Low-Order Finite Element Solver with Small Matrix-Matrix Multiplication Accelerated by AI-Specific Hardware for Crustal Deformation Computation

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ABSTRACT

This study proposes a fast low-order finite element solver for crustal deformation computations by applying Tensor Core, AI-specific hardware on a Volta GPU. Tensor Core can compute large matrix-matrix multiplications rapidly in half precision. We redesign a state-of-the-art solver algorithm so that lower-precision data types can be used and memory access costs can be reduced even when we use small matrices. With the proposed solver, we solved 13 billion degrees-of-freedom two-layered problems that mimicked the Earth's crust and mantle using 36 compute nodes of Summit. In the matrix-vector kernel, we obtained a 4.1-fold speedup over a standard kernel in a single-precision format. Our proposed solver increased the FLOP count of the entire solver; however, we reduced the time-to-solution by 1.7-fold since the Tensor Core provided a high effective performance.

CCS CONCEPTS

- **Computing methodologies** → **Massively parallel algorithms;**
- **Applied computing** → **Earth and atmospheric sciences.**

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KEYWORDS

Transprecision computing, GPU computation, Finite element analysis, Conjugate gradient method

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1 INTRODUCTION

1.1 Background of Crustal Deformation Computation

A better understanding of earthquakes and their generation processes is important for future disaster mitigation. References [32] discuss inverse analysis for the distribution of fault slip. This inversion connects to evaluation of long-term plate movements and future earthquake occurrence. It requires many Green's functions and GPS-based crustal deformation observation data to compare. Green's functions are the responses including post-earthquake deformation for given inter-plate slips, each of which is computed by forward analysis of crustal deformation computation. Crustal deformation observation data has been accumulated as observation systems have been developed (e.g., the GNSS Earth Observation Network System (GEONET) [35] in the area around Japan). Jointly

with observation networks, inverse analyses with a lot of crustal deformation computations will enhance the capability of monitoring and evaluating earthquake hazard.

To realize the inversion, we need to analyze the time history of the visco-elasto-plastic response of the 3D heterogeneous crust and mantle because of the slip on the plate boundary and faults. Previously, analytic solutions ([36] for elastic analysis and [10] for visco-elastic analysis) were widely used under the conditions that target domains were homogeneous or horizontally-layered. As crustal structure data has been accumulated [43], [6], [17], models of fault interactions considering fault friction with high resolution have been available. Therefore, the demand for crustal deformation computations using these high-resolution models has increased. Some studies have used low-order finite element modeling (FEM) to accommodate heterogeneity in material properties and complex geometry of the target domain [28], [37], [47]. It is implied that fine spatial resolution in the entire medium surrounding the faults using is necessary, given the heterogeneous nature of the Earth [30], [18]. In addition, FEM has the advantage to satisfy the traction-free condition at the ground surface automatically. Following previous studies, we use FEM for the target problem.

On the other hand, inversion using FEM requires massive computation cost. At least a km-scale resolution is required in the simulation, and visco-elasto-plastic loading and response of the faults and surrounding crust must be captured. When we compute for $1000 \text{ km} \times 1000 \text{ km} \times 100 \text{ km}$ domain with approximately 1 km resolution, 10^{10} to 10^{11} degrees of freedom (DOF) are necessary in the FEM. The analysis of 10^3 time steps is required to obtain the time-history response. Moreover, introduction of inverse analysis requires 10^2 to 10^3 case forward analyses.

To conduct these simulations within a realistic timeframe, we must reduce the computation cost. The visco-elasto-plastic response analysis can be realized by a small modification of the solver algorithm for the static response analyses of the linear elastic body (see [37] for visco-elastic analysis); hence discussion of computation cost in linear problems works for that in visco-elasto-plastic analysis. In this paper, we focus on the performance in finite element method for elasto-static problems.

1.2 Background of Finite Element Solver for Crustal Deformation Computation

The solver part accounts for the largest proportion of the computation cost in the entire analysis; thus, we focus on the acceleration of the solver. Multi-grid solvers are considered suitable for our targeting problems as they can attain good convergency even for larger-scale problems by reducing long wavelength errors efficiently on their coarsened grids.

There are many variants of multi-grid solver algorithms according to the problem characteristics of the crustal deformation problem (e.g., complexity in geometry, heterogeneity of material properties, and targeting resolution). For example, [39] is proposed for solving large-scale problems in global mantle convection using hexahedral elements and adaptive refinement, and [9] is proposed for solving large-scale static problems with tetrahedral elements. Therefore, it is not straightforward to define a single state-of-the-art solver for comparison in a general setting.

In this paper, we define a multi-grid solver [21] as the baseline solver that is robust for solving a wide variety of tetrahedral mesh problems which combines conjugate-gradient methods to improve robustness of the geometric multi-grid method. It is shown to attain high performance and scalability on large-scale CPU- and GPU-based computer systems [21], [20], [22]. In particular, we compare performance with the SC14 solver as it is the most general and robust. As explained in Section 4, most components are common to the SC14 solver and our proposed solver; thus, we can clearly see the factor of the performance improvement in our proposed method. A variant of this solver using algebraic multi-grid (AMG) exists [9], but regeneration of AMG-matrices in each nonlinear iteration of nonlinear problems leads to high cost in GPU environments, and thus we remove this from comparison. Although the solver performance is compared to a single solver algorithm, the basic algorithm concepts of coarsening and the computational components used in the variety of multi-grid solvers are similar (e.g., matrix-vector products used in smoothers); thus, the idea of acceleration of the key component in the solver is expected to be useful for a variety of problem settings.

1.3 Motivation for Using AI-specific Hardware

Considering the diversity of hardware, it is expected that algorithms can be developed for specific hardware that are faster than state-of-the-art algorithms designed for general hardware. Development of deep learning is expanding the demand to compute matrix-matrix multiplication rapidly. Learning is computationally expensive and can sometimes take up to 10^4 hours as noted in [42], in which matrix-matrix multiplication accounts for the largest proportion of the computational cost. To meet this demand, much hardware specialized for matrix-matrix multiplication has been developed. For example, the NVIDIA Volta GPU [34] has Tensor Core, which was designed to accelerate deep learning [29], while the entire GPU is compatible with general scientific computations. Peak FLOPS performance using single or double precision is 7.8 or 15.6 TFLOPS, respectively, whereas the performance of Tensor Core can reach 125 TFLOPS, so significant acceleration can be achieved. A Tensor Processing Unit (TPU) [12] by Google also targets deep learning, and the second-generation TPU has achieved 45 TOPS per chip. This tendency is expected to continue. In fact, chips designed for artificial intelligence (AI) [5], [4] have been announced by many corporations at Hot Chips, which is known as the conference where up-to-date information about newly developed chips is released.

