New Cosystolic Expanders from Tensors Imply Explicit Quantum LDPC Codes with $\Omega(\sqrt{n} \log^k n)$ Distance

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ABSTRACT

In this work we introduce a new notion of expansion in higher dimensions that is stronger than the well studied cosystolic expansion notion, and is termed *Collective-cosystolic expansion*.

We show that tensoring two cosystolic expanders yields a new cosystolic expander, assuming one of the complexes in the product, is not only cosystolic expander, but rather a collective cosystolic expander.

We then show that the well known bounded degree cosystolic expanders, the Ramanujan complexes are, in fact, collective cosystolic expanders. This enables us to construct new bounded degree cosystolic expanders, by tensoring of Ramanujan complexes.

Using our new constructed bounded degree cosystolic expanders we construct *explicit* quantum LDPC codes of distance $\sqrt{n} \log^k n$ for any k , improving a recent result of Evra et. al. [FOCS, 2020], and setting a new record for distance of explicit quantum LDPC codes.

The work of Evra et. al. [FOCS, 2020] took advantage of the high dimensional expansion notion known as cosystolic expansion, that occurs in Ramanujan complexes. Our improvement is achieved by considering tensor product of Ramanujan complexes, and using their newly derived property, the collective cosystolic expansion.

CCS CONCEPTS

• **Theory of computation** → **Error-correcting codes**; • **Computer systems organization** → **Quantum computing**.

KEYWORDS

Quantum Codes, Cosystolic Expanders, Ramanujan Complexes

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1 INTRODUCTION

A cosystolic expansion is a topological measure of expansion in high dimensions which is not equivalent to the spectral high dimensional

measure of expansion captured by the well studied notion of local-spectral expansion. This topological expansion is hard to obtain. There is essentially only one known family of bounded degree complexes that possesses this topological notion of expansion. This family is derived from the so called *Ramanujan complexes* arising from number theory [26, 27]. In dimension two another family of cosystolic expanders was recently found in [19]. Part of the interest in this topological notion of high dimensional expansion, is related to the strong connection between cosystolic expanders and quantum LDPC codes with good distance.

In this work we develop a theory showing conditions under which the tensor product of two cosystolic expanders yields a new cosystolic expander. Interestingly, to get cosystolic expansion for the product, we need to introduce a stronger notion of expansion than cosystolic expansion, that we term *collective cosystolic expansion*. In a nutshell, for the product of cosystolic expanders to yield a new cosystolic expander, one of the complexes in the product needs to possess this stronger expansion property, namely to be a collective cosystolic expander.

We further show that the well known Ramanujan complexes are not only cosystolic expanders, as was shown by [6, 17], but rather they are also collective cosystolic expanders. Hence by taking tensor products of them, we obtain many new families of bounded degree cosystolic expanders, in all dimensions.

We then use the new cosystolic expanders that we have constructed to construct *explicit* LDPC quantum code. The code is of distance $\sqrt{n} \log^k n$ for any k , thus improving over the state of the art explicit quantum LDPC codes obtained recently by [7]. The current distance record for not necessarily explicit LDPC quantum codes is $n^{\frac{3}{5}}$, achieved in the recent exciting work [14], which constructs a *random* LDPC code from some kind of a random combinatorial S^1 -bundle.

The [7] paradigm obtained explicit quantum codes from cosystolic expanders (and more precisely Ramanujan complexes); It further implied that if it were possible to derive the quantum codes from homologies and cohomologies of large dimensions, their distance could have also been $\sqrt{n} \text{polylog}(n)$, with powers in the polylog that are as large as one wishes. However, it is not known whether the cohomologies are non zero in dimensions greater than 2. This was the bottleneck in improving the distance of the code constructed by [7] from $\sqrt{n} \log n$ to $\sqrt{n} \text{polylog}(n)$. [7] leaves it as an open question, whether this bottleneck can be addressed. Our idea of obtaining the improved code, in a nutshell, is the following: by tensoring we can ensure non vanishing cohomologies in dimensions as high as we wish, while maintaining the cosystolic

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expansion. This results in the improved quantum LDPC codes that we get.

1.1 High Dimensional Expanders, Cosystolic Expanders, and Collective Cosystolic Expanders

In the following we discuss high dimensional expanders with emphasis on the topological notion of expansion that is known as cosystolic expansion. Then we introduce the stronger notion of collective cosystolic expansion. This new notion is going to play a key role in the tensoring machinery that we are developing in this work.

High dimensional expanders. High dimensional expansion is an emerging theory which gains considerable attention recently as it led to solutions of several important open questions in theoretical computer science and mathematics. To list two notable examples: In mathematics, the study of high dimensional expanders led to a resolution [6, 17] of an open question raised by Gromov [9, 12], as to whether topological overlapping property is possible in bounded degree high dimensional complexes. In computer science, this theory led to a solution of a famous conjecture of Mihail and Vazirani about the existence of an efficient algorithm to count the bases of a matroid for any matroid. This was recently achieved by [3], building on high dimensional random walks that were studied in [18].

