Ubiquity of the Exponent of Matrix Multiplication

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ABSTRACT

The asymptotic exponent of matrix multiplication is the smallest ω such that one may multiply two $n \times n$ matrices or invert an $n \times n$ matrix in $O(n^{\omega+\varepsilon})$ -complexity for $\varepsilon > 0$ arbitrarily small. One of the biggest open problem in complexity theory and numerical linear algebra is its conjectured value $\omega = 2$. This article is about the universality of ω . We will show that ω is not only the asymptotic exponent for the product operation in matrix algebras but also that for various infinite families of Lie algebras, Jordan algebras, and Clifford algebras. In addition, we will show that ω is not just the asymptotic exponent for matrix product and inversion but also that for the evaluation of any matrix-valued polynomial and rational functions of matrix variables.

CCS CONCEPTS

• Theory of computation $\rightarrow$ Algebraic complexity theory.

KEYWORDS

Matrix multiplication, bilinear complexity, Lie algebras, Jordan algebras, Clifford algebras, matrix polynomials

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1 INTRODUCTION

Fifty years ago, Strassen announced a surprising algorithm that gives the product of two $n \times n$ matrices in $O(n^{\log_2 7})$ multiplications [17]. He also showed that the number of additions may be bounded by a constant factor of the number of multiplications and that an $n \times n$ linear system can be solved in the same time complexity it takes to multiply two matrices. The best possible exponent ω so that one may multiply two $n \times n$ matrices in $O(n^{\omega+\varepsilon})$ -time for any $\varepsilon > 0$ is now known as the asymptotic *exponent of matrix multiplication* (the extra ε term is so that one may account for algorithms running in, say, $O(n^\omega \log n)$ -time).

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Strassen's upper bound for ω , i.e., $\log_2 7 \approx 2.8074$, has been improved over the years. Roughly speaking, there are three methods to sharpen the upper bound. The first is the line of argument introduced by Strassen in his *laser method* and generalized in [4, 16]. Coppersmith and Winograd employed it to improve the upper bound to 2.3755 [10], a record that held for twenty five years until it was improved by Vassilevska Williams [19] and Le Gall [14] to 2.373. Nevertheless, it is now known [1] that the laser method has its limitations and cannot yield an upper bound for ω that is better than 2.3. The second approach, based on algebraic geometry and pioneered by Landsberg about fifteen years ago [11] provides powerful techniques for obtaining *lower bounds* for the complexity of matrix multiplication [3, 6, 7, 11–13]. The third approach is the representation theoretic framework pioneered by Cohn and Umans [8, 9, 18] by embedding matrices into a group algebra of a suitably chosen group and using group representation theory to study the complexity of matrix multiplication.

The results in this article may be viewed as a fourth approach. We show that ω is the asymptotic exponent for the product operations in many different types of algebras including Lie, Jordan, and Clifford algebras. Consequently, the complexity of the product operation in any of these algebras provides an alternative route towards determining or bounding the value of ω . Furthermore, we show that the asymptotic complexity of evaluating univariate or multivariate matrix-valued functions of matrices is also given by ω , providing yet another route towards obtaining its value.

This article is an extended abstract of [15], intended only to highlight a few key results therein. The reader will find the proofs of these results as well as a far more extensive discussion in [15].

2 BILINEAR AND MULTIPLICATIVE COMPLEXITY

Let U, V, W be vector spaces over a field $\mathbb{F}$ and let $\beta : U \times V \rightarrow W$ be a bilinear map. The *bilinear complexity* of β is defined to be the minimum number of multiplications over $\mathbb{F}$ required to evaluate β . Equivalently, the bilinear complexity of β is the *rank* of its structure tensor $T_\beta \in U^* \otimes V^* \otimes W$, which is denoted by $\text{rank}(T_\beta)$ and is defined to be the minimal r such that

$$T_\beta = \sum_{j=1}^r u_j^* \otimes v_j^* \otimes w_j, \quad u_j^* \in U^*, v_j^* \in V^*, w_j \in W, j = 1, \dots, r.$$

We refer readers to [5, 20] for more information. Let

$$S := \{(U_n, V_n, W_n, \beta_n) : n \in \mathbb{N}\} \quad (1)$$

be a sequence of bilinear maps $\beta_n : U_n \times V_n \rightarrow W_n$. We define the *asymptotic exponent* of S to be

$$\omega(S) := \liminf_{n \rightarrow \infty} \log_n(\text{rank}(T_{\beta_n})). \quad (2)$$

For the special case when $U_n = V_n = W_n$ for all $n \in \mathbb{N}$, we will just write

$$S = \{(V_n, \beta_n) : n \in \mathbb{N}\}$$

instead of the more cumbersome $S = \{(V_n, V_n, V_n, \beta_n) : n \in \mathbb{N}\}$ as in (1). This will be case when V_n is an algebra and $\beta_n : V_n \times V_n \rightarrow V_n$ the product operation of the algebra.