By utilizing these computational resources, it is possible to benefit from the high peak performance of specialized cores. In this paper, we target acceleration of a solver based on multi-grid algorithms. On the other hand, to use this AI-specific hardware for our target application, the following two points must be resolved.

Firstly, such hardware only supports lower-precision data types because the main target application is deep learning. As described in [14], deep learning does not require very high accuracy. Hence, Tensor Core on a Volta GPU and a TPU only support data types with accuracy less than single precision. A common data type used in deep learning, half precision numbers are composed of a 1-bit sign part, a 5-bit exponent part, and a 10-bit fraction part, as defined in IEEE 754 [27]. Exponent bits and fraction bits are truncated from

single-precision data types, which are composed of a 1-bit sign part, an 8-bit exponent part, and a 23-bit fraction part. Accordingly, half-precision numbers have smaller dynamic ranges and less accuracy as compared to single- or double-precision. The minimum and maximum positive values that can be represented by half-precision data type are 10^{-8} and 65,504, respectively. This range is too small for general numerical simulations and heedless implementations easily lead to overflow or underflow; therefore, methods that avoid these conditions are essential. Half-precision data types are unable to guarantee accuracy of more than four digits, so the issue of accuracy is critical to convergence of solvers. A TPU uses another data type called bfloat16 [46] for multiplication, but its accuracy is less than that of half precision because it truncates many bits of the fraction part while keeping the same number of exponent bits as in single-precision data types. Hence, it is also a challenge to apply bfloat16 to iterative solvers.

Secondly, memory access costs are a bottleneck to the performance when small matrices are used. General matrix-matrix multiplications are dense computations and easily parallelized; hence, they are suitable for GPU computations. Methods to execute such multiplications on GPUs have been developed with cuBLAS [33] and MAGMA [45] as major libraries, which are tuned mainly for multiplications involving large matrices. They already have functions to use Tensor Core and these libraries are most straightforward ways to use AI-specific hardware. In these matrix-matrix multiplications, the computation and data access costs are of the orders, $O(N^3)$ and $O(N^2)$, respectively, for matrices of size N ; thus, as the matrix size increases, memory access costs relatively decrease. The aforementioned hardware has achieved good performance for multiplication operations that are common in typical deep learning applications that use matrices with sizes of more than several hundred. However, the sizes of our targeting matrices are only several dozen elements as we are targeting low-order FEM. Smaller matrix-matrix multiplications appear in various applications (e.g., sparse direct solver [49] and tensor contractions [25]). In these computations, data access costs can be a bottleneck that hampers the utilization of the hardware's full potential. A previous study [3] showed that it was difficult for these libraries to attain good performance for small matrix-matrix multiplication, so optimizations of the target matrix sizes were required.

To use hardware dedicated to matrix-matrix multiplication in lower-precision data types, we have the issues noted above.

In this study, we propose a low-order finite element solver with small matrix-matrix multiplications accelerated by AI-specific hardware. We focus especially on Tensor Core on an NVIDIA Tesla V100 GPU. By designing an appropriate multi-grid preconditioner, we obtain the results at double precision while most computations are executed at a lower precision. Also, we introduce methods to reduce data access costs so that access to the device memory or other memories does not limit performance of the matrix-matrix multiplications. We were able to introduce further optimizations of thread parallelization and memory access for our target computations; thus, our implementation achieves better performance than typical libraries at the expense of versatility. Our proposed algorithm increases the number of computations compared to a conventional solver or a state-of-the-art solver, but the time-to-solution

is shorter as most computations are matrix-matrix multiplications that can be computed rapidly on Tensor Core.

Many approaches to use transprecision computing in linear solvers have been proposed [7], [15], [40], [8], [16]. Especially, the solver in [16] can obtain the solution in double precision with refinement of the initial solution computed on Tensor Core. This solver targets dense matrix and its matrix size is 10^3 ; thus our target problem is assumed to be more difficult to attain reasonable performance. Also, an approach that computes matrix-vector multiplication as matrix-matrix multiplication is proposed in [2]. It uses high-order elements; however, our application uses low-order elements. Therefore, the matrix sizes are smaller and pose a greater challenge to good performance.

Considering the development trend of similar hardware, a successful case to use such hardware by incorporating computations of lower precision and reducing memory access costs is expected to have implications for improvements in the performance of other scientific applications. Our proposed solver is an extension of [23]. In this paper, we describe the methodology in detail and the additional performance measurements conducted in a smaller computation environment.

The remainder of this paper is organized as follows. Section 2 describes the solver algorithm that enables matrix-matrix multiplication on Tensor Core, and Section 3 describes performance tunings for small matrix-matrix multiplications. Section 4 discusses the effectiveness of our proposed method via performance measurements made on the supercomputer, Summit. Section 5 presents our conclusions.

2 SOLVER ALGORITHM TO USE TENSOR CORE

In the present study, we focus on the finite element method with second-order tetrahedral elements for the static response of a linear elastic body [19]. To calculate a displacement function u_i with double precision, the target equation is:

$$\frac{d}{dx_i} \left(c_{ijkl}(\mathbf{x}) \frac{d}{dx_k} u_l(\mathbf{x}) \right) = 0, \quad (1)$$

Here, c_{ijkl} denotes the stiffness tensor. $\mathbf{x}$ and $\frac{d}{dx_i}$ denote the spatial coordinate and differentiation with respect to x_i , respectively. By discretizing Eq. (1), it results in the following equation:

$$\mathbf{K}\mathbf{u} = \mathbf{f}, \quad (2)$$

where $\mathbf{K}$ is the global stiffness matrix, $\mathbf{u}$ is the displacement vector, and $\mathbf{f}$ is the force vector.

2.1 Overview of the solver

We target the acceleration of this equation-based modeling by utilizing AI-specific architecture such as Tensor Core. We convert the algorithm to a highly dense computational one consisting of matrix-matrix multiplications.

To use Tensor Core's operations, we have to reduce the accuracy of the variables in the solver. So, we use an adaptive conjugate gradient-type solver [11], which roughly solves the same target

equation by employing conjugate gradient solvers in the preconditioner. This solver has the capability of combining various methods into the preconditioner.