The high dimensional expansion theory generalizes the well studied and influential theory of graph expansion to higher dimensions. It turns out that high dimensional expansion is a much more rigid phenomenon than expansion in graphs, and this implies some of the important aspects of this theory. There are several non-equivalent definitions to what is a high dimensional expander. In this work we refer to the notion of cosystolic expansion that was defined in [6]. Cosystolic expansion is generalisation of the topological notion of expansion in graphs, known as the Cheeger constant of a graph, to higher dimensions. There is a very closely related definition of *coboundary expansion* that was defined by Gromov and by Linial and Meshulam [12, 23]. However, in this work we refer to the exact definition of cosystolic-expansion that is relevant for obtaining quantum CSS codes.

DEFINITION 1.1 (COSYSTOLIC EXPANSION [6]). Given $\eta, \mu > 0$, a d -dimension complex X with a weight function $\| - \|$ (e.g. the normalized Hamming norm) on its cochains is a (μ, η) -cosystolic expander if for every $i < d$

- The i -cofilling constant at most μ , i.e., for every non-zero i -cochain α , whose boundary is the $i + 1$ cochain $\delta(\alpha)$, there exists an i -cocycle α' such that $\| \alpha - \alpha' \| \leq \mu \| \delta(\alpha) \|$.
- The weight of the i -cosystole is at least η .

We now move to introduce our new notion of topological expansion that is stronger than cosystolic expansion and is termed collective cosystolic expansion:

DEFINITION 1.2 (COLLECTIVE COSYSTOLIC EXPANSION). Given $\eta, \mu^{\text{coll}} > 0$, a d -dimension complex X with a weight function $\| - \|$ on its cochains is a $(\mu^{\text{coll}}, \eta)$ -collective cosystolic expander if for every $i < d$

- The i -collective cofilling constant is at most μ .
The i -collective cofilling constant of a complex X is defined as

$$\mu_i^{\text{coll}}(X) = \max_{\beta_1, \dots, \beta_m \in B^{i+1}(X)} \min_{\substack{\alpha_1, \dots, \alpha_m \in C^i(X): \\ \delta(\alpha_i) = \beta_i}} \left\{ \frac{\| \bigcup_{a \in [m]} \alpha_a \|}{\| \bigcup_{a \in [m]} \beta_a \|} \right\}.$$

Note that restricting to the case that $m = 1$ gives rise to the usual cosystolic expansion.

- The weight of the i -cosystole is at least η .

REMARK 1.3. In both definitions, in case there are no cosystoles, the complexes are termed μ -coboundary expanders and μ^{coll} -collective coboundary expanders respectively. As usual in the context of expanders, when a family of complexes is said to be a (collective) cosystolic expander, it is with respect to uniform (μ, η) (or $(\mu^{\text{coll}}, \eta)$), and similarly for (collective) coboundary expanders.

1.2 Tensoring Cosystolic Expanders Via Collective Cosystolic Expansion Property

In the following we show how to obtain new cosystolic expanders from tensoring two cosystolic expanders where one of the complexes in the product possesses the stronger collective cosystolic expansion property. In fact we have a hierarchy of results, ordered by the strength of the assumptions on the expansions of the complexes in the tensor product. The first level in the hierarchy deals with the product of a cosystolic expander and a complex with linear cosystoles, the second level deals with the product of a cosystolic expander and a collective cosystolic expander, and the third level deals with the product of two collective cosystolic expanders.

We state an approximate version of the theorem now, referring the reader to Theorem 3.1 for the precise, and more general, statement.

THEOREM 1.4 (TENSORING COSYSTOLIC EXPANDERS THEOREM). Let X, Y be two complexes of bounded degree and dimension greater than d .

- Suppose in addition that X is a cosystolic expander and Y has cosystoles of linear size. Then the tensor product complex $X \otimes Y$ is of bounded degree and has linear cosystoles in any dimension smaller or equal to d .
- Suppose in addition that X is a cosystolic expander and Y is a collective cosystolic expander. Then the tensor product complex $X \otimes Y$ is a bounded degree cosystolic expander in any dimension smaller or equal to d .
- Suppose in addition that both X and Y are collective cosystolic expanders. Then the tensor product complex $X \otimes Y$ is a bounded degree collective cosystolic expander in any dimension smaller or equal to d .

This theorem shows that importance of collective cosystolic expansion, and that this is the correct notion when one wants to consider products of high dimensional expanders.

1.3 Ramanujan Complexes are Bounded Degree Collective Cosystolic Expanders

In the following we briefly discuss Ramanujan complexes, which until this work were essentially the only family of bounded degree high dimensional complexes that were proven to be cosystolic

expanders. In the recent work [18], another family of bounded degree complexes was studied, and, in dimension two this family was shown to be a family of cosystolic expanders.

Ramanujan Complexes. Bounded degree graphs that are expanding are abundant, e.g., a random d -regular graph is such. However, high dimensional and bounded degree cosystolic expanders seem to be rare. A well studied family of such complexes, known as the family of Ramanujan complexes, was constructed in [26, 27]. In the work of [6, 17] it was shown that the Ramanujan complexes are explicit bounded degree cosystolic expanders.