Before going further we shall look at two examples. Let

$$M(\mathbb{F}) := \{(\mathbb{F}^{n \times n}, \mu_n) : n \in \mathbb{N}\}, \quad (3)$$

where $\mathbb{F}^{n \times n}$ is the vector space of $n \times n$ matrices over $\mathbb{F}$ and μ_n is the matrix multiplication map sending a pair of matrices (X, Y) to their product XY . Then $\omega(M(\mathbb{F}))$ is exactly the exponent of matrix multiplication ω described in Section 1.

Take another example where the matrix product in (3) is replaced by commutator product $[X, Y] = XY - YX$,

$$\mathfrak{gl}(\mathbb{F}) := \{(\mathbb{F}^{n \times n}, [\cdot, \cdot]) : n \in \mathbb{N}\}.$$

Note that previously we have $M_n(\mathbb{F})$, the usual matrix algebra, i.e., the set $\mathbb{F}^{n \times n}$ equipped with standard matrix multiplication, but here we have $\mathfrak{gl}_n(\mathbb{F})$, i.e., the set $\mathbb{F}^{n \times n}$ equipped with commutator product. In this case $\mathfrak{gl}_n(\mathbb{F})$ is the Lie algebra of the general linear group $GL_n(\mathbb{F})$. These two products are of a very different nature — for example, the commutator product is not associative but it satisfies the Jacobi identity.

It is easy to see that $\omega(\mathfrak{gl}(\mathbb{F})) \leq \omega(M(\mathbb{F})) = \omega$ and, with a bit more work, that equality holds.

PROPOSITION 2.1. $\omega(\mathfrak{gl}(\mathbb{F})) = \omega(M(\mathbb{F})) = \omega$.

We will see in Section 4 that Proposition 2.1 extends to all infinite families of semisimple Lie algebras $\mathfrak{sl}_{n+1}(\mathbb{F})$, $\mathfrak{so}_{2n+1}(\mathbb{F})$, $\mathfrak{sp}_{2n}(\mathbb{F})$, $\mathfrak{so}_{2n}(\mathbb{F})$ equipped with the commutator product.

In order to discuss operations that are not bilinear, for example, evaluations of functions like $\mathbb{F}^{n \times n} \rightarrow \mathbb{F}^{n \times n}$, $X \mapsto X^2$, we require a notion similar to bilinear complexity. Let U, V be vector spaces over $\mathbb{F}$ and let $\varphi : U \rightarrow V$ be a map. The *multiplicative complexity* [5] of φ is defined to be the minimal number of multiplications and divisions over $\mathbb{F}$ required to evaluate φ and is denoted by $L(\varphi)$. We set $L(\varphi) = \infty$ if φ cannot be evaluated with just field arithmetic, i.e., additions, subtractions, multiplications, divisions. As in the case of bilinear maps, let

$$T := \{(U_n, V_n, \varphi_n) : n \in \mathbb{N}\}$$

be a sequence of maps $\varphi_n : U_n \rightarrow V_n$, we define its *asymptotic exponent* to be

$$L(T) := \liminf_{n \rightarrow \infty} \log_n(L(\varphi_n)). \quad (4)$$

While in general the multiplicative complexity of a bilinear map will not be the same as its bilinear complexity, we will see next that, in an appropriate sense, they are equivalent asymptotically. So there is no ambiguity in giving the same name to (2) and (4).

3 ASYMPTOTIC EXPONENTS

A bilinear map $\beta : U \times V \rightarrow W$ may also be regarded as a map $\tilde{\beta} : U \oplus V \rightarrow W$. In general, $\omega(\beta) \neq L(\tilde{\beta})$. Nevertheless, given any sequence $S = \{(U_n, V_n, W_n, \beta_n) : n \in \mathbb{N}\}$ of bilinear maps and its corresponding sequence $\tilde{S} = \{(U_n \oplus V_n, W_n, \tilde{\beta}_n) : n \in \mathbb{N}\}$ of maps, the following always holds [15].

