Some Properties of Multivariate Differential Dimension Polynomials and their Invariants

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ABSTRACT

In this paper we obtain new results on multivariate dimension polynomials of differential field extensions associated with partitions of basic sets of derivations. We prove that the coefficient of the summand of the highest possible degree in the canonical representation of such a polynomial is equal to the differential transcendence degree of the extension. We also give necessary and sufficient conditions under which the multivariate dimension polynomial of a differential field extension of a given differential transcendence degree has the simplest possible form. Furthermore, we describe some relationships between a multivariate dimension polynomial of a differential field extension and dimensional characteristics of subextensions defined by subsets of the basic sets of derivations.

CCS CONCEPTS

• Computing methodologies → Symbolic and algebraic manipulation.

KEYWORDS

Differential field extension, differential dimension polynomial, differential transcendence degree

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1 INTRODUCTION

Differential dimension polynomials introduced in [5] by E. Kolchin play the same role in differential algebra as Hilbert polynomials play in commutative algebra and algebraic geometry. An important feature of differential dimension polynomials is that they describe in exact terms the freedom degree of a continuous dynamic system as well as the number of arbitrary constants in the general solution of a system of algebraic partial differential equations. The following fundamental result introduces and describes main properties of a

(univariate) dimension polynomial of a finitely generated differential field extension. This theorem combines the results obtained in [6, Chapter II, Sections 12 and 13] adjusted to the case of differential fields of characteristic zero.

THEOREM 1.1. *Let K be a differential field of characteristic zero, that is, a field considered together with the action of a set $\Delta = \{\delta_1, \dots, \delta_m\}$ of mutually commuting derivations of K into itself. Let Θ denote the free commutative semigroup of all power products of the form $\theta = \delta_1^{k_1} \dots \delta_m^{k_m}$ ($k_i \geq 0$), let $\text{ord } \theta = \sum_{i=1}^m k_i$, and for any $r \geq 0$, let $\Theta(r) = \{\theta \in \Theta \mid \text{ord } \theta \leq r\}$. Let $L = K\langle\eta_1, \dots, \eta_n\rangle$ be a differential field extension of K generated by a finite set $\eta = \{\eta_1, \dots, \eta_n\}$. (As a field, $L = K(\{\theta\eta_j \mid \theta \in \Theta, 1 \leq j \leq n\})$.) Then there exists a polynomial $\omega_{\eta|K}(t) \in \mathbb{Q}[t]$ such that*

(i) $\omega_{\eta|K}(r) = \text{tr. deg}_K K(\{\theta\eta_j \mid \theta \in \Theta(r), 1 \leq j \leq n\})$ for all sufficiently large $r \in \mathbb{Z}$;

(ii) $\deg \omega_{\eta|K} \leq m$ and $\omega_{\eta|K}(t)$ can be represented as

$\omega_{\eta|K}(t) = \sum_{i=0}^m a_i \binom{t+i}{i}$ where $a_0, \dots, a_m \in \mathbb{Z}$;

(iii) $d = \deg \omega_{\eta|K}$, a_m and a_d do not depend on the choice of the system of Δ -generators η of the extension L/K . Moreover, a_m is equal to the differential transcendence degree of L over K (denoted by $\Delta\text{-tr. deg}_K L$), that is, to the maximal number of elements $\xi_1, \dots, \xi_k \in L$ such that the set $\{\theta\xi_i \mid \theta \in \Theta, 1 \leq i \leq k\}$ is algebraically independent over K .

The polynomial $\omega_{\eta|K}$ is called the *differential dimension polynomial* of the differential field extension L/K associated with the system of differential generators η . The invariants $d = \deg \omega_{\eta|K}$ and a_d in the last part of the theorem are called the *differential* (or Δ -) *type* and *typical differential* (or Δ -) *transcendence degree* of the extension L/K ; they are denoted by $\Delta\text{-type}_K L$ and $\Delta\text{-t. tr. deg}_K L$, respectively.

Differential dimension polynomials provide a power tool for the study of systems of algebraic differential equations. For a wide class of such systems, the dimension polynomial of the corresponding differential field extension expresses the strength of the system of equations in the sense of A. Einstein. This concept, that was introduced in [1] as an important qualitative characteristic of a system of PDEs, can be expressed as a certain differential dimension polynomial, as it is shown in [13]. Another important application of differential dimension polynomials is based on the fact that if P is a prime (in particular, linear) differential ideal of a finitely generated differential algebra R over a differential field K and L is the quotient field of R/P treated as a differential overfield of K , then the differential dimension polynomial of the extension L/K characterizes the ideal P ; assigning such polynomials to prime differential ideals has led to a number of new results on the Krull-type dimension of differential algebras and differential field extensions (see, for

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example, [3], [4], [12] and [7, Chapter 7]). It should be also added that the dimension polynomial associated with a finitely generated differential field extension carries certain differential birational invariants, that is, numbers that do not change when we switch to another finite system of generators of the extension. These invariants are closely connected to some other important characteristics; for example, one of them is the differential transcendence degree of the extension. Among recent works on univariate differential dimension polynomials one has to mention the work of O. Sanchez [14] on the evaluation of the coefficients of a differential dimension polynomial, the work of J. Freitag, O. Sanchez and W. Li on the definability of Kolchin polynomials, and works of M. Lange-Hegermann [8] and [9], where the author introduced a differential dimension polynomial of a characterizable (not necessarily prime) differential ideal and a countable differential polynomial that generalizes the concept of differential dimension polynomial.

In 2001 the author introduced a concept of a multivariate differential dimension polynomial of a finitely generated differential field extension associated with a partition of the set of basic derivations Δ (see [10]). The proof of the corresponding existence theorem that generalizes the first two parts of Theorem 1.1, was based on a special type of reduction in a ring of differential polynomials that takes into account the partition of Δ . It was also shown that a multivariate differential dimension polynomial carries essentially more differential birational invariants of the corresponding differential field extension than its univariate counterpart. As it is demonstrated in Example 3.18, a multivariate dimension polynomial associated with an algebraic differential equation with parameters can carry all this parameters, while the univariate dimension polynomial determines just some relation between the parameters. Therefore, there is a strong motivation for the study of multivariate differential dimension polynomials and their invariants. The main difficulty in this study is due to the fact that a multivariate dimension polynomial of a prime differential polynomial ideal is determined by a characteristic set with respect to several term orderings. Such sets were introduced in [10], but the corresponding theory is in its infancy. Another problem, that is partially solved in this paper, is to characterize invariants of multivariate dimension polynomials and to find relationships between invariants of such polynomials associated with different partitions of the basic set of derivations.

In this paper we obtain new results on multivariate differential dimension polynomials of differential field extensions associated with partitions of the basic sets of derivations. We give necessary and sufficient conditions under which the multivariate dimension polynomial of a differential field extension of a given differential transcendence degree has the simplest possible form. This result (Theorem 3.16) generalizes the corresponding property of univariate differential dimension polynomials proved in [15]. We also prove that the coefficient of the summand of the highest possible degree in the canonical representation of a multivariate dimension polynomial is equal to the differential transcendence degree of the extension. Furthermore, we obtain some relationships between a multivariate dimension polynomial of a differential field extension and dimensional characteristics of subextensions defined by subsets of the basic sets of derivations.

2 PRELIMINARIES

Throughout the paper $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{Q}$ denote the sets of all integers, all nonnegative integers and all rational numbers, respectively. If M is a finite set, then $\text{Card } M$ will denote the number of elements of M . By a ring we always mean an associative ring with unity. Every ring homomorphism is unitary (maps unity onto unity), every subring of a ring contains the unity of the ring, and every algebra over a commutative ring is unitary. Unless otherwise indicated, every field is supposed to have zero characteristic.

A *differential ring* is a commutative ring R considered together with a finite set Δ of mutually commuting derivations of R into itself. The set Δ is called a basic set of the differential ring R that is also called a Δ -ring. A subring (ideal) R_0 of a Δ -ring R is called a differential (or Δ -) subring of R (respectively, a differential (or Δ -) ideal of R) if $\delta(R_0) \subseteq R_0$ for any $\delta \in \Delta$. If a differential (Δ -) ring is a field, it is called a differential (or Δ -) field. In what follows, Θ (or Θ_Δ if we want to indicate the basic set) denotes the free commutative semigroup generated by Δ (that is, if $\Delta = \{\delta_1, \dots, \delta_m\}$, then $\Theta = \{\theta = \delta_1^{k_1} \dots \delta_m^{k_m} \mid k_1, \dots, k_m \in \mathbb{N}\}$).

If R is a Δ -ring and $S \subseteq R$, then the smallest Δ -ideal of R containing S is denoted by $[S]$ (as an ideal, it is generated by the set $\{\theta\xi \mid \xi \in S\}$). If the set S is finite, $S = \{\xi_1, \dots, \xi_q\}$, we say that the Δ -ideal $I = [S]$ is finitely generated, write $I = [\xi_1, \dots, \xi_q]$ and call $\xi_1, \dots, \xi_q$ differential (or Δ -) generators of I . If a Δ -ideal is prime (in the usual sense), it is called a *prime differential* (or Δ -) *ideal*.