One of the main findings of our work here is that the Ramanujan complexes are not only cosystolic expanders but rather they are collective cosystolic expanders. For showing that, we also show that their building blocks, the *spherical buildings* are also collective cosystolic expanders with vanishing cohomology, i.e. collective coboundary expanders. The main technical novelty in our proof that the Ramanujan complexes are collective cosystolic expanders stems in using the new notion of *locally minimal collection of cochains* which is a generalization of the local minimality notion introduced in [6, 17].

The basic definitions required for the proof that the Ramanujan complexes are collective cosystolic expanders, as well as the proof, will be given in Section 4. The proof of the analogous property for spherical buildings appears in Appendix A.

THEOREM 1.5 (RAMANUJAN COMPLEXES ARE COLLECTIVE COSYSTOLIC EXPANDERS THEOREM). *For any $d \in \mathbb{N}$, any large enough q , there exists a constant $\mu^{\text{coll}}(d, q)$ such that and any $0 \leq k < d$, all rank $d+1$ Ramanujan complexes whose codimension-1 cells touch $1+q$ top cells are collective cosystolic expanders with collective cofilling constant at most $\mu^{\text{coll}}(d, q)$.*

THEOREM 1.6 (SPHERICAL BUILDINGS ARE COLLECTIVE COBOUNDARY EXPANDERS THEOREM). *For any $0 \leq k \leq n-1$ there exists a constant $\mu^{\text{coll}}(k, n)$ such that any rank- $n+1$ spherical building is a collective coboundary expander whose collective cofilling constant at most $\mu^{\text{coll}}(k, n)$.*

On cosystolic expansion vs. collective cosystolic expansion. The collective cosystolic expansion is obviously stronger than the usual cosystolic expansion. We will show, however, that both spherical complexes and Ramanujan complexes have this property, and with the same proven constants. This raises the question

Open question. Does any cosystolic expander is also a collective cosystolic expander with same constants?

1.4 New Bounded Degree Cosystolic Expanders from Tensors of Ramanujan Complexes

In the following we describe new families of bounded degree complexes which are cosystolic expanders, and have non vanishing cohomology groups in dimensions greater than 2. Such families were not known to exist prior to this work, and are obtained from tensors of Ramanujan complexes. The non vanishing of the high cohomologies will be crucial to obtaining the improved quantum codes we present.

The new bounded degree cosystolic expanders that we construct here are obtained by taking a tensor product of a d dimensional Ramanujan complex with itself $\ell < d$ times. Since we have shown that

Ramanujan complexes are collective cosystolic expanders we can apply our theorem about tensoring collective cosystolic expanders to the Ramanujan complexes, to extract from them new bounded degree cosystolic expanders with cosystoles of linear size.¹

The new bounded degree complexes that are obtained by the tensoring operation have additional two properties that we wish to highlight. These additional two properties will be useful later on in the quantum code construction that we introduce.

The first property is the non vanishing of cohomologies up to level $k \leq \ell$. The second property is a systolic distance that grows with ℓ .

Non vanishing of cohomologies up to level $k \leq \ell$. In [17, Propositions 3.5, 3.6] it was shown that for any $d \geq 1$ there exist infinitely many Ramanujan complexes of dimension d for which the 1-st cohomology does not vanish, and for any $d \geq 2$ there exist infinitely many Ramanujan complexes of dimension d for which the i -th cohomology does not vanish for $i = 1, 2$.

By the properties of the Ramanujan complexes mentioned above, it has non-zero cohomologies at dimension 1. We use this to deduce that in our new co-systolic expander, that was obtained by taking the k -skeleton of the k -tensor-power of a Ramanujan complex, the cohomologies do not vanish up to level k .

The systolic distance of the tensored complex. The injectivity radius of the Ramanujan complex is the maximal r such that for any vertex in the complex, the ball of radius r around it, is a contractible subcomplex. Since the Ramanujan complexes are quotients of Bruhat-Tits buildings, this radius is also the minimal distance between vertices identifies by the quotient, minus 1. A consequence of [22, Proposition 3.3] (see also [7, Theorem 5.10]) is that Ramanujan complexes have injectivity radius which is logarithmic in the size of the complex. Using it [7, Theorem 5.11] showed that the i -systole of a Ramanujan complex of dimension $d > i$ is of length at least $\log^i n$, assuming the systole exists.

We show that the same estimate holds for the systoles of the k -tensor power of a Ramanujan complex with logarithmic injectivity radius, only that in this case the existence of systoles is guaranteed, as explained before.

The following theorem summarizes the main properties of the tensor powers of Ramanujan complexes.

THEOREM 1.7 (NEW BOUNDED DEGREE COSYSTOLIC EXPANDERS BY TENSORING RAMANUJAN COMPLEXES). *Let X be a Ramanujan complex with n vertices, dimension d , locality bounded by Q and injectivity radius at least $c \log n$ for some constants $c, Q > 0$, and non vanishing first cohomology. Let Y be the complex obtained by taking the tensor product of X with itself $\ell < d$ times. Then Y is a bounded degree cosystolic expander with the following properties:*

- **Bounded degree property.**

Y has n^ℓ vertices, locality at most ℓQ and hence all nonempty Y_i are of size $\Theta(n^\ell)$, where the coefficient in Θ depends only on ℓ, Q .