PROPOSITION 3.1. *Let $S, \tilde{S}$ be as above. Then $\omega(S) = L(\tilde{S})$.*

Let $\mathbb{L}$ be a finite extension of the field $\mathbb{F}$. If S is a sequence of bilinear maps over $\mathbb{L}$, then both $\omega_{\mathbb{F}}(S)$ and $\omega_{\mathbb{L}}(S)$ are well-defined. Fortunately, they are also unambiguous.

PROPOSITION 3.2. *If $\dim_{\mathbb{F}} \mathbb{L} < \infty$, then $\omega_{\mathbb{F}}(S) = \omega_{\mathbb{L}}(S)$.*

For a sequence $S = \{(U_n, V_n, W_n, \beta_n) : n \in \mathbb{N}\}$ of bilinear maps over $\mathbb{F}$, we define its $\mathbb{L}$ -extension by

$$S_{\mathbb{L}} := \{(U_n \otimes_{\mathbb{F}} \mathbb{L}, V_n \otimes_{\mathbb{F}} \mathbb{L}, W_n \otimes_{\mathbb{F}} \mathbb{L}, \beta_n^{\mathbb{L}}) : n \in \mathbb{N}\},$$

where the $\mathbb{L}$ -bilinear map $\beta_n^{\mathbb{L}} : (U_n \otimes_{\mathbb{F}} \mathbb{L}) \times (V_n \otimes_{\mathbb{F}} \mathbb{L}) \rightarrow (W_n \otimes_{\mathbb{F}} \mathbb{L})$ is the natural extension of the $\mathbb{F}$ -bilinear map $\beta_n : U_n \times V_n \rightarrow W_n$.

PROPOSITION 3.3. *If $\dim_{\mathbb{F}} \mathbb{L} < \infty$, then $\omega_{\mathbb{L}}(S_{\mathbb{L}}) = \omega_{\mathbb{F}}(S)$.*

As a consequence, we obtain the following:

COROLLARY 3.4. (i) *For any sequence of bilinear maps S over $\mathbb{C}$,*

$$\omega_{\mathbb{C}}(S) = \omega_{\mathbb{R}}(S).$$

(ii) *For any sequence of bilinear maps S over $\mathbb{R}$,*

$$\omega_{\mathbb{C}}(S \otimes_{\mathbb{R}} \mathbb{C}) = \omega_{\mathbb{R}}(S).$$

By Corollary 3.4, the exponent of matrix multiplication over $\mathbb{R}$ and over $\mathbb{C}$ are the same, i.e., $\omega(M(\mathbb{R})) = \omega(M(\mathbb{C}))$ and we may write ω without ambiguity.

4 ASYMPTOTIC EXPONENTS OF ALGEBRAS

We will now look into the asymptotic exponents of the products in various infinite families of algebras. The bottom line is that under some very mild conditions, they are all equal to ω . In this and the next section, the field $\mathbb{F}$ will either be $\mathbb{R}$ or $\mathbb{C}$.

4.1 Lie algebras

A *Lie algebra* is a vector space $\mathfrak{g}$ over a field $\mathbb{F}$ equipped with a skew symmetric bilinear map called the *Lie bracket*,

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g},$$

that satisfies the Jacobi identity:

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0, \quad x, y, z \in \mathfrak{g}.$$

We say a Lie algebra is *simple* if it does not contain any nontrivial ideal and *semisimple* if it is a direct sum of simple Lie algebras. For example, the vector space $\mathfrak{so}_n(\mathbb{R})$ consisting of $n \times n$ skew-symmetric matrices, equipped with the commutator product $[A, B] = AB - BA$ is a simple Lie algebra; on the other hand, $\mathfrak{gl}_n(\mathbb{R})$, as described in Section 2, is a semisimple Lie algebra.