Only an approximate solution is required in the preconditioner; therefore, we can combine the use of several methods to gradually solve the approximated equation at low cost. Especially, we can use lower precision numbers in the preconditioner (see [40] for similar approaches). Fig. 1 shows the developed solver algorithm. Here, all matrix-vector multiplications in the solver are computed by an element-by-element method [48] to reduce memory usage. We use a model mapped to a structured mesh $PreCG^v$, linear tetrahedral elements $PreCG^e$, and the target second-order tetrahedral elements $PreCG$. In addition, we used $PreCG_{part}$, which solves parts of the problem that are expected to have large differences from the original model $PreCG$ when using the abovementioned preconditioner. Here, $PreCG_{part}$ was extracted from $PreCG$ using an AI-based method [22]. Only this part was then solved in the preconditioner. Furthermore, instead of using a standard voxel finite element in $PreCG^v$, we designed an algorithm with a high mapping accuracy of the original problem $PreCG$ together with a high computational density.

2.2 Mapping of unstructured problem to structured mesh

For $PreCG^v$, we generate a 3D structured grid model $\bar{c}_{ijkl}$ equivalent to an unstructured grid based on c_{ijkl} of (1). Constant material properties are assigned to each structured grid, which has a size of $(ds/2)^3$, where ds is the size of each voxel element. Here, we target domains of arbitrary size. It is desirable to use a solver algorithm that can be applied robustly; therefore, we use conjugate gradient method for $PreCG^v$. 3×3 block Jacobi preconditioning [41] is used, which requires less memory consumption than other typical preconditioners. Obtaining an accurate solution with $PreCG^v$ leads to the reduction of the more costly $PreCG^e$ and $PreCG$, so we target the generation of $\bar{c}_{ijkl}$ with high mapping accuracy and low construction cost. Following the analyses of [24], we set $\bar{c}_{ijkl}$ as the volume average of c_{ijkl} in each structured grid with a size of $(ds/2^3)$.

Using this structured grid-based setting enables one to obtain low-spatial frequency solutions with considerable accuracy. These solutions are used as the initial solutions in the latter solvers in the preconditioner. While the low-frequency components of the solution are expected to become accurate because of the coarsening based on physical properties, local errors are expected to increase in places where the differences between c_{ijkl} and $\bar{c}_{ijkl}$ are large. Thus, following [22], we estimated the parts of the problem expected to have large errors using AI and solved only these parts using $PreCG_{part}$ (see footnote ¹ for details).

The mapping between solvers is conducted in every outer iteration of the adaptive conjugate gradient solver to enable the equation to be solved roughly many times. Thus, almost all of the computational costs of the conjugate gradient solver in the second-order

¹We used a small problem with characteristics similar to that of the target problem to train an artificial neural network (ANN) that estimates the difference between the solutions obtained using c_{ijkl} and those obtained using $\bar{c}_{ijkl}$. The differences in c_{ijkl} and $\bar{c}_{ijkl}$ and the number of nodal connectivity in the mesh were used as input for the ANN. We applied this ANN to the target problem to inference parts to be solved in $PreCG_{part}$.

tetrahedral mesh in double precision are transferred to the coarsened solvers in low-precision arithmetic. Note that even though coarsening and low-precision arithmetic are used in the preconditioner, the final double-precision results of the solver are not altered from those of the standard solvers.

2.3 Introduction of Tensor Core operations to matrix-vector kernel

$PreCG^v$ is a structured grid-based problem and uses uniform basis functions; hence, it is suitable for attaining shorter time-to-solution as its convergency is better than that in unstructured problems. In addition, matrix-vector multiplication in the $PreCG^v$ solver can be rearranged to matrix-matrix multiplication.

Stiffness matrix for ie -th element can be described as follows:

$$\mathbf{K}_{(ie)} = \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \mathbf{B}^T \mathbf{D}_{(ie)}(r_1, r_2, r_3) \mathbf{B} \det|J| dr_1 dr_2 dr_3, \quad (3)$$

where $\mathbf{B}$ is displacement-strain transformation matrix, and $\mathbf{D}_{(ie)}(r_1, r_2, r_3)$ is matrix derived from material properties; r_1 , r_2 , and r_3 are natural coordinate systems, respectively, and $\det|J|$ is a determinant of Jacobian matrix for transformation between natural and global coordinate systems. Note that all components of $\mathbf{B}$ and $\det|J|$ are constant in structured problems as basis functions are uniform. We use voxel elements that consists of eight smaller voxels of size $(ds/2)^3$ and possesses constant isotropic linear elastic material properties. Therefore, the matrix $\mathbf{D}_{(ie)}(r_1, r_2, r_3)$ is constant in each sub-voxel. We denote the matrix from the i -th of eight subvoxels in ie -th element as $\mathbf{D}_{(ie)}^i$. The matrix can be described as follows:

$$\mathbf{D}_{(ie)}^i = \mathbf{K}_{(ie)}^i \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \mathbf{G}_{(ie)}^i \begin{pmatrix} 4/3 & -2/3 & -2/3 & 0 & 0 & 0 \\ -2/3 & 4/3 & -2/3 & 0 & 0 & 0 \\ -2/3 & -2/3 & 4/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (4)$$

where $\mathbf{K}_{(ie)}^i$ and $\mathbf{G}_{(ie)}^i$ are bulk modulus and shear modulus for the i -th of eight subvoxels in ie -th element.

Considering the matrices in the right side of Eq. (4) are also constant, the element stiffness matrix in Eq. (3) is represented as linear combination of two matrices and coefficients derived from $\mathbf{K}_{(ie)}^i$ and $\mathbf{G}_{(ie)}^i$. Eventually, the multiplication of the element stiffness matrix and element vector $\mathbf{u}_{(ie)}$ becomes

$$\mathbf{f}_{(ie)} = \sum_{i=1}^8 \left(\alpha_{i(ie)} \mathbf{P}_i + \beta_{i(ie)} \mathbf{Q}_i \right) \mathbf{u}_{(ie)}. \quad (5)$$

Here, we redefine the scalar values corresponding to the material properties $\mathbf{K}_{(ie)}^i$ and $\mathbf{G}_{(ie)}^i$ as $\alpha_{i(ie)}$ and $\beta_{i(ie)}$. When we extend our method to visco-elastic analysis, we can apply the same procedure by setting appropriate coefficients $\alpha_{i(ie)}$ and $\beta_{i(ie)}$. $\mathbf{P}_i$ and $\mathbf{Q}_i$ denote the 24×24 matrices with constant values and $\mathbf{u}_{(ie)}$ is the ie -th element displacement data of size of 24. The result of the