Let R_1 and R_2 be two differential rings with the same basic set $\Delta = \{\delta_1, \dots, \delta_m\}$. (More rigorously, we assume that there exist injective mappings of the set Δ into the sets of mutually commuting derivations of the rings R_1 and R_2 . For convenience we will denote the images of elements of Δ under these mappings by the same symbols $\delta_1, \dots, \delta_m$.) A ring homomorphism $\phi : R \rightarrow S$ is called a *differential* (or Δ -) *homomorphism* if $\phi(\delta a) = \delta\phi(a)$ for any $\delta \in \Delta$, $a \in R$.

If K is a Δ -field and K_0 a subfield of K which is also a Δ -subring of K , then K_0 is said to be a differential (or Δ -) subfield of K , and K is called a differential (or Δ -) field extension or a Δ -overfield of K_0 . We also say that we have a Δ -field extension K/K_0 . In this case, if $S \subseteq K$, then the intersection of all Δ -subfields of K containing K_0 and S is the unique Δ -subfield of K containing K_0 and S and contained in every Δ -subfield of K containing K_0 and S . It is denoted by $K_0\langle S \rangle$ or by $K_0\langle S \rangle_\Delta$ if we want to indicate the set of basic derivations Δ . If $K = K_0\langle S \rangle$ and the set S is finite, $S = \{\eta_1, \dots, \eta_n\}$, then K is said to be a finitely generated Δ -field extension of K_0 with the set of Δ -generators $\{\eta_1, \dots, \eta_n\}$. In this case we write $K = K_0\langle \eta_1, \dots, \eta_n \rangle$. It is easy to see that the field $K_0\langle \eta_1, \dots, \eta_n \rangle$ coincides with the field $K_0(\{\theta\eta_i \mid \theta \in \Theta, 1 \leq i \leq n\})$.

Let L/K be a Δ -field extension. We say that a set $U \subseteq L$ is Δ -*algebraically dependent* over K , if the family $\{\theta(u) \mid u \in U, \theta \in \Theta\}$ is algebraically dependent over K . Otherwise, the family U is said to be Δ -*algebraically independent* over K . An element $u \in L$ is said to be Δ -*algebraic* over K if the set $\{u\}$ is Δ -algebraically dependent over K . A maximal Δ -algebraically independent over K subset of L is called a differential (or Δ -) transcendence basis of L over K (or of the extension L/K). It is known (see [6, Chapter II]) that every system of Δ -generators of L/K contains a Δ -transcendence basis of L over K and if L/K is finitely generated as a Δ -field extension,

then all Δ -transcendence bases have the same number of elements called the *differential (or Δ -) transcendence degree* of L over K ; it is denoted by $\Delta\text{-tr. deg}_K L$.

If K is a Δ -field and $Y = \{y_1, \dots, y_n\}$ is a finite set of symbols, then one can consider the countable set of symbols $\Theta Y = \{\theta y_j | \theta \in \Theta, 1 \leq j \leq n\}$ and the polynomial ring $R = K[\{\theta y_j | \theta \in \Theta, 1 \leq j \leq n\}]$ in the set of indeterminates ΘY . This polynomial ring is naturally viewed as a Δ -ring where $\delta(\theta y_j) = (\delta\theta)y_j$ ($\delta \in \Delta, \theta \in \Theta, 1 \leq j \leq n$) and the elements of Δ act on the coefficients of the polynomials as they act in K . The ring R is called the *ring of differential polynomials* in differential (Δ -) indeterminates $y_1, \dots, y_n$ over the Δ -field K . This ring is denoted by $K\{y_1, \dots, y_n\}$; its elements are called differential (or Δ -) polynomials.

MULTIVARIATE NUMERICAL POLYNOMIALS OF SUBSETS OF $\mathbb{N}^m$

DEFINITION 2.1. A polynomial $f(t_1, \dots, t_p)$ in p variables ($p \geq 1$) with rational coefficients is said to be numerical if $f(t_1, \dots, t_p) \in \mathbb{Z}$ for all sufficiently large $t_1, \dots, t_p \in \mathbb{Z}$, that is, there exists $(s_1, \dots, s_p) \in \mathbb{Z}^p$ such that $f(r_1, \dots, r_p) \in \mathbb{Z}$ whenever $(r_1, \dots, r_p) \in \mathbb{Z}^p$ and $r_i \geq s_i$ ($1 \leq i \leq p$).

Clearly, every polynomial with integer coefficients is numerical. As an example of a numerical polynomial in p variables with non-integer coefficients one can consider $\prod_{i=1}^p \binom{t_i}{m_i}$ ($m_1, \dots, m_p \in \mathbb{Z}$), where $\binom{t}{k} = \frac{t(t-1)\dots(t-k+1)}{k!}$ for any $k \in \mathbb{Z}, k \geq 1$, $\binom{t}{0} = 1$, and $\binom{t}{k} = 0$ if k is a negative integer.

If f is a numerical polynomial in p variables ($p > 1$), then $\deg f$ and $\deg_{t_i} f$ ($1 \leq i \leq p$) will denote the total degree of f and the degree of f relative to the variable t_i , respectively. The following theorem gives the "canonical" representation of a numerical polynomial in several variables.

THEOREM 2.2. Let $f(t_1, \dots, t_p)$ be a numerical polynomial in p variables $t_1, \dots, t_p$, and let $\deg_{t_i} f = m_i$ ($1 \leq i \leq p$). Then the polynomial $f(t_1, \dots, t_p)$ can be represented as

$$f(t_1, \dots, t_p) = \sum_{i_1=0}^{m_1} \dots \sum_{i_p=0}^{m_p} a_{i_1 \dots i_p} \binom{t_1 + i_1}{i_1} \dots \binom{t_p + i_p}{i_p} \quad (1)$$

with integer coefficients $a_{i_1 \dots i_p}$ that are uniquely defined by the numerical polynomial.

In the rest of this section we deal with subsets of $\mathbb{N}^m$ where the positive integer m is represented as a sum of p nonnegative integers $m_1, \dots, m_p$ ($p \geq 1$). In other words, we fix a partition $(m_1, \dots, m_p)$ of m .

If $A \subseteq \mathbb{N}^m$, then for any $r_1, \dots, r_p \in \mathbb{N}$, $A(r_1, \dots, r_p)$ will denote the subset of A that consists of all m -tuples $(a_1, \dots, a_m)$ such that $a_1 + \dots + a_{m_1} \leq r_1, a_{m_1+1} + \dots + a_{m_1+m_2} \leq r_2, \dots, a_{m_1+\dots+m_{p-1}+1} + \dots + a_m \leq r_p$. Furthermore, we shall associate with the set A a set $V_A \subseteq \mathbb{N}^m$ that consists of all m -tuples $v = (v_1, \dots, v_m) \in \mathbb{N}^m$ that are not greater than or equal to any m -tuple from A with respect to the product order on $\mathbb{N}^m$. (Recall that the product order on the set $\mathbb{N}^k$ ($k \in \mathbb{N}, k \geq 1$) is a partial order $\leq_p$ on $\mathbb{N}^k$ such that $c = (c_1, \dots, c_k) \leq_p c' = (c'_1, \dots, c'_k)$ if and only if $c_i \leq c'_i$ for all $i = 1, \dots, k$. If $c \leq_p c'$ and $c \neq c'$, we write $c <_p c'$). Clearly, an element $v = (v_1, \dots, v_m) \in \mathbb{N}^m$ belongs to V_A if and only if for any

element $(a_1, \dots, a_m) \in A$ there exists $i \in \mathbb{N}, 1 \leq i \leq m$, such that $a_i > v_i$.

The following two theorems proved in [7, Chapter 2] generalize the well-known Kolchin's result on the numerical polynomials of subsets of $\mathbb{N}^m$ (see [6, Chapter 0, Lemma 17]) and give an explicit formula for the numerical polynomials in p variables associated with a finite subset of $\mathbb{N}^m$.

THEOREM 2.3. Let A be a subset of $\mathbb{N}^m$ where $m = m_1 + \dots + m_p$ for some nonnegative integers $m_1, \dots, m_p$ ($p \geq 1$). Then there exists a numerical polynomial $\omega_A(t_1, \dots, t_p)$ in p variables with the following properties:

- (i) $\omega_A(r_1, \dots, r_p) = \text{Card } V_A(r_1, \dots, r_p)$ for all sufficiently large $(r_1, \dots, r_p) \in \mathbb{N}^p$ (i. e., there exist $(s_1, \dots, s_p) \in \mathbb{N}^p$ such that the equality holds for all $(r_1, \dots, r_p) \in \mathbb{N}^p$ such that $(s_1, \dots, s_p) \leq_p (r_1, \dots, r_p)$).
- (ii) $\deg \omega_A \leq m$ and $\deg_{t_i} \omega_A \leq m_i$ for $i = 1, \dots, p$.
- (iii) $\deg \omega_A = m$ if and only if the set A is empty. In this case $\omega_A(t_1, \dots, t_p) = \prod_{i=1}^p \binom{t_i + m_i}{m_i}$.
- (iv) ω_A is a zero polynomial if and only if $(0, \dots, 0) \in A$.

DEFINITION 2.4. The polynomial $\omega_A(t_1, \dots, t_p)$ is called the *dimension polynomial of the set $A \subseteq \mathbb{N}^m$ associated with the partition $(m_1, \dots, m_p)$ of m* .