- **Collective-cosystolic expansion property.**

¹We will work with the normalized Hamming norm. With this weight function, the second item in the definition of cosystolic expander implies that the cosystole is linear.

The k -cofilling constant of Y , for $k \leq l$ is at most μ , for a constant μ which depends only on Q, l , the cosystolic bounds of X and the cofilling constants of X .

- **Non vanishing cohomologies up to dimension ℓ .**

$H_k(Y), H^k(Y) \neq 0, k \leq l$.

- **Linear cosystole size.**

The Hamming norm of the k th cosystole of Y , for $k \leq l$ is $\Omega(|Y_k|)$, where the coefficient inside Ω depends only on Q, l , the cosystolic bounds and cofilling constants of X .

- **Systoles grow with the tensor power.**

$\text{Sys}_k(Y) = \Omega(\log^k(|Y_k|))$ for $k \leq l$, where the constant inside Ω depends only on k, l, d, c .

Again we prove a more general version below, see Theorem 5.2

1.5 New Quantum LDPC Codes from Tensoring Ramanujan Complexes

In the following we construct explicit quantum LDPC codes (CSS codes) using the new bounded degree cosystolic expanders that we have constructed by tensoring Ramanujan complexes. Our codes achieve distance of $\sqrt{n} \log^k n$ for any k . This sets a new record on the distance for *explicit* quantum LDPC codes. The very recent breakthrough [14] provides a random quantum LDPC code of distance $n^{3/5}$, which is the current record for quantum LDPC codes, but that construction is non explicit.

The work of [7] introduced the usefulness of high dimensional expanders and, in particular, of the Ramanujan complexes to the construction of quantum LDPC codes. Our improvement is essentially derived by turning to consider tensor products of high dimensional expanders, and in particular tensor product of Ramanujan complexes.

Below we discuss what a CSS quantum codes are and the [7] paradigm for constructing LDPC quantum code from co-systolic expanders. We then discuss a bottleneck in their paradigm and how we overcome it to get our improved codes.

On CSS codes and i -(homology, cohomology)-pair. A quantum CSS code can be naturally obtained from pairs of dual spaces, the i -th homology and cohomology spaces, associated with a d dimensional complex (for any $0 < i < d$). The dimension of the derived CSS code is the dimension of the i -th homology, which equals the dimension of the i -th cohomology. The minimal length element in the i -cohomology is called the i -cosystole and the minimal length element in the i -homology is called the i -systole. The distance of the quantum CSS code associated to a pair of i -homology and i -cohomology is the minimum of the length of the i -systole and i -cosystole. For detailed definition of these topological notions: homology, cohomology, systoles and cosystoles, we refer the reader to Section 2.1. For exact definition of quantum CSS codes we refer the reader to Section 2.3.

The [7] paradigm, its bottleneck and our insight. The discussion above, makes it clear that for constructing improved quantum codes one has to seek complexes with i -systoles and i -cosystole as large as possible. The first idea of [7] is that high dimensional expanders are going to provide complexes with linear length cosystoles and mildly large systoles. Their second idea, based on Hastings's work

[15], is that it is possible to balance the distances between the i -systoles and i -cosystole of a certain complex to get a quantum LDPC code, whose distance is the geometric average of the two, and its rate is large if the i -th cohomology does not vanish. The relevant balancing theorem is the following

THEOREM 1.8 (BALANCING THEOREM OF [7] BASED ON [15]). *A complex of size $\Theta(m)$ with non zero i -th cohomology, with i -systole of size Sys_i , and i -cosystole of size $\text{CoSys}_i^i \gg \text{Sys}_i$, gives rise to a quantum LDPC code of length $n = \Theta(m \frac{\text{CoSys}_i^i}{\text{Sys}_i})$, distance CoSys_i^i , and dimension $\Omega(\frac{\text{CoSys}_i^i}{\text{Sys}_i})$.*

In the special case that the i -cosystole is also linear in m , the resulting code has length $n = \Theta(m^2/\text{Sys}_i)$, distance

$$\Theta(m) = \Theta(\sqrt{n \cdot \text{Sys}_i}),$$

and dimension

$$\Omega(\frac{m}{\text{Sys}_i}) = \Omega\left(\sqrt{\frac{n}{\text{Sys}_i}}\right).$$

[7] observed that as one turns to i -th homology and cohomology pair of larger and larger i , in a Ramanujan complex of dimension $d > i$, the distance of the derived quantum code improves, as long as its dimension does not vanish. However, by increasing i one loses control on the rate, and thus can not ensure that it is non zero. This is the bottleneck that [7] faced, and this what prevented them from improving the quality of the constructed codes. They leave it as an open question whether this bottleneck can be addressed. So [7] are facing the so called "High Cohomologies Challenge", namely if they knew that Ramanujan complexes have non vanishing cohomologies in large dimensions they could have used this knowledge to derive improved explicit quantum LDPC codes.