Figure 1: Solver algorithm. The linear set of the equations of the unstructured second-order tetrahedral mesh is computed in double precision using an adaptive conjugate gradient-based preconditioner, which consists of a Tensor Core-accelerated structured grid solver, a first-order tetrahedral element solver, and a second-order tetrahedral element solver. Point-to-point communication is overlapped with the computations following previous studies [31] so that it does not become a bottleneck of the performance.

matrix-vector product $\mathbf{f}_{(ie)}$ is added into the global left-hand side vector. Although we use an eight-node voxel element, it reflects the heterogeneous material properties of the eight subvoxels, and so, can be considered a special type of coarsening that reflects c_{ijkl} with small data access costs. Considering that $\mathbf{P}_i$ and $\mathbf{Q}_i$ are constant among the elements, we can gather $\mathbf{u}_{(1)}, \mathbf{u}_{(2)}, \dots$ to generate a matrix to conduct a matrix-matrix multiplication: $\mathbf{P}_i \begin{pmatrix} \mathbf{u}_{(1)}, \mathbf{u}_{(2)}, \dots \end{pmatrix}$ and $\mathbf{Q}_i \begin{pmatrix} \mathbf{u}_{(1)}, \mathbf{u}_{(2)}, \dots \end{pmatrix}$. We use Tensor Core for this matrix-matrix multiplication part. Peak FLOP performance on Tensor Core is 4-fold of that in FP16 computations on Volta GPUs. Therefore, it is faster to compute for each constant matrix $\mathbf{P}_i$ and $\mathbf{Q}_i$ as matrix-matrix multiplication on Tensor Core even though the number of required floating-point number operations increases.

Tensor Core is designed to accelerate computations requiring accuracies less than FP32; hence, special care is required to use these arithmetic units for the above computations. Compared to deep learning etc., the conjugate gradient method is prone to numerical errors; therefore, we design an algorithm that is stable even when using reduced precision data types. We add two methods such that the solver converges even when half-precision arithmetic is used by Tensor Core on a Volta GPU. First, we introduce local normalization. We rescale vector $\mathbf{u}_{(ie)}$ (originally in FP32) such that its values are within the range of FP16. After the computation, we reflect the scaling factors when adding $\mathbf{f}_{(ie)}$ in FP32. Secondly, we remove the rigid body motion from the input vectors to circumvent the rounding errors incurred in the computing process. This removal results in the convergence of the solver with an iteration count slightly higher than in FP32.

3 PERFORMANCE TUNING FOR TENSOR CORE COMPUTATION

We design an algorithm that computes much of the computation cost in Eq. (5) on Tensor Core, AI-specific hardware on a Volta GPU. On the other hand, to reduce the time-to-solution of the solver, we have to consider the computational characteristics required in our target computations, which are different to those of AI. Firstly, this kernel involves the read/write operations of the element vectors $\mathbf{p}_{(ie)}$ and $\mathbf{q}_{(ie)}$; hence, it may become memory bound. Second, the matrix size is much smaller than those used in typical deep learning applications and leads to relatively large memory access costs. Therefore, thorough reductions of the data access costs are essential for utilizing the potential of Tensor Core for this kernel. We explain below how we develop the algorithm to reduce access cost for global memory and shared memory.

3.1 Reduction of the device memory access costs

Reductions of the read/write costs of vectors $\mathbf{u}_{(ie)}$ and $\mathbf{f}_{(ie)}$ before and after Tensor Core's computations are required for higher performance. So, we introduce warp specialization and overlap the data movements with Tensor Core's computations. First, we decompose the domain allocated to each GPU into 80 subdomains, each for the 80 streaming multiprocessors of the V100 GPUs. One thread block synchronously computes one $x - y$ plane at a time and moves in the z direction. As shown in Fig. 2, we separate the threads into two groups: the first group computes the matrix vector products by using the data in the shared memory and writes the results back to the shared memory, whereas the other group moves the data between the shared memory and the device memory. While

Figure 2: Rough image of warp specialization in the kernel.

```

__device__ void tensor_core_matmul(
    float *dmat, half *amat, half *bmat, float *cmat)
{
    wmma::fragment<wmma::matrix_a, 16, 16, 16, half, wmma::col_major> A;
    wmma::fragment<wmma::matrix_b, 16, 16, 16, half, wmma::col_major> B;
    wmma::fragment<wmma::accumulator, 16, 16, 16, float> C;
    wmma::load_matrix_sync(A, amat, 16);
    wmma::load_matrix_sync(B, bmat, 16);
    wmma::fill_fragment(C, 0.0f);

    wmma::mma_sync(C, A, B, C); // C+=A*B

    wmma::store_matrix_sync(dmat, C, 16, wmma::row_major);
}

```

Figure 3: Example code for using Tensor Cores in CUDA C.

the computation group is computing the matrix-vector products of elements between $z = i$ and $z = i + 1$, the data movement group adds the results of plane $z = i - 1$ in the shared memory to the device memory via `atomicAdd` and reads the element vectors for plane $z = i + 2$ to the shared memory. This procedure improves the utilization ratio of Tensor Core's computations.

3.2 Efficient data mapping for high performance computations on small matrices

We use the warp-level matrix multiply and accumulate (`wmma`) API, which is capable of low-level programming, to utilize Tensor Core's computations (Fig. 3). This API synchronously computes the products of $m \times k$ with $k \times n$ matrices using 32 threads. The current `wmma` API supports sizes $(m, n, k) = (16, 16, 16), (8, 32, 16), (32, 8, 16)$. The data must be moved from the device or shared memory to the registers via `wmma::load_matrix_sync`. The data are computed by `wmma::mma_sync`. Afterward, the results on the registers are stored to the device or shared memory by `wmma::store_matrix_sync`.