THEOREM 2.5. Let $A = \{a_1, \dots, a_n\}$ be a finite subset of $\mathbb{N}^m$ where n is a positive integer and $m = m_1 + \dots + m_p$ for some nonnegative integers $m_1, \dots, m_p$ ($p \geq 1$). Let $a_i = (a_{i1}, \dots, a_{im})$ ($1 \leq i \leq n$) and for any $l \in \mathbb{N}, 0 \leq l \leq n$, let $\Gamma(l, n)$ denote the set of all l -element subsets of the set $\mathbb{N}_n = \{1, \dots, n\}$. Furthermore, for any $\sigma \in \Gamma(l, p)$, let $\bar{a}_{\sigma j} = \max\{a_{ij} | i \in \sigma\}$ ($1 \leq j \leq m$) and $b_{\sigma j} = \sum_{h \in \sigma_j} \bar{a}_{\sigma h}$. Then

$$\omega_A(t_1, \dots, t_p) = \sum_{l=0}^n (-1)^l \sum_{\sigma \in \Gamma(l, n)} \prod_{j=1}^p \binom{t_j + m_j - b_{\sigma j}}{m_j} \quad (2)$$

REMARK 2.6. Clearly, if $A \subseteq \mathbb{N}^m$ and A' is the set of all minimal elements of the set A with respect to the product order on $\mathbb{N}^m$, then the set A' is finite and $\omega_A(t_1, \dots, t_p) = \omega_{A'}(t_1, \dots, t_p)$. Thus, Theorem 2.5 gives an algorithm that allows one to find the dimension polynomial of any subset of $\mathbb{N}^m$ (with a given representation of m as a sum of p positive integers): one should first find the set of all minimal points of the subset and then apply Theorem 2.5.

3 MULTIVARIATE DIFFERENTIAL DIMENSION POLYNOMIALS AND THEIR INVARIANTS

Let K be a differential (Δ -) field whose basic set of derivations Δ is represented as the union of p nonempty disjoint subsets ($p \geq 1$):

$$\Delta = \Delta_1 \cup \dots \cup \Delta_p \quad (3)$$

where $\Delta_i = \{\delta_{i1}, \dots, \delta_{im_i}\}$ for $i = 1, \dots, p$ ($m_1 + \dots + m_p = m$ where $m = \text{Card } \Delta$). Thus, we fix a partition of Δ .

Let Θ_i denote the free commutative semigroup generated by Δ_i ($1 \leq i \leq p$) and let Θ be the free commutative semigroup generated by the whole set Δ . For any $\theta = \delta_{11}^{k_{11}} \dots \delta_{1m_1}^{k_{1m_1}} \delta_{21}^{k_{21}} \dots \delta_{pm_p}^{k_{pm_p}} \in \Theta$, the numbers $\text{ord}_i \theta = \sum_{j=1}^{m_i} k_{ij}$ ($i = 1, \dots, p$) and $\text{ord } \theta = \sum_{i=1}^p \text{ord}_i \theta$

will be called the *order of θ with respect to Δ_i* and the *order of θ* , respectively. If $\theta, \theta' \in \Theta$, we say that θ' divides θ (or that θ is a multiple of θ') and write $\theta' \mid \theta$ if there exists $\theta'' \in \Theta$ such that $\theta = \theta''\theta'$. As usual, the least common multiple of elements $\theta_1 = \prod_{i=1}^p \prod_{j=1}^{m_i} \delta_{ij}^{k_{ij1}}, \dots, \theta_q = \prod_{i=1}^p \prod_{j=1}^{m_i} \delta_{ij}^{k_{ijq}} \in \Theta$ is the element $\theta = \prod_{i=1}^p \prod_{j=1}^{m_i} \delta_{ij}^{k_{ij}}$, where $k_{ij} = \max\{k_{ijl} \mid 1 \leq l \leq q\}$ ($1 \leq i \leq p, 1 \leq j \leq m_i$), denoted by $\text{lcm}(\theta_1, \dots, \theta_q)$.

If $r_1, \dots, r_p, r \in \mathbb{N}$, the sets $\{\theta \in \Theta \mid \text{ord}_i \theta \leq r_i \text{ for } i = 1, \dots, p\}$ and $\{\theta \in \Theta \mid \text{ord} \theta \leq r\}$ will be denoted by $\Theta(r_1, \dots, r_p)$ and $\Theta(r)$, respectively. If $\xi \in K$ and $\Theta' \subseteq \Theta$, then $\Theta'\xi$, will denote the set $\{\theta(\xi) \mid \theta \in \Theta'\}$.

We consider p orderings $<_1, \dots, <_p$ of the semigroup Θ defined as follows. If $\theta = \delta_{11}^{k_{11}} \dots \delta_{pm_p}^{k_{pm_p}}$ and $\theta' = \delta_{11}^{l_{11}} \dots \delta_{pm_p}^{l_{pm_p}}$ are elements of Θ , then $\theta <_i \theta'$ if and only if the vector

$(\text{ord}_i \theta, \text{ord} \theta, \text{ord}_1 \theta, \dots, \text{ord}_{i-1} \theta, \text{ord}_{i+1} \theta, \dots, \text{ord}_p \theta, k_{i1}, \dots, k_{im}, k_{11}, \dots, k_{1m_1}, k_{21}, \dots, k_{i-1, m_{i-1}}, k_{i+1, 1}, \dots, k_{pm_p})$

is less than the vector

$(\text{ord}_i \theta', \text{ord} \theta', \text{ord}_1 \theta', \dots, \text{ord}_{i-1} \theta', \text{ord}_{i+1} \theta', \dots, \text{ord}_p \theta', l_{i1}, \dots, l_{im}, l_{11}, \dots, l_{1m_1}, l_{21}, \dots, l_{i-1, m_{i-1}}, l_{i+1, 1}, \dots, l_{pm_p})$

with respect to the lexicographic order on $\mathbb{N}^{m+p+1}$.

Let $K\{y_1, \dots, y_n\}$ be the ring of Δ -polynomials in Δ -indeterminates $y_1, \dots, y_n$ over K . Then the elements θy_i ($\theta \in \Theta, 1 \leq i \leq n$) will be called terms, and the set of all terms ΘY will be considered together with p orderings that correspond to the orderings of Θ and are denoted by the same symbols $<_1, \dots, <_p$. These orderings of ΘY are defined as follows. $\theta y_j <_i \theta' y_k$ ($\theta, \theta' \in \Theta, 1 \leq j, k \leq n, 1 \leq i \leq p$) if and only if $\theta <_i \theta'$ or $\theta = \theta'$ and $j < k$. By the i th order of a term $u = \theta y_j$ we mean the number $\text{ord}_i u = \text{ord}_i \theta$. The number $\text{ord} u = \text{ord} \theta$ is called the order of u .

We say that a term $u = \theta y_j$ is divisible by a term $v = \theta' y_j$ and write $v \mid u$, if $i = j$ and $\theta' \mid \theta$. For any terms $u_1 = \theta_1 y_j, \dots, u_q = \theta_q y_j$ with the same Δ -indeterminate y_j , the term $\text{lcm}(\theta_1, \dots, \theta_q) y_j$ is called the least common multiple of $u_1, \dots, u_q$, it is denoted by $\text{lcm}(u_1, \dots, u_q)$.

If $A \in K\{y_1, \dots, y_n\}$, $A \notin K$, and $1 \leq i \leq p$, then the highest with respect to the ordering $<_i$ term that appears in A is called the i -leader of the Δ -polynomial A . It is denoted by $u_A^{(i)}$. If A is written as a polynomial in one variable $u_A^{(1)}$, $A = I_d(u_A^{(1)})^d + I_{d-1}(u_A^{(1)})^{d-1} + \dots + I_0$ ($I_d, I_{d-1}, \dots, I_0$ do not contain $u_A^{(1)}$), then I_d is called the *leading coefficient* of the Δ -polynomial A and the partial derivative $\partial A / \partial u_A^{(1)} = dI_d(u_A^{(1)})^{d-1} + (d-1)I_{d-1}(u_A^{(1)})^{d-2} + \dots + I_1$ is called the *separant* of A . The leading coefficient and the separant of A are denoted by I_A and S_A , respectively.

DEFINITION 3.1. Let A and B be Δ -polynomials in $K\{y_1, \dots, y_n\}$. We say that A has lower rank than B and write $\text{rk } A < \text{rk } B$ if either $A \in K, B \notin K$, or the vector $(u_A^{(1)}, \deg_{u_A^{(1)}} A, \text{ord}_2 u_A^{(2)}, \dots, \text{ord}_p u_A^{(p)})$ is less than the vector $(u_B^{(1)}, \deg_{u_B^{(1)}} B, \text{ord}_2 u_B^{(2)}, \dots, \text{ord}_p u_B^{(p)})$ with respect to the lexicographic order ($u_A^{(1)}$ and $u_B^{(1)}$ are compared with respect to $<_1$ and all other coordinates are compared with respect to the natural order on $\mathbb{N}$). If the two vectors are equal (or $A \in K$ and $B \in K$) we say that the Δ -polynomials A and B are of the same rank and write $\text{rk } A = \text{rk } B$.