Our insight is that when we turn to tensor products of Ramanujan complexes, we can enjoy the effect of improving the code distance by turning to the i -th homology and cohomology pair of larger and larger i , while being able to ensure non vanishing rate. This is how we get our improved codes. Namely our idea is the following: We consider a k -tensor power of a Ramanujan complex of degree $d > k$. We argue that by the Künneth formula in the resulting complex the k -th cohomology will not vanish if the 1-st cohomology of the original complex did not vanish. Thus, we get a new complex whose k -th cohomology is non zero. We use the collective cosystolic expansion property of Ramanujan complexes to deduce that the tensored complex is a cosystolic expander up to dimension k , and hence its k -cosystole's length is linear in n , where n now stands for the number of vertices in the tensor power complex, and that its k -systole's length is $\Omega(\log^k n)$. Then, after applying the balancing procedure of Theorem 1.8, we obtain the promised quantum codes.

COROLLARY 1.9 (IMPROVED QUANTUM CODES THEOREM). *For every k there exists a LDPC quantum CSS code with following properties. The code distance is $\Omega(\sqrt{n} \log^{k/2}(n))$ and its dimension is $\frac{\sqrt{n}}{\log n^{k/2}}$; Where n is the size of the code.*

REMARK 1.10. *Tensor products of quantum codes, as well as other closely related notions of products, were considered in literature (see,*

for example, [2, 4, 10, 30]). Neither the systolic distance, nor the cosystolic distance are expected to be well behaved with respect to these operations in general. As explained above we exploit the properties of cosystolic expanders in order to analyze the cosystoles in the product code. For the analysis of systoles we need the special properties of the Ramanujan complex.

1.6 Plan of the Paper

This paper is structured as follows. In Section 2 we provide basic background in homological algebra, CSS codes and Ramanujan complexes. We also define systoles, cosystoles, cosystolic expansion and the new property of collective cosystolic expansion. In Section 3 we prove Theorem 5.2, which generalizes Theorem 1.4, and discusses the cosystolic behavior of tensor products of complexes. In Section 4 we give a criterion for simplicial complexes which guarantees collective cosystolic expansion, and prove that Ramanujan complexes satisfy it, hence proving Theorem 1.5. Finally in Section 5 we prove Theorem 1.7 and construct the promised quantum LDPC codes. Appendix A briefly discusses spherical complexes, and proves a generalized version of Theorem 1.6.

2 BACKGROUND AND DEFINITIONS

2.1 Complexes, Homology, and Cohomology

A (bounded) *chain complex* over a coefficient ring R is a collection of R -modules $C_\bullet = (C_m, C_{m+1}, \dots, C_n)$ and *boundary maps* $\partial_k : C_k \rightarrow C_{k-1}$ (C_{m-1} is taken to be 0), which satisfy

$$\partial_k \circ \partial_{k+1} = 0, \forall k.$$

We will restrict ourselves to $R = \mathbb{F}_2$, the field with two elements, so that each C_k will be a $\mathbb{F}_2$ -vector space, and ∂_k a linear map between $\mathbb{F}_2$ -vector spaces. We also consider the dual *coboundary maps*,

$$\delta^k = \partial_{k+1}^* : C_k^* \rightarrow C_{k+1}^*.$$

We call the elements of C_k *k-chains*, and those of C_k^* *k-cochains*.

Write

$$Z_k(C) = \ker(\partial_k), B_k(C) = \text{im}(\partial_{k+1}),$$

$$Z^k(C) = \ker(\delta^k), B^k(C) = \text{im}(\delta^{k-1}).$$

Elements of these sets are called *k-cycles*, *k-boundaries*, *k-cocycles* and *k-coboundaries* respectively.

The equations

$$\partial_k \circ \partial_{k+1} = 0, \delta^{k+1} \circ \delta^k = 0$$

(the second is obtained by dualizing the first) imply that

$$B_k(C) \subseteq Z_k(C), B^k(C) \subseteq Z^k(C).$$

We define the *k-homology* and *k-cohomology* group of C (with $\mathbb{F}_2$ -coefficients) as

$$H_k(C) = Z_k(C)/B_k(C), H^k(C) = Z^k(C)/B^k(C)$$

respectively.

The *k-th* Betty number is defined as $h_k(C) = \dim_{\mathbb{F}_2} H_k(C)$, which is readily seen to be equal $\dim_{\mathbb{F}_2} H^k(C)$ as well.

A *based chain complex* is a chain complex where each C_k comes equipped with a distinguished basis. This is equivalent to writing $C_k = \mathbb{F}_2^{X_k}$, for a set X_k to which we will refer as the set of *k-cells*.