To compute Eq. (5), Tensor Core's operations for the multiplication of $P_i(u_{(1)}, u_{(2)}, \dots)$ and $Q_i(u_{(1)}, u_{(2)}, \dots)$ must be issued for 32 threads. In contrast, our element-wise computations, including the scaling/rescaling of the element vectors $u_{(ie)}$ and $f_{(ie)}$, as well as the multiplication of the material properties $\alpha_{i(ie)}$ and $\beta_{i(ie)}$, must assign one thread to a specific element. These assignments lead to a large difference in the thread allocation between Tensor Core's computation part and the other parts. When we conduct the element-wise operations before or after Tensor Core's operations, each thread must retrieve the corresponding data via shared

$$\sum_{i=1}^8 \left(\begin{bmatrix} P_i^{1,1} & P_i^{1,2} & P_i^{1,3} \\ P_i^{2,1} & P_i^{2,2} & P_i^{2,3} \\ P_i^{3,1} & P_i^{3,2} & P_i^{3,3} \end{bmatrix} \begin{bmatrix} \alpha_{i(1)}u_{(1)}^1, \dots, \alpha_{i(32)}u_{(32)}^1 \\ \alpha_{i(1)}u_{(1)}^2, \dots, \alpha_{i(32)}u_{(32)}^2 \\ \alpha_{i(1)}u_{(1)}^3, \dots, \alpha_{i(32)}u_{(32)}^3 \end{bmatrix} + \begin{bmatrix} Q_i^{1,1} & Q_i^{1,2} & Q_i^{1,3} \\ Q_i^{2,1} & Q_i^{2,2} & Q_i^{2,3} \\ Q_i^{3,1} & Q_i^{3,2} & Q_i^{3,3} \end{bmatrix} \begin{bmatrix} \beta_{i(1)}u_{(1)}^1, \dots, \beta_{i(32)}u_{(32)}^1 \\ \beta_{i(1)}u_{(1)}^2, \dots, \beta_{i(32)}u_{(32)}^2 \\ \beta_{i(1)}u_{(1)}^3, \dots, \beta_{i(32)}u_{(32)}^3 \end{bmatrix} \right)$$

$$= \sum_{i=1}^8 \sum_{j=1}^3 \left(\begin{bmatrix} P_i^{1,j} & Q_i^{1,j} \\ P_i^{2,j} & Q_i^{2,j} \\ P_i^{3,j} & Q_i^{3,j} \end{bmatrix} \begin{bmatrix} \alpha_{i(1)}u_{(1)}^j, \dots, \alpha_{i(32)}u_{(32)}^j \\ \beta_{i(1)}u_{(1)}^j, \dots, \beta_{i(32)}u_{(32)}^j \\ \alpha_{i(1)}u_{(1)}^j, \dots, \alpha_{i(32)}u_{(32)}^j \\ \beta_{i(1)}u_{(1)}^j, \dots, \beta_{i(32)}u_{(32)}^j \end{bmatrix} \right)$$

$\begin{matrix} \xleftrightarrow{8} \\ \updownarrow 8 \end{matrix}$
 $\begin{bmatrix} P_i^{1,1} \end{bmatrix}$

Figure 4: Decomposition of the matrices in the multiplication to compute $\sum_{i=1}^8 (P_i(\alpha_{i(ie)}u_{(ie)}) + Q_i(\beta_{i(ie)}u_{(ie)}))$ on Tensor Core. Constant matrices have a size of 8×16 and the matrices composed of vectors have a size of 16×32 .

(a) Memory alignment of the 8×16 matrix seen from users

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47
48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63
64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95
96	97	98	99	100	101	102	103	104	105	106	107	108	109	110	111
112	113	114	115	116	117	118	119	120	121	122	123	124	125	126	127

$\downarrow$
`wmma::load_matrix_sync`

(b) Register mapping of the 8×16 matrix for the Tensor Core computation

	registers 0, 1, ..., 15															
<code>tid%32=0, 4, 8, 12</code>	0	1	2	3	32	33	34	35	64	65	66	67	96	97	98	99
<code>tid%32=1, 5, 9, 13</code>	8	9	10	11	40	41	42	43	72	73	74	75	104	105	106	107
<code>tid%32=2, 6, 10, 14</code>	16	17	18	19	48	49	50	51	80	81	82	83	112	113	114	115
<code>tid%32=3, 7, 11, 15</code>	24	25	26	27	56	57	58	59	88	89	90	91	120	121	122	123
<code>tid%32=16, 20, 24, 28</code>	4	5	6	7	36	37	38	39	68	69	70	71	100	101	102	103
<code>tid%32=17, 21, 25, 29</code>	12	13	14	15	44	45	46	47	76	77	78	79	108	109	110	111
<code>tid%32=18, 22, 26, 30</code>	20	21	22	23	52	53	54	55	84	85	86	87	116	117	118	119
<code>tid%32=19, 23, 27, 31</code>	28	29	30	31	60	61	62	63	92	93	94	95	124	125	126	127

Figure 5: Relationship of the data allocation in the `wmma` API. When we call `wmma::load_matrix_sync`, each thread refers to a specific portion of the matrix according to its index and stores the data in the registers.

memory. We can call `wmma` API for this data remapping; however, doing so increases the amount of shared memory access. If we used Tensor Core for large matrices, we could decompose the matrices into many sub-matrices and reuse the same sub-matrix many times, thereby covering the costs of storing the results into the shared memory and remapping to the registers. However, our matrix is so small that the performance of the kernel easily becomes shared memory bandwidth bound when we request overly frequent remapping. Therefore, we minimize the number of data remapping


```
asm(
    ".reg .f32 f<1>;/n/t"
    "mov.f32 f0, 0f00000000;/n/t"
    "mma.mma.sync.col.col.m8n32k16.f32.f32"
    "{%0,%1,%2,%3,%4,%5,%6,%7}, {%8,%9,%10,%11,%12,%13,%14,%15},"
    "{%16,%17,%18,%19,%20,%21,%22,%23}, {f0,f0,f0,f0,f0,f0,f0,f0};/n/t"
    "1"
    : "=f"(cv[0]), "=f"(cv[1]), "=f"(cv[2]), "=f"(cv[3]),"
    : "=f"(cv[4]), "=f"(cv[5]), "=f"(cv[6]), "=f"(cv[7])"
    : "r"(av[0]), "r"(av[1]), "r"(av[2]), "r"(av[3]),"
    : "r"(av[4]), "r"(av[5]), "r"(av[6]), "r"(av[7]),"
    : "r"(bv[0]), "r"(bv[1]), "r"(bv[2]), "r"(bv[3]),"
    : "r"(bv[4]), "r"(bv[5]), "r"(bv[6]), "r"(bv[7]);

```

Figure 6: Example code for using Tensor Core with the PTX assembly in CUDA C. Note that two FP16 values, namely av and bv , are packed into an INT32 register; thus, only eight registers are designated in the assembly.

operations by the efficient decomposition of the matrices and by the direct input of the necessary data into the registers after decoding the data remapping in the wmma API. The details of this implementation are described below.