DEFINITION 3.2. Let A and B be Δ -polynomials in $K\{y_1, \dots, y_n\}$ and $A \notin K$. We say that B is reduced with respect to A if the following two conditions hold.

(i) B does not contain any term $\theta u_A^{(1)}$ ($\theta \in \Theta, \theta \neq 1$) such that $\text{ord}_i(\theta u_A^{(1)}) \leq \text{ord}_i u_B^{(i)}$ for $i = 2, \dots, p$.

(ii) If B contains $u_A^{(1)}$, then either there exists $j, 2 \leq j \leq p$, such that $\text{ord}_j u_B^{(j)} < \text{ord}_j u_A^{(j)}$ or $\text{ord}_j u_A^{(j)} \leq \text{ord}_j u_B^{(j)}$ for all $j = 2, \dots, p$ and $\deg_{u_A^{(1)}} B < \deg_{u_A^{(1)}} A$.

A Δ -polynomial B is said to be reduced with respect to a set $\mathcal{A} \subseteq K\{y_1, \dots, y_n\}$ if B is reduced with respect to every element of $\mathcal{A}$.

REMARK 3.3. The last definition shows that a Δ -polynomial B is not reduced with respect to a Δ -polynomial A ($A \notin K$) if either B contains a term $\theta u_A^{(1)}$ ($\theta \in \Theta, \theta \neq 1$) such that $\text{ord}_i(\theta u_A^{(1)}) \leq \text{ord}_i u_B^{(i)}$ for $i = 2, \dots, p$ or B contains $u_A^{(1)}$ and in this case $\text{ord}_j u_A^{(j)} \leq \text{ord}_j u_B^{(j)}$ for $j = 2, \dots, p$ and $\deg_{u_A^{(1)}} A \leq \deg_{u_A^{(1)}} B$. This observation is helpful if one would like to show that a Δ -polynomial is not reduced with respect to some other Δ -polynomial.

DEFINITION 3.4. A set of Δ -polynomials $\mathcal{A}$ is called *autoreduced* if $\mathcal{A} \cap K = \emptyset$ and every element of $\mathcal{A}$ is reduced with respect to any other element of this set.

The following two statements are proved in [10] (see [10, Theorem 4.5] and [10, Theorem 4.6]).

PROPOSITION 3.5. Every autoreduced set is finite.

PROPOSITION 3.6. Let $\mathcal{A} = \{A_1, \dots, A_r\}$ be an autoreduced set in $K\{y_1, \dots, y_n\}$ and $B \in K\{y_1, \dots, y_n\}$. Then there are a Δ -polynomial B_0 and nonnegative integers p_i, q_i ($1 \leq i \leq r$) such that B_0 is reduced with respect to $\mathcal{A}$, $\text{rk } B_0 \leq \text{rk } B$, and $\prod_{i=1}^r I_{A_i}^{p_i} S_{A_i}^{q_i} B \equiv B_0 \pmod{[\mathcal{A}]}$.

In what follows, while considering an autoreduced set $\mathcal{A} = \{A_1, \dots, A_r\}$, we always assume that its elements are arranged in order of increasing rank: $\text{rk } A_1 < \dots < \text{rk } A_r$.

DEFINITION 3.7. Let $\mathcal{A} = \{A_1, \dots, A_r\}$ and $\mathcal{B} = \{B_1, \dots, B_s\}$ be two autoreduced sets. Then $\mathcal{A}$ is said to have lower rank than $\mathcal{B}$ if one of the following two cases holds:

(i) There exists $k \in \mathbb{N}$ such that $k \leq \min\{r, s\}$, $\text{rk } A_i = \text{rk } B_i$ for $i = 1, \dots, k-1$ and $\text{rk } A_k < \text{rk } B_k$.

(ii) $r > s$ and $\text{rk } A_i = \text{rk } B_i$ for $i = 1, \dots, s$.

If $r = s$ and $\text{rk } A_i = \text{rk } B_i$ for $i = 1, \dots, r$, then $\mathcal{A}$ is said to have the same rank as $\mathcal{B}$.

The statements of Propositions 3.8 and 3.10 below can be obtained by mimicking the proofs of the corresponding statements for classical Ritt-Kolchin autoreduced sets (see [7, Proposition 5.3.10 and Lemma 5.3.12]).

PROPOSITION 3.8. Every nonempty family of autoreduced sets contains an autoreduced set of lowest rank.

DEFINITION 3.9. Let J be a Δ -ideal of the ring of Δ -polynomials $K\{y_1, \dots, y_n\}$. Then an autoreduced subset of J of lowest rank is called a characteristic set of the ideal J .

PROPOSITION 3.10. Let $\mathcal{A} = \{A_1, \dots, A_d\}$ be a characteristic set of a Δ -ideal J of the ring of Δ -polynomials $R = K\{y_1, \dots, y_n\}$. Then

an element $B \in J$ is reduced with respect to the set $\mathcal{A}$ if and only if $B = 0$. In particular, $I_A \notin J$ and $S_A \notin J$ for every $A \in \mathcal{A}$.

Let K be a Δ -field and $L = K\langle\eta_1, \dots, \eta_n\rangle$ a finitely generated Δ -extension of K with a set of Δ -generators $\eta = \{\eta_1, \dots, \eta_n\}$. Then there exists a natural Δ -homomorphism ϕ_η of the ring of Δ -polynomials $K\{y_1, \dots, y_n\}$ onto the Δ -subring $K\{\eta_1, \dots, \eta_n\}$ of L such that $\phi_\eta(a) = a$ for any $a \in K$ and $\phi_\eta(y_j) = \eta_j$ for $j = 1, \dots, n$. If $A \in K\{y_1, \dots, y_n\}$, then $\phi_\eta(A)$ is called the *value* of A at η and is denoted by $A(\eta)$. Obviously, $P = \text{Ker } \phi_\eta$ is a prime Δ -ideal of $K\{y_1, \dots, y_n\}$. It is called the *defining ideal* of η . If we consider the quotient field Q of $R = K\{y_1, \dots, y_n\}/P$ as a Δ -field (where $\delta(\frac{u}{v}) = \frac{v\delta(u) - u\delta(v)}{v^2}$ for any $u, v \in R$), then this quotient field is naturally Δ -isomorphic to the field L . The Δ -isomorphism of Q onto L is identical on K and maps the images of the Δ -indeterminates $y_1, \dots, y_n$ in the factor ring R onto the elements $\eta_1, \dots, \eta_n$, respectively.

Let K be a differential (Δ -) field, $\text{Card } \Delta = m$, and let a partition (3) of Δ be fixed: $\Delta = \Delta_1 \cup \dots \cup \Delta_p$ ($p \geq 1$), where $\Delta_i = \{\delta_{i1}, \dots, \delta_{im_i}\}$ ($1 \leq i \leq p$). Furthermore, let $L = K\langle\eta_1, \dots, \eta_n\rangle$ be a Δ -field extension of K generated by a finite set $\eta = \{\eta_1, \dots, \eta_n\}$. Let P be the defining ideal of η and $\mathcal{A} = \{A_1, \dots, A_d\}$ a characteristic set of P .

For any $r_1, \dots, r_p \in \mathbb{N}$, let

$U'_{r_1 \dots r_p} = \{u \in \Theta Y \mid \text{ord}_i u \leq r_i \text{ for } i = 1, \dots, p \text{ and } u \text{ is not a derivative of any } u_{A_i}^{(1)} \text{ (that is, } u \neq \theta u_{A_i}^{(1)} \text{ for any } \theta \in \Theta; i = 1, \dots, d) \text{ and let}$

$U''_{r_1 \dots r_p} = \{u \in \Theta Y \mid \text{ord}_i u \leq r_i \text{ for } i = 1, \dots, p \text{ and for every } \theta \in \Theta, A \in \mathcal{A} \text{ such that } u = \theta u_A^{(1)}, \text{ there exists } i \in \{2, \dots, p\} \text{ such that } \text{ord}_i(\theta u_A^{(1)}) > r_i\}$.

(If $p = 1$, $U'_{r_1} = \{u \in \Theta Y \mid \text{ord}_1 u \leq r_1 \text{ and } u \text{ is not a derivative of any } u_{A_i}^{(1)}\}$ and $U''_{r_1} = \emptyset$.) Furthermore, for any $(r_1 \dots r_p) \in \mathbb{N}^p$, let $U_{r_1 \dots r_p} = U'_{r_1 \dots r_p} \cup U''_{r_1 \dots r_p}$.

The following theorem proved in [10, Section 5] establishes the existence and describes the form of a multivariate dimension polynomial associated with a finite system of Δ -generators of a Δ -field extension and with a partition of the set Δ . We give an extended version of this result that follows from the proof of [10, Theorem 5.1].