We shall usually denote the based chain complex by the sets $X = (X_m, \dots, X_n)$, and put $C_k(X) = \mathbb{F}_2^{X_k}$. We will sometimes lighten notations by writing $\sigma \in X$, instead of $\sigma \in \bigcup X_i$, for a cell σ . Working with bases allows identifying C_k with its dual, hence considering both the *k-chains* and *k-cochains* as living in the same space. In addition it provides a natural definition of a *support* and *norm* of a chain or cochain α . The support of α , $\text{supp}(\alpha)$ is the set of cells X_k which appear with coefficient 1 when α is written according to the X_k basis. We identify chain or cochains with their supports, and use it to define set theoretic operations such the union or intersection of chains or cochains, just by the corresponding operators on the support. We also sometimes write $\sigma \in \alpha$ to denote $\sigma \in \text{supp}(\alpha)$; in this case we say that the cell σ belongs to α . The norm of α , $|\alpha|$ is the cardinality of its support, or equivalently the Hamming weight of α written in the X_k basis. For a cochain or a chain α denote by α_k its $C^k(X)$ -part.

We say that a cell $\sigma \in X_k$ is *contained* in a cell X_{k+l} if

- $l = 1$ and $\sigma \in \text{supp}(\partial\tau)$.
- $l > 1$ and $\sigma \in \text{supp}(\partial\sigma')$ for some σ' which is contained in τ .

A complex is *pure* if each cell is contained in some top dimensional cell.

Many examples of based complexes come from topology. Two examples are simplicial and polyhedral complexes, another example we shall not need is CW-complexes. A simplicial complex $X = (X_0, \dots, X_d)$ is a based chain complex where the elements of X_k correspond to different subsets S of X_0 , of size $k + 1$. The boundary map takes S to the sum of its $k + 1$ subsets of size k . A simplicial complex always has lower locality $k + 1$ in dimension k . For simplicial complexes, a cell σ is contained in a cell τ precisely if σ is contained in τ as a set. A generalization of a simplicial complex which we call polyhedral complex is² a based chain complex where any $c \in X_k$ corresponds to subset of the *vertex set* X_0 . And if $c \in X_k$ corresponds to a set S , then $\partial_k c$ is a sum over several subsets of S which correspond to elements of X_{k-1} . We also require that all cells in X_1 correspond to pairs of elements of X_0 , which we call *edges*, while *k-cells* for $k > 1$ correspond to sets of vertices with more than 2 elements. In these two examples we identify the cells with the sets of vertices to which they correspond. Polyhedral complexes arise naturally, as cubical complexes, and, more importantly for this work, as tensor products of simplicial complexes. The dimension of a simplicial or polyhedral complex is the largest d for which $X_d \neq \emptyset$.

We say that the based complex X has *upper locality* p_k in dimension k if

$$\sup_{\sigma \in C^k(X)} \frac{|\partial\sigma|}{|\sigma|} \leq p_k,$$

this is equivalent to $\sup_{\sigma \in X_k} |\partial\sigma| \leq p_k$. We similarly say that X has *lower locality* q_k in dimension k if

$$\sup_{\sigma \in C_k(X)} \frac{|\partial\sigma|}{|\sigma|} \leq q_k,$$

and again it is equivalent to requiring $\sup_{\sigma \in X_k} |\partial\sigma| \leq q_k$.

We say that the complex is *Q-local* if it has upper and lower localities Q in each dimension.

²this is not the standard definition of a polyhedral complex, but it will suffice for our needs

A weighted complex is a based complex X together with a weight function

$$\| \cdot \|: C^\bullet(X) \rightarrow \mathbb{R}_{\geq 0},$$

which vanishes only on the 0-cochain and is additive in the sense that

$$\| \alpha \| = \sum_{\sigma \in \text{supp}(\alpha)} \| \sigma \|.$$

The Hamming norm is an example of a weight function, another example is the normalized Hamming norm, defined by

$$\| \alpha \| = \frac{|\text{supp}(\alpha)|}{|X_k|}, \alpha \in C^k(X).$$

A third example that will be relevant to Section 4 and in Appendix A, is the weight function on d -dimensional complexes, given by

$$\| \alpha \| = \frac{\sum_{\sigma \in \text{supp}(\alpha)} |\{\tau \in X_d \mid \tau \text{ contains } \sigma\}|}{\binom{d+1}{k+1} X_d}, \alpha \in C^k(X). \quad (1)$$

2.2 Cosystoles, Systoles, Cosystolic Expansion and Collective Cosystolic Expansion

In the following we define systoles and cosystoles and then we will be ready to present the main definitions of this work: the well studied cosystolic expansion notion as well as a stronger notion than cosystolic expansion, that we introduce in this work, and we term Collective-cosystolic expansion.

We define the k -systole and k -cosystole of a weighted complex as

$$\text{Sys}_k(X) = \min_{\alpha \in Z_k \setminus B_k} \| \alpha \|, \text{CoSys}_k(X) = \min_{\alpha \in Z^k \setminus B^k} \| \alpha \|.$$

We also define the k -th cosystolic bound by

$$\eta_k(X) = \text{CoSys}_k(X) / \| X_k \|,$$

where we consider X_k as the cochain whose support is X_k . We make the convention that $\eta_k(X) = 1$ if $h_k(X) = 0$. We define the k -th cofilling constant by

$$\mu_k(X) = \sup_{\alpha \in B^{k+1}(X) \setminus \{0\}} \inf_{\substack{\beta \in C^k(X) \\ \partial\beta = \alpha}} \frac{\| \beta \|}{\| \alpha \|}.$$

A simple generalization of the cofilling constant that will play a key role in this work is the following notion.