THEOREM 4.1 (LIE ALGEBRAS). *For each $n \in \mathbb{N}$, let $\mathfrak{g}_n$ be a semisimple Lie algebra over $\mathbb{F}$ with $\dim \mathfrak{g}_n \leq cn^2$ for some constant $c > 0$. Then for the sequence of Lie brackets*

$$\mathfrak{g}(\mathbb{F}) = \{(\mathfrak{g}_n, [\cdot, \cdot]) : n \in \mathbb{N}\},$$

we have

$$\omega(\mathfrak{g}(\mathbb{F})) \leq \omega,$$

with equality as long as $\mathfrak{g}_n$ is not a direct sum of exceptional Lie algebras for sufficiently large n .

It follows that the Lie brackets given by the commutator product $[A, B] = AB - BA$ on the Lie algebras $\mathfrak{gl}_n(\mathbb{R})$, $\mathfrak{sl}_{n+1}(\mathbb{F})$, $\mathfrak{so}_{2n+1}(\mathbb{F})$, $\mathfrak{sp}_{2n}(\mathbb{F})$, $\mathfrak{so}_{2n}(\mathbb{F})$ all have the same asymptotic complexity as matrix multiplication.

4.2 Jordan algebras

A *Jordan algebra* is a symmetric analogue of a Lie algebra, and is defined to be a nonassociative algebra $\mathcal{J}$ satisfying

$$xy = yx, \quad (xy)x^2 = x(yx^2), \quad x, y \in \mathcal{J}.$$

Similar to Lie algebras, a Jordan algebra is called *simple* if it does not have a nontrivial ideal and *semisimple* if it can be decomposed as a direct sum of simple Jordan algebras. Moreover, a Jordan algebra $\mathcal{J}$ is said to be *formally real* if

$$x_1^2 + \dots + x_n^2 = 0 \implies x_1 = \dots = x_n = 0$$

for any positive integer n and $x_1, \dots, x_n \in \mathcal{J}$. For example, the vector space $\text{Sym}_n(\mathbb{R})$ (resp. $\text{Herm}_n(\mathbb{C})$) of $n \times n$ symmetric (resp. Hermitian) matrices equipped with the Jordan product $A \circ B := (AB + BA)/2$ is a formally real simple Jordan algebra.

THEOREM 4.2 (JORDAN ALGEBRAS). *For each $n \in \mathbb{N}$, let $\mathcal{J}_n$ be a Jordan algebra that is either semisimple over $\mathbb{C}$ or formally real over $\mathbb{R}$ with the following properties:*

- (i) *there exists constant $c_1 > 0$ such that for sufficiently large n , $\dim_{\mathbb{R}}(\mathcal{J}_n) \leq c_1 n^2$;*
- (ii) *there exist constants $0 < c_2 < c_3$ such that for sufficiently large n , $\mathcal{J}_n$ contains at least one simple ideal $\mathcal{I}_n$ where $c_2 n^2 \leq \dim_{\mathbb{R}}(\mathcal{I}_n) \leq c_3 n^2$.*

Then for the sequence of Jordan products

$$\mathcal{J}(\mathbb{F}) = \{(\mathcal{J}_n, \circ) : n \in \mathbb{N}\},$$

we have

$$\omega(\mathcal{J}(\mathbb{F})) = \omega.$$

It follows that the Jordan product $A \circ B = (AB + BA)/2$ on $\text{Sym}_n(\mathbb{R})$ or $\text{Herm}_n(\mathbb{C})$ has the same asymptotic complexity as matrix multiplication.

4.3 Clifford algebras

A *Clifford algebra* $\text{Cl}(V, q)$ is an algebra associated with a vector space V and a quadratic form q on V , defined by

$$\text{Cl}(V, q) := \mathcal{T}(V)/\mathcal{I}_q,$$

where $\mathcal{T}(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n}$ is the tensor algebra of V and $\mathcal{I}_q$ is the ideal generated by elements of the form $v \otimes v - q(v)$, $v \in V$. The most common example of Clifford algebras [2] is $\text{Cliff}(n) := \text{Cl}(\mathbb{R}^n, q)$ where

$$q(x) := x_1^2 + \dots + x_n^2$$

for any $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. Then $\text{Cliff}(n)$ is a real vector space of dimension 2^n , equipped with a noncommutative product $x \cdot y$ for any $x, y \in \mathbb{R}^n$ that satisfies