THEOREM 3.11. *With the above notation,*

(i) *For all sufficiently large $(r_1 \dots r_p) \in \mathbb{N}^p$, the set $U_{r_1 \dots r_p}$ is a transcendence basis of $K(\bigcup_{j=1}^n \Theta(r_1, \dots, r_p)\eta_j)$ over K .*

(ii) *There exist numerical polynomials $\omega_{\eta|K}(t_1, \dots, t_p)$ and $\phi_{\eta|K}(t_1, \dots, t_p)$ in p variables such that $\omega_{\eta|K}(r_1, \dots, r_p) = \text{Card } U'_{r_1 \dots r_p}$ and $\phi_{\eta|K}(r_1, \dots, r_p) = \text{Card } U''_{r_1 \dots r_p}$ for all sufficiently large $(r_1 \dots r_p) \in \mathbb{N}^p$, so that the polynomial $\Phi_{\eta|K}(t_1, \dots, t_p) = \omega_{\eta|K}(t_1, \dots, t_p) + \phi_{\eta|K}(t_1, \dots, t_p)$ has the property that $\Phi_{\eta}(r_1, \dots, r_p) = \text{tr. deg}_K K(\bigcup_{j=1}^n \Theta(r_1, \dots, r_p)\eta_j)$ for all sufficiently large $(r_1, \dots, r_p) \in \mathbb{N}^p$.*

(iii) $\deg_{t_i} \Phi_{\eta|K} \leq m_i$ ($1 \leq i \leq p$), so that $\deg \Phi_{\eta|K} \leq m$ and the polynomial $\Phi_{\eta|K}(t_1, \dots, t_p)$ can be represented as

$$\Phi_{\eta|K}(t_1, \dots, t_p) = \sum_{i_1=0}^{m_1} \dots \sum_{i_p=0}^{m_p} a_{i_1 \dots i_p} \binom{t_1 + i_1}{i_1} \dots \binom{t_p + i_p}{i_p} \quad (4)$$

where $a_{i_1 \dots i_p} \in \mathbb{Z}$ for all $i_1 \dots i_p$.

(iv) $\phi_{\eta|K}(t_1, \dots, t_p)$ is an alternating sum of polynomials in p variables of the form

$$\begin{aligned} \phi_{j; k_1, \dots, k_q} &= \binom{t_1 + m_1 - b_{1j}}{m_1} \dots \binom{t_{k_1-1} + m_{k_1-1} - b_{k_1-1,j}}{m_{k_1-1}} \\ &\quad \left[\binom{t_{k_1} + m_{k_1} - a_{k_1,j}}{m_{k_1}} - \binom{t_{k_1} + m_{k_1} - b_{k_1,j}}{m_{k_1}} \right] \\ &\quad \binom{t_{k_1+1} + m_{k_1+1} - b_{k_1+1,j}}{m_{k_1+1}} \dots \binom{t_{k_q-1} + m_{k_q-1} - b_{k_q-1,j}}{m_{k_q-1}} \\ &\quad \left[\binom{t_{k_q} + m_{k_q} - a_{k_q,j}}{m_{k_q}} - \binom{t_{k_q} + m_{k_q} - b_{k_q,j}}{m_{k_q}} \right] \dots \\ &\quad \binom{t_p + m_p - b_{pj}}{m_p}, \text{ so that } \deg \phi_{\eta|K} < m \end{aligned}$$

DEFINITION 3.12. *Numerical polynomial $\Phi_{\eta|K}(t_1, \dots, t_p)$, whose existence is established by the last theorem, is called a differential (or Δ -) dimension polynomial of the differential field extension $L = K\langle\eta_1, \dots, \eta_n\rangle$ associated with the set of Δ -generators $\eta = \{\eta_1, \dots, \eta_n\}$ and with partition (3) of the basic set of derivations Δ .*

REMARK 3.13. With the notation of the last theorem, if $\eta_1, \dots, \eta_n$ are Δ -algebraically independent over K , then

$$\Phi_{\eta|K}(t_1, \dots, t_p) = n \prod_{i=1}^p \binom{t_i + m_i}{m_i}. \quad (5)$$

Indeed, all elements $\delta_{11}^{k_{11}} \dots \delta_{1m_1}^{k_{1m_1}} \delta_{21}^{k_{21}} \dots \delta_{pm_p}^{k_{pm_p}}$ such that $\sum_{j=1}^n k_{ij} \leq r_i$ ($1 \leq i \leq p$) form a transcendence basis of $K(\bigcup_{j=1}^n \Theta(r_1, \dots, r_p)\eta_j)$ over K . By Theorem 2.3 (iii), the number of such elements is $n \prod_{i=1}^p \binom{r_i + m_i}{m_i}$, so we arrive at formula (5).

REMARK 3.14. Theorem 3.11 shows that the main problem in computing the multivariate Δ -dimension polynomial $\Phi_{\eta|K}$ is constructing a characteristic set of the defining Δ -ideal of the Δ -field extension. If this ideal is linear (that is, the defining system of differential equations on the generators of the extension is linear), then this problem was solved in [11] by an algorithm for constructing a Gröbner basis with respect to several term orderings (see [11, Algorithm 1] and [11, Theorem 3.10]). In the nonlinear case the problem of generalizing the Ritt-Kolchin algorithm to the case of autoreduced sets with respect to several term orderings defined above is still open.

For any permutation $(j_1, \dots, j_p)$ of the set $\{1, \dots, p\}$ ($p \geq 1$), let $\leq_{j_1, \dots, j_p}$ denote the corresponding lexicographic order on $\mathbb{N}^p$ such that $(r_1, \dots, r_p) \leq_{j_1, \dots, j_p} (s_1, \dots, s_p)$ if and only if either $r_{j_1} < s_{j_1}$ or there exists $k \in \mathbb{N}$, $1 \leq k \leq p-1$ such that $r_{j_v} = s_{j_v}$ for $v = 1, \dots, k$ and $r_{j_{k+1}} < s_{j_{k+1}}$. If E is a finite subset of $\mathbb{N}^p$, then E' will denote the set of all p -tuples $e \in E$ that are maximal

elements of E with respect to one of the $p!$ orders $\leq_{j_1, \dots, j_p}$. Say, if $E = \{(3, 0, 2), (2, 1, 1), (0, 1, 4), (1, 0, 3), (1, 1, 6), (3, 1, 0), (1, 2, 0)\} \subseteq \mathbb{N}^3$, then $E' = \{(3, 0, 2), (3, 1, 0), (1, 1, 6), (1, 2, 0)\}$.

The following result gives differential birational invariants carried by a multivariate dimension polynomial of a differential field extension. In particular, it shows that multivariate differential dimension polynomials carry essentially more such invariants than their univariate counterparts.

THEOREM 3.15. *Let K be a differential field with a basic set of derivations Δ and let partition (3) of the set Δ into the union of p disjoint sets ($p \geq 1$) be fixed. Let $L = K\langle\eta_1, \dots, \eta_n\rangle$ be a Δ -field extension of K with the finite set of Δ -generators $\eta = \{\eta_1, \dots, \eta_n\}$ and let*

$$\Phi_{\eta|K}(t_1, \dots, t_p) = \sum_{i_1=0}^{m_1} \dots \sum_{i_p=0}^{m_p} a_{i_1 \dots i_p} \binom{t_1 + i_1}{i_1} \dots \binom{t_p + i_p}{i_p} \quad (6)$$

be the corresponding differential dimension polynomial. Let $E_\eta = \{(i_1 \dots i_p) \in \mathbb{N}^p \mid 0 \leq i_k \leq m_k (k = 1, \dots, p) \text{ and } a_{i_1 \dots i_p} \neq 0\}$. Then $d = \deg \Phi_{\eta|K}$, $a_{m_1 \dots m_p}$, the elements $(k_1, \dots, k_p) \in E'_\eta$, the corresponding coefficients $a_{k_1 \dots k_p}$, and the coefficients of the terms of total degree d in $\Phi_{\eta|K}$ do not depend on the system of Δ -generators η .

PROOF. The fact that the elements $(k_1, \dots, k_p)$ of the set E'_η and the corresponding coefficients $a_{k_1 \dots k_p}$ do not depend on the system of Δ -generators η of L/K is established in the proof of Theorem 5.3 of [10] using the observation that if $\zeta = \{\zeta_1, \dots, \zeta_q\}$ is another system of Δ -generators of L/K , then there exists $(s_1, \dots, s_p) \in \mathbb{N}^p$ such that $\Phi_{\eta|K}(r_1, \dots, r_p) \leq \Phi_{\zeta|K}(r_1 + s_1, \dots, r_p + s_p)$ and $\Phi_{\zeta|K}(r_1, \dots, r_p) \leq \Phi_{\eta|K}(r_1 + s_1, \dots, r_p + s_p)$ for all sufficiently large $(r_1, \dots, r_p) \in \mathbb{N}^p$. Clearly, these inequalities show that $\deg \Phi_{\zeta|K} = \deg \Phi_{\eta|K}$. Let $d = \deg \Phi_{\eta|K}$. Let us order the terms of total degree d in $\Phi_{\eta|K}$ and $\Phi_{\zeta|K}$ using the lexicographic order $\leq_{p, p-1, \dots, 1}$ and for sufficiently large $r \in \mathbb{N}$, set $x_1 = r$, $x_2 = 2^{x_1}$, $x_3 = 2^{x_2}$, $\dots$, $x_p = 2^{x_{p-1}}$, $R = 2^{x_p}$ and $r_i = x_i R$ ($1 \leq i \leq p$). If $r \rightarrow \infty$, then the last two inequalities immediately imply that $\Phi_{\eta|K}$ and $\Phi_{\zeta|K}$ have the same coefficients of the corresponding terms of total degree d . $\square$

The next theorem characterizes one of the invariants of polynomial (6).