DEFINITION 2.1. *The k -th collective cofilling constant of the weighted complex X is defined by*

$$\mu_k^{\text{coll}}(X) = \sup_{\alpha_1, \dots, \alpha_m \in B^{k+1}(X) \setminus \{0\}} \inf_{\substack{\beta_1, \dots, \beta_m \in C^k(X) \\ \forall a \partial\beta_a = \alpha_a}} \frac{\| \bigcup_a \beta_a \|}{\| \bigcup_a \alpha_a \|}.$$

Clearly $\mu_k \leq \mu_k^{\text{coll}}$.

We say that X has the (collective) cofilling property in dimension k with constant c_k if its (collective) cofilling constant is bounded by c_k . We say that X has the (collective) cofilling property with constants $\{c_k\}_k$ if c_k upper bounds its (collective) cofilling constant in dimension k . In the special case that all (collective) cofilling constants are bounded by c we say that X has the (collective) cofilling property with constant c .

We can define also systolic bounds and filling constants, but they will not be needed in this work.

REMARK 2.2. *If the ratio between weight functions is bounded from above and from below by two positive constants $R > r$, then the ratios of the cosystolic bound, cofilling and collective cofilling constants with respect to the two weight functions are bounded from above and below by R/r and r/R respectively.*

For pure d dimensional complexes of locality q , the Hamming and normalized Hamming norms, and the weight function (1), give rise to cosystolic bounds, cofilling and collective cofilling constant whose ratios are bounded from below and from above by explicit positive constants which depend on q, d . Thus, if we consider q, d as constants themselves, we can estimate the cosystolic bounds, cofilling and collective cofilling constant for one weight function using the other, up to a multiplicative constant.

In fact, the cosystolic bounds calculated with respect to the Hamming and normalized Hamming weights are the same, the filling and cofilling constants are proportional with proportionality constant in dimension k being $|X_k|/|X_{k+1}|$.

The only complexes which we will examine in this work, for which q cannot be considered as a constant, are the spherical buildings discussed in the appendix. There it will be important to work with the weight function (1).

2.3 Codes, CSS, etc

A quantum CSS code [5, 29] of length n is defined by two binary matrices H_X and H_Z , each with n columns, and such their row-spaces W_X and W_Z are orthogonal. Equivalently, $H_Z^T H_X = 0$. The matrices H_X and H_Z can be thought of as the parity-check matrices of classical codes, $C_X = W_X^\perp$ and $C_Z = W_Z^\perp$ respectively. The dimension of the quantum code is given by $n - \dim W_X - \dim W_Z$, equivalently it is the dimension of either of the quotient spaces C_X/W_Z or C_Z/W_X . The Hamming distance d_X (respectively d_Z) is defined as the smallest weight of a vector of C_X not in W_Z (respectively C_Z not in W_X). The minimum distance d of the quantum code is defined as $d = \min(d_X, d_Z)$. A quantum CSS code is said to be Low Density Parity Check (LDPC) if both matrices H_X and H_Z have row and column weights bounded from above by a constant.

If we define X_0, X_1, X_2 as the sets of rows of H_X , columns of H_X (or equivalently of H_Z) and rows of H_Z respectively, then H_Z^T and H_X are the matrices representing two linear maps ∂_2 and ∂_1

$$\mathbb{F}_2^{X_2} \xrightarrow{\partial_2} \mathbb{F}_2^{X_1} \xrightarrow{\partial_1} \mathbb{F}_2^{X_0}$$

such that $\partial_1 \partial_2 = 0$. This process can be inverted, starting from a based chain complex of length 2 we obtain the data of H_X , H_Z and hence of the CSS code. Thus, given any based chain complex $X = (X_0, \dots, X_d)$ with $d \geq 2$, taking any 2-subcomplex of the form

$$\mathbb{F}_2^{X_{k+1}} \xrightarrow{\partial_{k+1}} \mathbb{F}_2^{X_k} \xrightarrow{\partial_k} \mathbb{F}_2^{X_{k-1}},$$

gives rise to a CSS code. Unwinding the definitions, it is straight forward that

$$d_Z = \text{CoSys}_k(X), \quad d_X = \text{Sys}_k(X),$$

and that the dimension of the code is the k -th Betty number $h_k(X)$. In this sense a pair of k -th homology and cohomology of a complex gives rise to a quantum CSS code. When X is local the code is LDPC.

The prototypical example of a LDPC quantum CSS code is the influential work [21].

