers decreases, i.e., $d_{i,k-1} > d_{i,k}$. Algorithm $\mathcal{A}_2$ determines states for $d_{i,k-1} - d_{i,k}$ servers that may power down. This is done as follows. Initially, $\mu := d_{i,k}$. While $\mu < d_{i,k-1}$, $\mathcal{A}_2$ finds the next time $t_{k'}$ such that $d_{i,k'} \geq \mu + 1$, i.e., at least $\mu + 1$ servers are active. It chooses an optimal state to be assumed by servers numbered $\mu + 1, \dots, \min\{d_{i,k-1}, d_{i,k'}\}$ at time t_k . This is the state s_{i,j^*} with $j^* = \arg \min_{1 \leq j \leq \sigma_i} \{r_{i,j}(t_{k'} - t_k) + \Delta_{i,j}\}$. Then $\mu := \min\{d_{i,k-1}, d_{i,k'}\}$.

By Lemmas 4.6 and 4.7, the energy consumed by Σ_i^* is upper bounded by the cost incurred by f^3 in component C_i . Thus, the energy consumed by the combined schedule Σ^* is at most $\text{cost}(f^3) \leq \tau \text{cost}(f^*)$. Schedule Σ^* is feasible because, by Lemma 4.4, $\sum_{i=1}^{\tau} d_{i,k} \geq d_k$. The proof of Theorem 4.1 is complete.

5 CONCLUSION

In this article, we have introduced the problem DPM that models an important energy and capacity management problem in data centers. For the scenario that each server has exactly two states—an active state and a sleep state—we have developed a combinatorial, polynomial time algorithm for computing optimal schedules. For the setting where each server has an arbitrary number of states, we have devised a τ -approximation algorithm, where τ is the number of server types.

There are several interesting directions for future research. A challenging problem is to settle the complexity of DPM if each server has an arbitrary number of states. We conjecture that the problem is NP-hard but have not been successful in developing a proof. Another working direction is to improve the approximation factor of τ , assuming that the problem is indeed NP-hard. Otherwise, a polynomial time algorithm for computing optimal schedules has to be developed. Furthermore, it is sensible to study the online scenario where the computing demands are revealed over time.

In our problem DPM, the servers have equal computation power. It is interesting to explore a refined setting in which the servers have different computation power, which is related to their energy consumption. Specifically, if a server has a high computation power, then its energy consumption rate in the active state is high. Last but not least, in Section 1, we mentioned multicore architectures as a possible application of DPM. In these environments, the processors often have several active states. It would be interesting to model and algorithmically investigate such settings as well.

APPENDIX

PROOF OF PROPOSITION 2.1. Consider any schedule. Let $[t, t')$ be a time interval such that S_i powers down at time t and is in low-power mode until time t' . Let j , $1 \leq j \leq \sigma_i$, be the largest integer such that S_i visits $s_{i,j}$ between t and t' . Considering that the state transition energies satisfy the triangle inequality, the total energy needed to transition from the active state to $s_{i,j}$ is at least $\Delta_{i,0,j}$. The total energy consumed in transitioning from $s_{i,j}$ to the active state is at least $\Delta_{i,j,0}$. The server's states are numbered in order of decreasing power consumption rates. Hence, between t and t' , S_i consumes the least energy if it always resides in $s_{i,j}$ and transitions between the active state and $s_{i,j}$ at times t and t' , respectively. $\square$

PROOF OF PROPOSITION 2.2. Let Σ be any optimal schedule that fulfills the property of Proposition 2.1. Suppose that it performs state transitions at times not contained in T . We show how to repeatedly modify Σ so that it satisfies the properties of Proposition 2.2. Let $t \in [t_1, t_n)$ be the earliest time such that $t \notin T$ and a server S_i performs a state transition at this time. Assume that $t \in (t_k, t_{k+1})$, for some $1 \leq k \leq n - 1$.

In a first step, we argue that if S_i powers up at time t , then there must be another server that powers down at time t . Analogously, if S_i powers down at time t , then there must be a server that powers up at time t . First, assume that S_i powers up and that there is no other server that powers down at time t . Let t' be the earliest time such that $t' > t$ and a state transition is performed at time t' . Schedule Σ meets the computing demand during $[t_k, t)$, which do not change until t_{k+1} . Furthermore, no server powers down during (t, t') . Hence, at time t , server S_i can remain in its current low-power state until time $\min\{t_{k+1}, t'\}$ and then power up. The modified schedule is feasible and has a strictly smaller energy consumption than Σ . This contradicts the fact that Σ is optimal. Next, suppose that S_i powers down and that no other server powers up at time t . Schedule Σ meets the computing demand at time t , which is identical to that in $[t_k, t)$. By the choice of t , no state transitions occur during (t_k, t) . Thus, S_i can power down already at time t_k . The resulting schedule is feasible and consumes less energy than Σ .