THEOREM 3.16. *With the notation of the last theorem, $a_{m_1 \dots m_p} = \Delta\text{-tr. deg}_K L$.*

PROOF. Let $\Delta\text{-tr. deg}_K L = d$. Then, as it was mentioned in section 2, one can choose a Δ -transcendence basis of L/K from the set η , so we can assume that $\eta_1, \dots, \eta_d$ form such a basis. Since the family $\{\theta\eta_i \mid \theta \in \Theta, 1 \leq i \leq d\}$ is algebraically independent over K , it follows from Remark 3.13 that $\text{tr. deg}_K K(\bigcup_{j=1}^d \Theta(r_1, \dots, r_p)\eta_j) = d \prod_{i=1}^p \binom{r_i + m_i}{m_i}$ for all $(r_1, \dots, r_p) \in \mathbb{N}^p$. Let $F = K\langle\eta_1, \dots, \eta_d\rangle$. Then every element η_j , $d+1 \leq j \leq n$, is Δ -algebraic over K . It means that there exists a Δ -polynomial $A_j \in F\{y_j\}$ ($F\{y_j\}$ is the ring of Δ -polynomials in one Δ -indeterminate y_j over F) such that $A_j(\eta_j) = 0$. Taking such a polynomial of the smallest possible

degree we can assume that $S_{A_j}(\eta_j) \neq 0$. Let $A_j = \sum_{k=0}^{q_j} I_{jk}(u_{A_j}^{(1)})^k$ where all terms of all I_{jk} are less than $u_{A_j}^{(1)}$ with respect to $<_1$.

If $\delta \in \Delta$, then $\delta A_j(\eta_j) = 0$, so

$$S_{A_j}(\eta_j) \delta(u_{A_j}^{(1)}(\eta_j)) + \sum_{k=0}^{q_j} \delta(I_{jk}(\eta_j)) u_{A_j}^{(1)}(\eta_j) = 0.$$

The term $\delta u_{A_j}^{(1)}$ has the form $\theta_j y_j$ for some $\theta_j \in \Theta$ and one can easily see that for any term v in any S_{A_j} or $\delta(I_{jk})$, we have $v <_1 \theta_j y_j$ and $\text{ord}_i v \leq \text{ord}_i u_{A_j}^{(1)} + 1$ ($i = 1, \dots, p$). It follows that $\theta_j \eta_j \in F(\{\theta\eta_k \mid \theta \in \Theta, \theta y_k <_1 \theta_j y_j \text{ and } \text{ord}_i \theta y_k \leq a_{ji} \text{ for } i = 1, \dots, p\})$ where $a_{ji} = \text{ord}_i u_{A_j}^{(1)} + 1$ ($1 \leq i \leq p$).

Since $F = K(\bigcup_{k=1}^d \bigcup_{(l_1, \dots, l_p) \in \mathbb{N}^p} \Theta(l_1, \dots, l_p)\eta_k)$, there exist

$$h_1, \dots, h_p \in \mathbb{N} \text{ such that } \theta_j \eta_j \in K(\bigcup_{k=1}^d \Theta(h_1, \dots, h_p)\eta_k \cup \{\theta\eta_k \mid \theta \in \Theta, \theta y_k <_1 \theta_j y_j, \text{ord}_i \theta y_k \leq a_{ji} (i = 1, \dots, p)\}).$$

Let $\theta' \in \Theta$ and $\theta_j \mid \theta'$. For any $i = 1, \dots, p$, let $s_i = \text{ord}_i \theta'$ (clearly, $s_i \geq a_{ji}$). Then

$$\theta' \eta_j \in K(\bigcup_{k=1}^d \Theta(s_1 + h_1, \dots, s_p + h_p)\eta_k \cup \{\theta\eta_k \mid \theta \in \Theta, \theta y_k <_1 \theta' y_j, \text{ord}_i \theta \leq \text{ord}_i \theta' (1 \leq i \leq p) \text{ and } \theta_j \nmid \theta\}).$$

Therefore, if $r_i \in \mathbb{N}$, $r_i \geq \max_{d+1 \leq j \leq n} \{a_{ji}\}$ ($i = 1, \dots, p$), then $K(\bigcup_{k=1}^n \Theta(r_1, \dots, r_p)\eta_k) \subseteq K(\bigcup_{k=1}^d \Theta(r_1 + h_1, \dots, r_p + h_p)\eta_k \cup \bigcup_{j=d+1}^n [\Theta(r_1, \dots, r_p) \setminus \Theta(r_1 - a_{j1}, \dots, r_p - a_{jp})]\eta_j)$.

It follows that $\Phi_{\eta|K}(r_1, \dots, r_p) \leq d \prod_{i=1}^p \binom{r_i + m_i}{m_i} + \sum_{j=d+1}^n \left[\prod_{i=1}^p \binom{r_i + m_i}{m_i} - \prod_{i=1}^p \binom{r_i - a_{ji} + m_i}{m_i} \right]$ for all sufficiently large $(r_1, \dots, r_p) \in \mathbb{N}^p$. Since the total degree of the polynomial $\sum_{j=d+1}^n \left[\prod_{i=1}^p \binom{r_i + m_i}{m_i} - \prod_{i=1}^p \binom{r_i - a_{ji} + m_i}{m_i} \right]$ is less than m , we obtain that $a_{m_1 \dots m_p} \leq d$.

On the other hand, for all sufficiently large $(r_1, \dots, r_p) \in \mathbb{N}^p$, $\Phi_{\eta|K}(r_1, \dots, r_p) = \text{tr. deg}_K K(\bigcup_{k=1}^d \Theta(r_1, \dots, r_p)\eta_k) \geq \text{tr. deg}_K K(\bigcup_{k=1}^d \Theta(r_1, \dots, r_p)\eta_k) = d \prod_{i=1}^p \binom{r_i + m_i}{m_i}$, hence $a_{m_1 \dots m_p} \geq d$.

Thus, $a_{m_1 \dots m_p} = d = \Delta\text{-tr. deg}_K L$. $\square$

With the notation of Theorem 3.15, let $p \geq 2$, $1 \leq k < p$, $\Delta^{(k)} = \Delta_1 \cup \dots \cup \Delta_k$, and $F_{r_{k+1}, \dots, r_p}$ denote the $\Delta^{(k)}$ -field extension of K generated by the set $\bigcup_{j=1}^n \Theta(0, \dots, 0, r_{k+1}, \dots, r_p)\eta_j$ ($r_i \in \mathbb{N}$), that is, $F_{r_{k+1}, \dots, r_p} = K(\bigcup_{j=1}^n \Theta_{\Delta \setminus \Delta^{(k)}}(r_{k+1}, \dots, r_p)\eta_j)_{\Delta^{(k)}}$. Since $\Theta(r_1, \dots, r_p) = \Theta(r_1, \dots, r_k, 0, \dots, 0)\Theta(0, \dots, 0, r_{k+1}, \dots, r_p)$, we can combine the results of Theorems 3.11 and 3.16 to get the following statement.

COROLLARY 3.17. *With the above notation, and the Δ -dimension polynomial (6) of the extension $K\langle\eta_1, \dots, \eta_n\rangle/K$, the numerical polynomial in $p - k$ variables $\phi(t_{k+1}, \dots, t_p) =$*

$$\sum_{i_{k+1}=0}^{m_{k+1}} \dots \sum_{i_p=0}^{m_p} a_{m_1 \dots m_k i_{k+1} \dots i_p} \binom{t_{k+1} + i_{k+1}}{i_{k+1}} \dots \binom{t_p + i_p}{i_p}$$

describes the growth of $\Delta^{(k)}\text{-tr. deg}_K F_{r_{k+1}, \dots, r_p}$, that is,

$$\phi(r_{k+1}, \dots, r_p) = \Delta^{(k)}\text{-tr. deg}_K F_{r_{k+1}, \dots, r_p} \text{ for all sufficiently large } r_{k+1}, \dots, r_p \in \mathbb{N}^{p-k}.$$

This corollary, in particular, shows that if $L = K\langle\eta_1, \dots, \eta_n\rangle$ is a finitely generated differential field extension of a differential field K with a basic set Δ , $\Delta' \subseteq \Delta$, $\Delta'' = \Delta \setminus \Delta'$, and $m_1 = \text{Card } \Delta'$, $m_2 = \text{Card } \Delta''$ ($m_1 + m_2 = m$ where $m = \text{Card } \Delta$), then there exists

a univariate numerical polynomial $\phi(t) = \sum_{i=0}^{m_2} c_i \binom{t+i}{i}$ ($c_i \in \mathbb{Z}$) such that $\phi(r) = \Delta' \text{-tr. deg}_K K(\bigcup_{k=1}^n \Theta_{\Delta''}(r) \eta_k)_{\Delta'}$ and $c_{m_2} = \Delta' \text{-tr. deg}_K L$. Furthermore, if $\Phi_{\eta|K}(t_1, t_2) = \sum_{i=0}^{m_1} \sum_{j=0}^{m_2} a_{ij} \binom{t_1+i}{i} \binom{t_2+j}{j}$ is the bivariate Δ -dimension polynomial of L/K associated with the partition $\Delta = \Delta' \cup \Delta''$ and $d = \deg_{t_1} \Phi_{\eta|K} < m_1$, then $\Delta' \text{-tr. deg}_K K(\bigcup_{k=1}^n \Theta_{\Delta''}(r) \eta_k)_{\Delta'} = \sum_{i=0}^{m_2} a_{dj} \binom{r+j}{j}$ (in this case $d = \Delta' \text{-type}_K K(\bigcup_{k=1}^n \Theta_{\Delta''}(r) \eta_k)_{\Delta'}$).