First, we modify our algorithm such that we can reuse the same matrix as many times as possible and obtain the results by the simple summation of the matrix-matrix multiplications (Fig. 4). We compute the multiplication on the order of $\sum_{i=1}^8 (P_i (\alpha_{i(i)} \mathbf{u}_{(ie)})) + Q_i (\beta_{i(i)} \mathbf{u}_{(ie)})$. The computations of the 32 elements are subdivided into 72 Tensor Core operations with the block size $(m, n, k) = (8, 32, 16)$. Here, sub-matrices $P_i^{ix, iy}$ and $Q_i^{ix, iy}$ with indexes ix and iy are of size 8×8 and satisfies

$$P_i = \begin{pmatrix} P_i^{1,1} & P_i^{1,2} & P_i^{1,3} \\ P_i^{2,1} & P_i^{2,2} & P_i^{2,3} \\ P_i^{3,1} & P_i^{3,2} & P_i^{3,3} \end{pmatrix}, \quad Q_i = \begin{pmatrix} Q_i^{1,1} & Q_i^{1,2} & Q_i^{1,3} \\ Q_i^{2,1} & Q_i^{2,2} & Q_i^{2,3} \\ Q_i^{3,1} & Q_i^{3,2} & Q_i^{3,3} \end{pmatrix},$$

and vector $\mathbf{u}_{(ie)}^{ix}$ with index ix is of size 8 and satisfies $\mathbf{u}_{(ie)}^T = (\mathbf{u}_{(ie)}^{1,T} \ \mathbf{u}_{(ie)}^{2,T} \ \mathbf{u}_{(ie)}^{3,T})$. Thanks to this reorganization, the element vector matrix of size 16×32 ($\alpha_{i(i)} \mathbf{u}_{(ie)}^i$ and $\beta_{i(i)} \mathbf{u}_{(ie)}^j$ in Fig. 4) can be reused several times. We can also compute by extracting the multiplication results in the shared memory just once per 32 elements with `wmma::store_matrix_sync`, thereby significantly reducing the number of accesses to the shared memory.

Second, when we construct the matrices out of the element vectors ($\alpha_{i(i)} \mathbf{u}_{(ie)}^j$ and $\beta_{i(i)} \mathbf{u}_{(ie)}^j$ in Fig. 4), we omit the redundant `wmma::load_matrix_sync` calls. For example, in the wmma API of the current CUDA version, it is found that each thread uses 16 FP16 numbers, thereby leading to $16 \times 32 = 512$ values among the 32 threads with duplication among threads as shown in [38] (the 8×16 matrix mapping is shown in Fig. 5). Using this property, we can call the wmma API from the PTX assembly (Fig. 6) and remove the data movement via shared memory, i.e., element-wise vectors $\alpha_{i(i)} \mathbf{u}_{(ie)}^j$ and $\beta_{i(i)} \mathbf{u}_{(ie)}^j$ can be always in the registers.

4 PERFORMANCE MEASUREMENT

4.1 Problem settings

We target a two-layered problem that mimicks the Earth's crust and mantle (Fig. 7). The interface of the layers has a wave-shaped geometry with sharp material discontinuity at the interface. To see the effects of transprecision computing in different problem settings,

Case 1	Vp (m/s)	Vs (m/s)	Density (kg/m³)
Layer 1	Ranging from 351 to 400	Ranging from 693 to 1600	1850
Layer 2	Ranging from 1867 to 8734	Ranging from 645 to 3912	2700
Case 2	Vp (m/s)	Vs (m/s)	Density (kg/m³)
Layer 1	Ranging from 3438 to 4500	Ranging from 2081 to 2400	2450
Layer 2	Ranging from 5715 to 8734	Ranging from 3322 to 3912	2700

Figure 7: Measurement model consisting of the unstructured second-order tetrahedral elements of each GPU. The model is duplicated in the x - and y -directions according to the number of compute nodes.

we use two settings. We compute the displacement response to a given fault slip in Case 1 and the displacement response to gravity in Case 2. The material properties change in the depth directions inside the crust and the mantle; hence, the material within each layer of this model is changed as well. The bottom of the model is fixed. Figure 7 shows a model per GPU. We duplicate this model in the x - and y -directions to create a model for 216 GPUs. The total degrees of freedom (DOF) are 13,257,281,664 and the DOFs in the preconditioning solvers *PreCG*, *PreCG^c*, *PreCG^v*, and *PreCG_{part}* are 13,257,281,664, 2,011,200,192, 1,683,540,936, and 472,350,744, respectively.

We use this problem to compare the performance of our proposed solver, the standard PCGE, and the SC14 solver. PCGE is a standard conjugate gradient solver with 3×3 block Jacobi conditioning [41] and an element-by-element [48] matrix-vector kernel tuned for the V100 architecture. The SC14 solver is a tuned version of the state-of-the-art solver in [21] and it is our proposed solver without *PreCG^v* and *PreCG_{part}* while PCGE is our proposed solver without *PreCG^v*, *PreCG^c*, *PreCG*, and *PreCG_{part}*. Here, all the major computations of the solver are ported and tuned for the V100 architecture. FP64 is used for the computation throughout the PCGE solver, whereas both FP64 and FP32 are used in SC14 solver (FP32 is used for the multi-grid preconditioner). To clarify the effect of the differences among the solver algorithms, all techniques to improve the performance of preconditioning solvers are also implemented on the SC14 solver and PCGE, including the reduction of the memory access costs in an element-by-element method by the use of the shared memory and the overlapping of point-to-point communications and matrix-vector multiplications. Namely, the difference in performance of solvers results from only whether each preconditioning solver (*PreCG^v*, *PreCG^c*, *PreCG*, or *PreCG_{part}*) is enabled, and there is no difference in detailed implementations. Note that the relative error tolerances of all solvers are set to $\epsilon = 1.0 \times 10^{-8}$.