Therefore, suppose that S_i powers up and that some server $S_{i'}$ powers down at time t . Let $s_{i,j}$, with $1 \leq j \leq \sigma_i$, be the state in which S_i resides just before powering up at time t . Let $s_{i',j'}$, $1 \leq j' \leq \sigma_{i'}$, be the state to which $S_{i'}$ transitions at time t . A first observation is that $r_{i,0} - r_{i,j} = r_{i',0} - r_{i',j'}$. If $r_{i,0} - r_{i,j} < r_{i',0} - r_{i',j'}$ held true, then S_i and $S_{i'}$ could perform their state transitions already at time $t - \delta$, where $\delta > 0$ is chosen such that $t_k \leq t - \delta$. The energy consumed by the new schedule is equal to that of the original one plus $\delta(r_{i,0} - r_{i,j} + r_{i',j'} - r_{i',0})$. The last expression is smaller than 0. If $r_{i,0} - r_{i,j} > r_{i',0} - r_{i',j'}$ held true, then S_i and $S_{i'}$ could postpone their state transitions to time $t + \delta$, where $\delta > 0$ is chosen such that neither S_i nor $S_{i'}$ performs a state transition during $(t, t + \delta)$. The energy consumption of the new schedule is equal to that of the original one plus $\delta(r_{i',0} - r_{i',j'} + r_{i,j} - r_{i,0})$. Again, the last expression is smaller than 0.

Let $t(i)$ be the earliest time such that $t(i) > t$ and S_i powers down. Similarly, let $t(i') > t$ be the earliest time such that $S_{i'}$ powers up. If $S_{i'}$ does not power up again, let $t(i') = t_n$. We modify Σ . First, assume that $t(i) > t(i')$. In this case, at time t , server S_i remains in $s_{i,j}$ until $t(i')$ and then powers up. Server $S_{i'}$ remains in the active state until $t(i')$. Considering that $r_{i,0} - r_{i,j} = r_{i',0} - r_{i',j'}$ and $S_{i'}$ does not perform state transitions at times t and $t(i')$, the energy consumption of the modified schedule cannot be larger than that of the original one.

Next, assume that $t(i) \leq t(i')$. The schedule modifications are as follows. Server $S_{i'}$ remains in the active state until time $t(i)$. It then powers down to $s_{i',j'}$ if $t(i) < t(i')$ or if $t(i) = t(i') = t_n$. Let $s_{i,j''}$, with $1 \leq j'' \leq \sigma_i$, be the state selected by S_i at time $t(i)$. If $j \geq j''$, then S_i remains in state $s_{i,j}$ at time t until the next time S_i powers up, if such a time exists. If $j < j''$, then at the most recent time at which S_i transitioned to $s_{i,j}$, it selects $s_{i,j''}$ instead. It remains in this state until the next powering up. The total energy consumed by S_i and $S_{i'}$ while residing in the various states does not increase. For the interval from t to $t(i)$, this holds true because, again, $r_{i,0} - r_{i,j} = r_{i',0} - r_{i',j'}$ and $r_{i,j''} < r_{i,j}$ if $j < j''$. In fact, if $j < j''$, then an energy of $r_{i,j} - r_{i,j''}$ is saved during time units prior to $t(i)$. If $j > j''$, then $r_{i,j} < r_{i,j''}$ and an energy of $r_{i,j''} - r_{i,j}$ is saved during time units after $t(i)$. The energy incurred by state transitions does not increase either. For $S_{i'}$, this is obvious. If $S_{i'}$ originally performed a power-up operation at time $t(i)$, then the corresponding energy is saved in the modified schedule. Server S_i does not perform state transitions at times t and $t(i)$. Hence, if $j \geq j''$, then an energy of $\Delta_{i,j'',0}$ is saved. If $j < j''$, then an energy of $\Delta_{i,j,0}$ is saved.

By repeating the modifications described in the last paragraph, we obtain a schedule in which no state transitions occur at time t . State transitions performed at times $t' < t$ still satisfy $t' \in T$. Handling further times $t \notin T$ with state transitions in the same way, we obtain a final schedule with the desired properties. $\square$

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