The following theorem provides necessary and sufficient conditions on generators of a differential field extension of a given differential transcendence degree d under which the corresponding multivariate dimension polynomial has the simplest possible form.

THEOREM 3.18. *With the notation of Theorem 3.15, the following conditions are equivalent.*

- (i) $\Phi_{\eta|K}(t_1, \dots, t_p) = d \prod_{i=1}^p \binom{t_i + m_i}{m_i}$.
- (ii) $\Delta \text{-tr. deg}_K K(\eta_1, \dots, \eta_n) = \text{tr. deg}_K(\eta_1, \dots, \eta_n) = d$.

PROOF. (i) $\Rightarrow$ (ii). By Theorem 3.16, $d = \Delta \text{-tr. deg}_K L$ where $L = K(\eta_1, \dots, \eta_n)$. Without loss of generality we can assume that $\eta_1, \dots, \eta_d$ is a Δ -transcendence basis of L over K . Then for all sufficiently large $(r_1, \dots, r_p) \in \mathbb{N}^p$, $\Phi_{\eta|K}(r_1, \dots, r_p) = \text{tr. deg}_K K(\bigcup_{j=1}^n \Theta(r_1, \dots, r_p) \eta_j) = \text{tr. deg}_K K(\bigcup_{j=1}^d \Theta(r_1, \dots, r_p) \eta_j)$, hence $\text{tr. deg}_K K(\bigcup_{j=1}^d \Theta(r_1, \dots, r_p) \eta_j) = 0$. Therefore, every element η_j , $d+1 \leq j \leq n$, is algebraic over the field $F = K(\eta_1, \dots, \eta_d)$, so if η' denotes the $(n-d)$ -tuple $(\eta_{d+1}, \dots, \eta_n)$, then $\Phi_{\eta'|F}(t_1, \dots, t_p) = 0$.

Let P be the defining Δ -ideal of η' in the ring of Δ -polynomials $F\{y_1, \dots, y_{n-d}\}$. Let $\mathcal{A}$ be a characteristic set of P (we use the terminology and term orderings $<_1, \dots, <_p$ introduced in the beginning of this section). For every $j = 1, \dots, n-d$, let E_j denote the set of all $(k_1, \dots, k_m) \in \mathbb{N}^m$ such that $\delta_1^{k_1} \dots \delta_m^{k_m} y_j$ is a 1-leader of an element of $\mathcal{A}$. Since $\Phi_{\eta'|F} = 0$, we also have $\omega_{\eta'|F} = 0$ where $\omega_{\eta'|F}$ is the polynomial in p variables defined in Theorem 3.11(ii). Furthermore, it follows from Theorem 2.3(iv) that $\omega_{\eta'|F} = 0$ if and only if $E_j = \{(0, \dots, 0)\}$ for $j = 1, \dots, n-d$.

Since $y_1 <_1 y_j$ for $j = 2, \dots, n-d$, a Δ -polynomial in $\mathcal{A}$ with leader y_1 is a usual polynomial in y_1 with coefficients in F . Therefore, η_{d+1} and all $\theta \eta_{d+1}$ ($\theta \in \Theta$) are algebraic over F . If $\eta'' = (\eta_{d+2}, \dots, \eta_n)$, then $\Phi_{\eta''|F}(r_1, \dots, r_p) \leq \Phi_{\eta'|F}(r_1, \dots, r_p)$ for all $(r_1, \dots, r_p) \in \mathbb{N}^p$, so $\Phi_{\eta''|F} = 0$ and we can repeat the above arguments and obtain that every $\theta \eta_j$ ($\theta \in \Theta$, $d+1 \leq j \leq n$) is algebraic over F .

Since the elements $\eta_{d+1}, \dots, \eta_n$ are algebraic over the field $F = K(\eta_1, \dots, \eta_d)$, there exist $h_1, \dots, h_p \in \mathbb{N}$ such that $\eta_{d+1}, \dots, \eta_n$ are algebraic over $K(\bigcup_{j=1}^d \Theta(h_1, \dots, h_p) \eta_j)$. It follows that if $(h_1, \dots, h_p) \leq_p (r_1, \dots, r_p)$, then the field extension $K(\bigcup_{j=1}^n \Theta(r_1, \dots, r_p) \eta_j) / K(\bigcup_{j=1}^d \Theta(h_1, \dots, h_p) \eta_j)$ is algebraic.

Suppose η_{d+1} is not algebraic over $K(\eta_1, \dots, \eta_d)$. Let $q_1, \dots, q_p$ be a minimal (with respect to the product order $<_p$) element of $\mathbb{N}^p$ such that η_{d+1} is algebraic over the field $K(\bigcup_{j=1}^d \Theta(q_1, \dots, q_p) \eta_j)$.

(By the assumption, $(q_1, \dots, q_p) \neq (0, \dots, 0)$). Without loss of generality we can assume that $q_1 \geq 1$. Then η_{d+1} is transcendental over the field

$K(\bigcup_{j=1}^d \Theta(q_1 - 1, \dots, q_p) \eta_j)$. Then there exists a term v in the ring of Δ -polynomials $K\{y_1, \dots, y_d\}$ such that $\text{ord}_i v = q_i$, $\text{ord}_i v \leq q_i$ for $i = 2, \dots, p$, η_{d+1} is transcendental over the field $K' = K(\{\theta \eta_j \mid \theta \in \Theta(q_1, \dots, q_p), 1 \leq j \leq d, \theta y_j <_1 v\})$ and algebraic over the field $K'(v(\eta))$. It follows that $v(\eta)$ is algebraic over $K(\bigcup_{j=1}^d \Theta(q_1, \dots, q_p) \eta_j \setminus \{\eta_{d+1}\} \cup \{v(\eta)\})$.

Therefore, if $\theta' \in \Theta(r_1, \dots, r_p)$ where $(h_1, \dots, h_p) \leq_p (r_1, \dots, r_p)$, then $\theta' v(\eta)$ is algebraic over $K(\bigcup_{j=1}^d \Theta(r_1 + q_1, \dots, r_p + q_p) \eta_j \setminus \{\theta' \eta_{d+1}\} \cup \{\theta' v(\eta)\})$.

Since η_{d+1} is algebraic over $K(\bigcup_{j=1}^d \Theta(q_1, \dots, q_p) \eta_j)$, $\theta' \eta_{d+1}$ is algebraic over $K(\bigcup_{j=1}^d \Theta(s_1 + q_1, \dots, s_p + q_p) \eta_j)$ where $s_i = \text{ord}_i \theta'$, $1 \leq i \leq p$ (clearly, $s_i \leq r_i$ for $i = 1, \dots, p$). Therefore, $\theta' v(\eta)$ is algebraic over $K(\bigcup_{j=1}^d \Theta(s_1 + q_1, \dots, s_p + q_p) \eta_j \setminus \{\theta' v(\eta)\})$, hence the set $\bigcup_{j=1}^d \Theta(r_1 + q_1, \dots, r_p + q_p) \eta_j$ is algebraically dependent over K that contradicts the fact that $\eta_1, \dots, \eta_d$ are Δ -algebraically independent over K .

Thus, η_{d+1} is algebraic over $K(\eta_1, \dots, \eta_d)$ and similarly every η_j , $d+1 \leq j \leq n$, is algebraic over $K(\eta_1, \dots, \eta_d)$, so $d = \Delta \text{-tr. deg}_K K(\eta_1, \dots, \eta_n) = \text{tr. deg}_K(\eta_1, \dots, \eta_n)$.

(ii) $\Rightarrow$ (i). As in the proof of Theorem 3.16, without loss of generality we can assume that $\eta_1, \dots, \eta_d$ is a Δ -transcendence basis of the Δ -field $L = K(\eta_1, \dots, \eta_n)$ over K . Then the elements $\eta_1, \dots, \eta_d$ are algebraically independent over K , so $K(\eta_1, \dots, \eta_n)$ is an algebraic extension of $K(\eta_1, \dots, \eta_d)$. Thus, $K(\bigcup_{j=1}^n \Theta(r_1, \dots, r_p) \eta_j)$ is an algebraic extension of the field $K(\bigcup_{j=1}^d \Theta(r_1, \dots, r_p) \eta_j)$ for any $(r_1, \dots, r_p) \in \mathbb{N}^p$.