$$x \cdot x = \sum_{j=1}^n x_j^2.$$

In this case, we have

$$\begin{aligned} \text{Cliff}(0) &= \mathbb{R}, & \text{Cliff}(1) &= \mathbb{C}, & \text{Cliff}(2) &= \mathbb{H}, \\ \text{Cliff}(3) &= \mathbb{H}^2, & \text{Cliff}(4) &= \mathbb{H}^{2 \times 2}, & \text{Cliff}(5) &= \mathbb{C}^{4 \times 4}, \\ \text{Cliff}(6) &= \mathbb{R}^{8 \times 8}, & \text{Cliff}(7) &= \mathbb{R}^{8 \times 8} \oplus \mathbb{R}^{8 \times 8}, \end{aligned}$$

and the recursion $\text{Cliff}(n+8) = \text{Cliff}(n) \otimes \mathbb{R}^{16 \times 16}$ determines all subsequent $\text{Cliff}(n)$, $n \geq 8$.

THEOREM 4.3 (CLIFFORD ALGEBRAS). *For each $n \in \mathbb{N}$, let $\text{Cl}_{2^n} := \text{Cl}(V_n, q_n)$ be a Clifford algebra with the following properties:*

- (i) *there exist constants $0 < c_1 < c_2$ such that for sufficiently large n , V_n is a vector space over $\mathbb{R}$ or $\mathbb{C}$ with $c_1 n \leq \dim V_n \leq c_2 n$;*
- (ii) *q_n is a non-degenerate quadratic form on V_n .*

Then for the sequence of Clifford products

$$\text{Cl}(\mathbb{F}) = \{(\text{Cl}_{2^n}, \cdot) : n \in \mathbb{N}\},$$

we have

$$\omega(\text{Cl}(\mathbb{F})) = \omega.$$

5 ASYMPTOTIC EXPONENT OF MATRIX FUNCTIONS

We may view the matrix product AB , Lie bracket $[A, B] = AB - BA$, Jordan product $A \circ B = (AB + BA)/2$ as bivariate polynomial functions of the matrix variables (A, B) . From the previous sections, we know that they all share the same asymptotic exponent. It is natural to ask if this extends to other polynomial functions of matrices. We remind the reader that $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ in this section. $\mathbb{F}\langle X_1, \dots, X_k \rangle$ will denote the *free algebra* in the noncommuting variables $X_1, \dots, X_k$ with coefficients in $\mathbb{F}$.

Let $f \in \mathbb{F}\langle X_1, \dots, X_k \rangle$ be a polynomial in the noncommuting variables $X_1, \dots, X_k$. For any matrices $A_1, \dots, A_k \in \mathbb{F}^{n \times n}$, one may evaluate f to obtain a matrix $f(A_1, \dots, A_k) \in \mathbb{F}^{n \times n}$. We denote the *evaluation map* by

$$\text{ev}_n : \mathbb{F}\langle X_1, \dots, X_k \rangle \times \underbrace{\mathbb{F}^{n \times n} \times \dots \times \mathbb{F}^{n \times n}}_{k \text{ copies}} \rightarrow \mathbb{F}^{n \times n}.$$

In other words,

$$\text{ev}_n(f, A_1, \dots, A_k) = f(A_1, \dots, A_k).$$

THEOREM 5.1. *Let $f \in \mathbb{F}\langle X_1, \dots, X_k \rangle$ be fixed. If*

$$\text{ev}_f := \{(\mathbb{F}^{n \times n} \times \dots \times \mathbb{F}^{n \times n}, \mathbb{F}^{n \times n}, \text{ev}_n(f, \cdot, \dots, \cdot)) : n \in \mathbb{N}\},$$

then

$$L(\text{ev}_f) = \omega.$$

A noteworthy point is that Theorem 5.1 holds regardless of the degree of f and the number of variables k involved. However complicated f is, the asymptotic exponent of evaluating f on k matrices is the same regardless, namely, equal to that of multiplying two matrices.

To extend Theorem 5.1 to matrix rational functions, we first observe that for $f, g \in \mathbb{F}\langle X_1, \dots, X_k \rangle$ and $A_1, \dots, A_k \in \mathbb{F}^{n \times n}$, inverting the matrix $g(A_1, \dots, A_k)$ also has asymptotic exponent ω , and thus

$$L(\text{ev}_{f/g}) \leq \omega, \tag{5}$$

where

$$\text{ev}_n(f/g, A_1, \dots, A_k) := f(A_1, \dots, A_k)g(A_1, \dots, A_k)^{-1}.$$

The reverse inequality and thus equality also holds in (5) but the argument is more intricate and is deferred to [15].

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