Table 1: Performance of kernel on one V100 GPU of Summit. Peak bandwidths are extracted from [26].

	elapsed time	FLOPS	device mem. throughput		L2 cache throughput		unified cache throughput	shared mem. throughput	
			read	write	read	write	load+store	load	store
FP32	3.830 ms	12.1 TFLOPS	62 GB/s	8.6 GB/s	189 GB/s	29 GB/s	11343 GB/s	11179 GB/s	86 GB/s
FP16	2.527 ms	18.3 TFLOPS	114 GB/s	29 GB/s	349 GB/s	77 GB/s	10227 GB/s	9934 GB/s	135 GB/s
typical TC	3.327 ms	13.9 TFLOPS	76 GB/s	14 GB/s	424 GB/s	61 GB/s	3267 GB/s	1498 GB/s	9323 GB/s
tuned TC	0.930 ms	49.7 TFLOPS	264 GB/s	37 GB/s	541 GB/s	123 GB/s	8155 GB/s	7435 GB/s	905 GB/s
peak bandwidths			750 GB/s		2155 GB/s		12080 GB/s	12080 GB/s	

For our proposed solver, we use 0.01, 0.08, 0.1, and 1.0×10^{-3} as the error tolerances of the *PreCG^v*, *PreCG^c*, *PreCG*, and *PreCG_{part}* preconditioning solvers, respectively. Also, we use 2000, 50, 50, and 50 as the maximum numbers of iterations in the *PreCG^v*, *PreCG^c*, *PreCG*, and *PreCG_{part}* solvers, respectively. We compare the performances of the solvers on Summit [44]. We use herein one GPU per MPI process (six MPI processes per node) and MPI_Wtime to measure the elapsed time. nvprof is used to measure FP32 and FP64 FLOPS while Tensor Core's instructions are counted from the code. All the measurements include outputting the displacement results on the ground surface to the file system. All measurements were confirmed to have the same double-precision results regardless of the method used (our proposed solver, SC14 solver, and PCGE) within a relative error tolerance of 10^{-8} .

4.2 Computation environment

Summit comprises 4608 IBM Power System AC922 nodes, each with two IBM POWER9 (22 SIMD multi-core) processors and six NVIDIA Tesla V100 [34] accelerators. Each node has 512 GB of DDR4 memory for use by the POWER9 processors. Each of the six accelerators has 16 GB of 900 GB/s high bandwidth memory (HBM2). The peak FP64 performance of a node is $7.8 \text{ TFLOPS} \times 6 = 46.8 \text{ TFLOPS}$, which yields a total of 215 PFLOPS for the entire system. The FP32 peak performance is double that of FP64, while the FP16 peak performance is quadruple that of FP64 when using double-width arithmetic units. Tensor Cores with mixed FP16 and FP32 arithmetic are also available and lead to 125 TFLOPS per GPU. The compute nodes are connected to a dual-rail EDR InfiniBand network with a node injection bandwidth of 25 GB/s in both send/receive directions. GPUDirect RDMA [13] is available for circumventing memory copies between the device memory and the host memory during MPI communications. We used CUDA version 10.1.168 and PGI compiler version 19.7 for performance measurement.

4.3 Performance of the structured grid matrix-vector product kernel

Firstly, we evaluated the performance of the matrix-vector kernel in the structured problem by using one Volta GPU on Summit. We targeted the inner domain of Fig. 7. That is, we computed multiplication for all elements except those on the outer faces of the domain, which accounted for the largest proportion of the computation costs in the *PreCG^v* solver and was required for the overlapping of the point-to-point communications and matrix-vector multiplications when multiple GPUs were used. The number of elements to compute was 2,457,000. Here, we compared four versions in total.

First, one version computed the multiplication in single precision, whereas the second version computed in half precision. Hereinafter, these versions are referred to as FP32 kernel and FP16 kernel, respectively. Regarding Tensor Core-enhanced kernels, we used a typical TC kernel and a tuned TC kernel. The typical TC kernel always called wmma API to map the components of the matrices between the shared memory and registers while the tuned TC kernel reduced the access costs to shared memory by explicit mapping with PTX assembly. All these kernels had the same access patterns to vectors $\mathbf{u}_{(ie)}$ and $\mathbf{f}_{(ie)}$; thus, the access patterns to the device memory were the same. We decomposed the 24×24 matrices into 8×8 sub-matrices for all the kernels as was the case with the tuned TC kernel. Except for the FP32 kernel, we normalized each element vector and removed the rigid body motion to circumvent numerical errors. The elapsed time, FLOPS, and measured memory bandwidth are summarized in Table 1.

With the FP32 kernel, we attained 77.6% of peak FLOPS in single precision; thus, we confirmed that our target kernel had achieved dense computation. Regarding the measured bandwidths, those of the shared memory and unified cache were close to the peak bandwidth noted in [26]. Compared to these values in the FP32 kernel, the FP16 kernel reduced the footprints of the shared memory and unified cache by using half precision instead of single precision for the matrix components. The FP16 kernel attained a 1.52-fold speedup over the FP32 kernel. This ratio was lower than that of the peak FLOPS between single and half precision on a Volta GPU. The normalization of the element vectors and data conversion were considered as possible reasons. As for the kernels using Tensor Core, we confirmed via a comparison between the typical TC kernel and tuned TC kernel that the costs of data access to shared memory had a large effect on the performance. High bandwidth in shared memory was measured in both kernels and was assumed to be the bottleneck of the performance. These kernels had lower bandwidths of the unified cache than did the other kernels because each thread in FP32 kernel and FP16 kernel had to access all the components of the 24×24 matrices, whereas each thread in the typical TC kernel and tuned TC kernel had access only to portions of the matrices. The typical TC kernel frequently used shared memory for mapping between the registers and shared memory; thus, much storage to shared memory was required and the elapsed time was longer than for the FP16 kernel. In contrast, the tuned TC kernel greatly reduced the access cost to shared memory and relieved the bottleneck. The tuned TC kernel attained 49.7 TFLOPS, which was high as compared to the performance for typical matrix-matrix multiplication. For example, [29] reported that a batched GEMM using 16×16 matrices had achieved 5 TFLOPS on a Volta GPU. When we compared the

Table 2: Number of iteration in each solvers. Proposed solver uses four types of data types in matrix-vector multiplication part of $PreCG^v$ solver.

Case 1 solver	the number of iterations				
	CG	$PreCG$	$PreCG^c$	$PreCG^v$	$PreCG_{part}$
PCGE	5226	-	-	-	-
SC14	10	219	2412	-	-
prop. (FP32)	8	103	108	1917	154
prop. (FP16)	8	102	108	1920	154
prop. (tuned TC)	8	102	108	1920	154
prop. (bfloat16)	8	102	105	1876	172

Case 2 solver	the number of iterations				
	CG	$PreCG$	$PreCG^c$	$PreCG^v$	$PreCG_{part}$
PCGE	3520	-	-	-	-
SC14	8	124	4086	-	-
prop. (FP32)	8	104	143	2604	47
prop. (FP16)	8	98	196	2721	97
prop. (tuned TC)	8	96	186	2448	100
prop. (bfloat16)	8	112	162	2720	45

elapsed time of each kernel, the tuned TC kernel exhibited a 4.1-fold speedup over the FP32 kernel and a 2.7-fold speedup over the FP16 kernel.