Since $\Phi_{(\eta_1, \dots, \eta_d)|K}(t_1, \dots, t_p) = d \prod_{i=1}^p \binom{t_i + m_i}{m_i}$ and $K(\bigcup_{j=1}^n \Theta(r_1, \dots, r_p) \eta_j)$ and $K(\bigcup_{j=1}^d \Theta(r_1, \dots, r_p) \eta_j)$ have the same transcendence degree over K , we obtain the equality of statement (i). $\square$

PROPOSITION 3.19. *Let $L = K(\eta_1, \dots, \eta_n)$ be a Δ -field extension generated by a finite set $\eta = \{\eta_1, \dots, \eta_n\}$ and let partition (3) of the set Δ be fixed. Suppose that $\Delta \text{-tr. deg}_K L = 0$ or that $\Delta \text{-tr. deg}_K L = d \geq 1$, $\eta_1, \dots, \eta_d$ form a Δ -transcendence basis of L over K and $\eta' = \{\eta_{d+1}, \dots, \eta_n\}$. Then*

$$\Phi_{\eta'|K(\eta_1, \dots, \eta_d)}(r_1, \dots, r_p) \leq \Phi_{\eta|K}(r_1, \dots, r_p) - d \prod_{i=1}^p \binom{r_i + m_i}{m_i} \quad (7)$$

for all sufficiently large $(r_1, \dots, r_p) \in \mathbb{N}^p$.

PROOF. If $\Delta \text{-tr. deg}_K L = 0$, the statement is obvious. Let $d = \Delta \text{-tr. deg}_K L \geq 1$ and $\{\eta_1, \dots, \eta_d\}$ a Δ -transcendence basis of L/K . Let $K' = K(\eta_1, \dots, \eta_d)$ and for any $r = (r_1, \dots, r_p) \in \mathbb{N}^p$, $\Lambda_1(r) = \bigcup_{k=1}^d \Theta(r_1, \dots, r_p) \eta_k$, $\Lambda_2(r) = \bigcup_{l=d+1}^n \Theta(r_1, \dots, r_p) \eta_l$ and $\Lambda_3(r) = \Lambda_1(r) \cup \Lambda_2(r)$.

Then $\Phi_{\eta'|K'}(r_1, \dots, r_p) = \text{tr. deg}_{K'} K'(\Lambda_2(r)) \leq \text{tr. deg}_{K(\Lambda_1(r))} K(\Lambda_3(r)) \text{tr. deg}_{K(\Lambda_1(r))} K(\Lambda_3(r)) = \text{tr. deg}_K K(\Lambda_3(r)) - \text{tr. deg}_K K(\Lambda_1(r)) = \Phi_{\eta|K}(r_1, \dots, r_p) - d \prod_{i=1}^p \binom{r_i + m_i}{m_i}$ for all sufficiently large $(r_1, \dots, r_p)$. $\square$

The next example shows that a multivariate dimension polynomial of a Δ -field extension carries essentially more information about the extension than its univariate counterpart.

EXAMPLE 3.20. Let K be a differential field with a basic set of derivations $\Delta = \{\delta_1, \delta_2, \delta_3\}$ and let L be a Δ -field extension of K generated by a single Δ -generator η with the defining equation

$$\delta_1^a \delta_2^b \delta_3^c \eta + \delta_1^a \eta + \delta_2^b \eta + \delta_3^{b+c} \eta = 0 \quad (8)$$

where a, b and c are some positive integers. In other words, $L = K\langle\eta\rangle$ is Δ -isomorphic to the quotient field of the factor ring $K\{y\}/P$ where P is the linear (and therefore prime) Δ -ideal of the ring of differential (Δ -) polynomials $K\{y\}$ generated by the Δ -polynomial $f = \delta_1^a \delta_2^b \delta_3^c y + \delta_1^a y + \delta_2^b y + \delta_3^{b+c} y$. (P is the defining ideal of η over K .)

By [6, Chapter II, Theorem 6], the univariate Kolchin differential dimension polynomial $\omega_{\eta/K}(t)$ of L/K is equal to the univariate dimension polynomial of the subset $\{(a, b, c)\}$ of $\mathbb{N}^3$. Using formula (2) for $p = 1$, we obtain that

$$\begin{aligned} \omega_{\eta/K}(t) &= \binom{t+3}{3} - \binom{t+3-(a+b+c)}{3} = \\ &= \left(\frac{a+b+c}{2}\right)t^2 + \left(\frac{(a+b+c)(4-a-b-c)}{2}\right)t + \\ &= \frac{(a+b+c)[(a+b+c)^2 - 6(a+b+c) + 11]}{6}. \end{aligned} \quad (9)$$

Now, let us fix a partition $\Delta = \Delta_1 \cup \Delta_2$ with $\Delta_1 = \{\delta_1, \delta_2\}$. Let $\Delta_2 = \{\delta_3\}$, and $\Phi_{\eta}(t_1, t_2)$ denote the Δ -dimension polynomial of L/K associated with this partition and the Δ -generator η . With the notation of the first part of this section, we obtain that $u_f^{(1)} = \delta_1^a \delta_2^b \delta_3^c y$ and $u_f^{(2)} = \delta_3^{b+c} y$. Using the notation of Theorem 3.11 and formula (2) we obtain that for all sufficiently large $(r_1, r_2) \in \mathbb{N}^2$, $\text{Card } U'_{r_1, r_2} = \binom{r_1+2}{2}(r_2+1) - \binom{r_1+2-(a+b)}{2}(r_2+1-c)$. Expanding the last expression and using symbols t_1 and t_2 for the variables representing r_1 and r_2 , respectively, we obtain the polynomial $\omega_{\eta/K}(t_1, t_2)$ (see Theorem 3.11) that describes the size of $\text{Card } U'_{r_1, r_2}$: $\omega_{\eta/K}(t_1, t_2) = \frac{c}{2}t_1^2 + (a+b)t_1t_2 + \frac{2a+2b+3c-2ac-2bc}{2}t_1 + \frac{(a+b)(3-a-b)}{2}t_2 + \frac{1}{2}[(a+b-2)(a+b-1)(c-1)+2]$.

For all sufficiently large $(r_1, r_2) \in \mathbb{N}^2$, $\text{Card } U''_{r_1, r_2} = \text{Card}\{\delta_1^{a+k_1} \delta_2^{a+k_2} \delta_3^{a+k_3} \mid k_1, k_2, k_3 \in \mathbb{N}, k_1+k_2 \leq r_1-(a+b), r_3-(b+c) < k_3 \leq r_3-c\} = \binom{r_1+2-(a+b)}{2}b$. Thus, with the notation of Theorem 3.11, $\phi_{\eta/K}(t_1, t_2) = \frac{b}{2}t_1^2 + \frac{b(3-2a-2b)}{2}t_1 + \frac{b(a+b-2)(a+b-1)}{2}$. It follows that the bivariate differential dimension polynomial of the extension L/K corresponding to the partition $\Delta = \{\delta_1, \delta_2\} \cup \{\delta_3\}$ is

$$\begin{aligned} \Phi_{\eta/K}(t_1, t_2) &= \left(\frac{b+c}{2}\right)t_1^2 + (a+b)t_1t_2 + \frac{1}{2}[2a+5b+3c-2ab-2ac \\ &\quad -2bc-2b^2]t_1 + \frac{(a+b)(3-a-b)}{2}t_2 + \\ &\quad \frac{1}{2}[(a+b-2)(a+b-1)(b+c-1)+2]. \end{aligned} \quad (10)$$

Finally, let us fix a partition $\Delta = \Delta_1 \cup \Delta_2 \cup \Delta_3$ with $\Delta_i = \{\delta_i\}$ ($i = 1, 2, 3$). Proceeding as before (with the notation of Theorem 3.11), we obtain that $\omega_{\eta/K}(t_1, t_2, t_3) = ct_1t_2 + bt_1t_3 + at_2t_3 + (b+c-bc)t_1 + (a+c-ac)t_2 + (a+b-ab)t_3 + a+b+c-ab-ac-bc+abc$

and $\phi_{\eta/K}(t_1, t_2) = bt_1t_2 + (b-b^2)t_1 + (b-ab)t_2 + (b-ab-b^2+ab^2)$, so

$$\begin{aligned} \Phi_{\eta/K}(t_1, t_2, t_3) &= (b+c)t_1t_2 + bt_1t_3 + at_2t_3 \\ &\quad + (2b+ac-bc-b^2)t_1 + (a+b+c-ab-ac)t_2 + (a+b-ab)t_3 \\ &\quad + (a+2b+c-2ab-ac-bc-b^2+ab^2+abc). \end{aligned} \quad (11)$$

It follows from Theorem 3.15 that the dimension Δ -polynomial in three variables given by (11) carries four invariants of the extension L/K : the total degree 2 and the coefficients $b+c$, b and a of the terms t_1t_2 , t_1t_3 and t_2t_3 , respectively. The dimensional polynomial (10) carries three invariants, the total degree 2 and the coefficients $b+c$ and $a+b$, while the univariate Kolchin polynomial (9) carries only two invariants of the extension, the total degree 2 and the sum of the parameters $a+b+c$. Therefore, the Δ -dimension polynomial (11) corresponding to the partition of Δ into the union of three disjoint subsets determines all three parameters a , b and c of the defining differential equation (8) while the univariate dimension polynomial gives just the sum of the parameters.

Also, in accordance with the above considerations, the dimension polynomial (10) (with $\Delta_1 = \{\delta_1, \delta_2\}$ and $\Delta_2 = \{\delta_3\}$) shows that Δ_2 -tr. $\deg_K K\langle\{\delta_1^{k_1} \delta_2^{k_2} \eta \mid k_1+k_2 \leq r\}\rangle_{\Delta_2} = (a+b)r + \frac{(a+b)(3-a-b)}{2}$ for all sufficiently large $r \in \mathbb{N}$.

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