4.4 Convergence of the solver

We used 36 compute nodes of Summit (216 GPUs) and evaluated the convergence and time-to-solution of the entire solver. In this subsection, we focused on the convergence of the solver. Here we additionally checked the convergence when using bfloat16, which is expected to be supported on some future architecture. This data type was not supported by a Volta GPU on hardware, so we emulated the convergence with the following procedures. First, we used the FP32 kernel with our proposed algorithm. The bfloat16 data type corresponded to a single-precision data type whose fraction part was truncated to seven bits; thus, we removed the lower 16 bits in the fraction parts of the single-precision numbers. When we used this data type, normalization of element vector $\mathbf{u}_{(ie)}$ and removal of the rigid body motion were applied to reduce numerical errors. These bit operations were conducted to $\mathbf{u}_{(ie)}$ before multiplication and $\mathbf{f}_{(ie)}$ after multiplication, respectively. In our problem settings, all the solvers converged in both Case 1 and Case 2. The number of iterations in each solver did not significantly change even when we used the lower-precision data types including bfloat16 as shown in Table 2. Therefore, the effectiveness of using lower-precision data types by our proposed method was demonstrated. For comparison, we also measured the convergence when we used the proposed solver without $PreCG^v$ solver. We used tuned TC kernel for matrix-vector kernel in iv . The number of iterations was $(CG, PreCG, PreCG^c, PreCG_{part}) = (8, 127, 2572, 278)$ in Case 1 and $(CG, PreCG, PreCG^c, PreCG_{part}) = (8, 123, 3901, 110)$ in Case 2; therefore, we can see that the introduction of $PreCG^v$ facilitated reducing the number of iteration especially in $PreCG^c$.

We confirmed that the usage of lower-precision data types in the matrix-vector kernel had different effects on the convergence

of the solver for Case 1 and Case 2. Its effect for the computation depends on the problem settings; therefore, we should choose the appropriate precision by considering arithmetic performance and convergence of the solver. In addition, the tolerance and the maximum number of the iterations in each preconditioning solver would have significant impacts on the entire computation cost. By optimizing these parameters, we can expect further performance improvement. This approach, including automatic tuning, would be one of our future tasks.

4.5 Time to solution of the solver

We compared the performance of our proposed solver to those of previous solvers. In addition to the tuned TC kernel, we measured the performance of the solver with the FP32 kernel and FP16 kernel. The FLOP counts and elapsed times of the entire solver are shown in Fig. 8. Compared to PCGE, the SC14 solver reduced the FLOP count by using adaptive preconditioning and the multi-grid method. The $PreCG^c$ solver accounted for the largest proportion of the whole computation cost; thus, most computations were done in single precision, which led to further speedup than a decrease in the FLOP count. When we applied our proposed algorithm with the FP32 kernel, the total FLOP count was higher than that of the SC14 solver because the number of computations in the structured mesh was much higher. However, the rearrangement to a structured problem avoided the complex connectivity of the unstructured mesh and reduced the memory access costs, leading to dense computation and the high effective performance of the matrix-vector kernel; therefore, the time-to-solution did not increase as much as did the FLOP count. In fact, our proposed solver with the FP32 kernel required 6.8-fold and 7.9-fold FLOP count for Case 1 and Case 2 over the SC14 solver, but the time-to-solution was just 1.03 times and 1.10 times longer for each case. Much of the time-to-solution of the solver was spent on the $PreCG^v$ part. For instance, matrix-matrix multiplication in $PreCG^v$ in Case 1 took 7.43 s, which accounted for 60% of the time-to-solution in the entire solver. Therefore, by switching the matrix-vector kernel from the FP32 kernel to the faster FP16 kernel or tuned TC kernel, we were able to improve the performance. We confirmed that the FLOP count did not change significantly with the FP16 kernel or tuned TC kernel, instead of the FP32 kernel. By using the tuned TC kernel for matrix-vector multiplication, we achieved a 1.7-fold speedup in both Case 1 and Case 2 in terms of time-to-solution as compared to the SC14 solver. The FLOPS in the entire solver reached 24.4 PFLOP/8.07 s/216 GPU = 14.0 TFLOPS per GPU in Case 2. When we skipped $PreCG^v$ for comparison, the elapsed time of the solver with tuned TC kernel was 10.20 s in Case 1 and 13.24 s in Case 2, respectively. Judging from these results, we confirmed that the introduction of $PreCG^v$ was a major factor in performance improvement. In summary, our proposed method required a much higher FLOP count than did previous solvers; however, the time-to-solution was shorter than that of the state-of-the-art SC14 solver because the matrix-matrix multiplication in our solver was rapidly computed by Tensor Core. Also, the speedup is insignificant when we did not compute on Tensor Core. Thus, we confirmed effectiveness of our proposed algorithm using AI-specific hardware.

Figure 8: FLOP count and time-to-solution of the solvers.

5 CONCLUSION

The demand for crustal deformation computation methods using high-fidelity models has increased. Especially, methods to consider heterogeneous crustal structure (e.g., finite element modeling) are required. The combination of these crustal deformation computations and observed crustal deformation data will lead to a better understanding of earthquake processes. However, massive computation cost is required because the analyses target large domain with high resolution. In this study, we focused on AI-specific hardware and designed a low-order finite element solver that used Tensor Core on a Volta GPU for fast crustal deformation computations. We rearranged the solver algorithm so that most computations were executed as matrix-matrix multiplication in half precision. The sizes of our matrices in multiplication were smaller than those in typical applications of AI; thus, memory access costs stood out. However, by the optimization of matrix decomposition and thread parallelization, we obtained highly effective performance from Tensor Core even when we used 24×24 matrices. Our proposed solver required a much higher FLOP count than did previous solvers, but the time-to-solution was shorter as Tensor Core exhibited extremely high effective performance. Further experiments with more complex geometry would be required as future work.

Our proposed solver has been used in actual problems, such as those presented in [23]. The utility of our solver algorithm is expected to have broad impacts, such as the improvement of unstructured finite element solvers based on multi-grid methods. Furthermore, the highly accelerated *PreCG^v* solver could be used for voxel finite element methods [1] that are computationally costly analyses typically used in the design optimizations of industrial products.

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