2.4 Künneth Theorem

There is a natural and well known notion of tensor product of chain complexes (see e.g. [2]), which we now describe in the based case. Let X, Y be two based complexes. The tensor product $(X \otimes Y)$, is the based complex given by

$$(X \otimes Y)_k = \bigsqcup_{i=0}^k X_i \times Y_{k-i}$$

hence the chain spaces are direct sums of tensor products

$$C^k(X \otimes Y) = \oplus_{i=0}^k C^i(X) \otimes C^{k-i}(Y). \quad (2)$$

The boundary and coboundary maps act on the element $v \otimes u \in C^i(X) \otimes C^{k-i}(Y)$, by

$$\partial_k^{X \otimes Y}(v \otimes u) = \partial_i^X(v) \otimes u + v \otimes \partial_{k-i}^Y(u),$$

$$\delta_k^{X \otimes Y}(v \otimes u) = \delta_i^X(v) \otimes u + v \otimes \delta_{k-i}^Y(u),$$

and are extended linearly to $C^k(X \otimes Y)$. In other words,

$$\delta^{X \otimes Y} = \delta^X \otimes Id_Y + Id_X \otimes \delta^Y,$$

and similarly for the boundary map. By slight abuse of notations we write δ^X instead of $\delta^X \otimes Id_Y$ and δ^Y instead of $Id_X \otimes \delta^Y$. Note that

$$\delta^X(C^{i,k-i}(X \otimes Y)) \subseteq C^{i+1,k-i}(X \otimes Y),$$

$$\delta^Y(C^{i,k-i}(X \otimes Y)) \subseteq C^{i,k-i+1}(X \otimes Y).$$

It is also simple to see that if X is Q local and Y is Q' local, then their tensor product is $Q + Q'$ local.

Observe that

$$|(X \otimes Y)_k| = \sum_{i=0}^k |X_i| |Y_{k-i}| = \sum_{i=0}^k |(X \otimes Y)_{i,k-i}|,$$

where $(X \otimes Y)_{i,k-i}$ denotes $X_i \times Y_{k-i}$. We also write $C^{i,k-i}(X \otimes Y) = C^i(X) \otimes C^{k-i}(Y)$, and we think of the vector space both as a subspace of $(X \otimes Y)_k$, and as a direct summand of it. For a chain or cochain α we write $\alpha_{i,k-i}$ for its $C^{i,k-i}(X \otimes Y)$ -component.

It is useful to identify the vector space $C^i(X) \otimes C^{j-i}(Y)$ with the space of $|X_i| \times |Y_{j-i}|$ matrices, with rows which correspond to $(j-i)$ -cochains of Y and columns to i -cochains of X . With this identifications the X -differential acts on columns while the Y -differential on rows. We shall use this identification throughout the paper without further notice.

A simple observation which motivates our introduction of polyhedral complexes is this family of complexes is closed under taking tensor products, and that tensor products of simplicial complexes are polyhedral complexes.

The well known Künneth theorem relates the homology and cohomology of $X \otimes Y$ with those of X, Y

THEOREM 2.3. *Let X and Y be complexes which are tensor products of simplicial complexes. The decomposition (2) induces isomorphisms*

$$H_k(X \otimes Y) \cong \oplus_{i=0}^k H_i(X) \otimes H_{k-i}(Y)$$

$$H^k(X \otimes Y) \cong \oplus_{i=0}^k H^i(X) \otimes H^{k-i}(Y)$$

In particular, it holds that

$$h_k(X \otimes Y) = \sum_{i=0}^k h_i(X) \times h_{k-i}(Y).$$

Thus, if for some i , $H^i(X)$ and $H^{k-i}(Y)$ are non trivial, then also the k -th homology and cohomology of $X \otimes Y$ are non trivial.

The proof can be found in any standard text on algebraic topology. For example, the case of homology appears in [16, Section 3.B], while the case of cohomology is handled in [16, Theorem 3.15]. In both cases it is proven for singular homologies and cohomologies, but these agree with polyhedral homology and cohomology when the polyhedral chain complex is a tensor product of simplicial complexes.

2.5 Ramanujan Complexes

Ramanujan graphs were discovered in the seminal work [25]. They have remarkable properties and, in particular, they are asymptotically the best possible expanders, at least spectrally. The Ramanujan complexes, which were constructed in [26, 27] as quotients of Bruhat-Tits buildings, are a high dimensional generalization of Ramanujan graphs. They are a family of high dimensional, bounded degree, simplicial complexes, and, like their one dimensional relatives, they also have many desired properties. In particular, the work of [6, 17] has shown that the d -skeletons of Ramanujan complexes of dimension $d+1$ are explicit bounded degree cosystolic expanders, and they possess Gromov's topological overlapping property [12].

The theory of Ramanujan complexes develops rapidly, and we refer the interested reader to the mentioned papers for precise definitions and a more detailed study.

In the following theorem we summarize the main properties of Ramanujan complexes that we shall need for this work, as were found by [6, 7, 17, 26, 27]. In Lemma 5.1 we will need some additional properties, mainly of Bruhat-Tits buildings, but we will list them in the course of the proof. Additional properties of spherical buildings, that will be required for the proof of Theorem 1.6 will be mentioned in Appendix A.

THEOREM 2.4 (KNOWN PROPERTIES OF RAMANUJAN COMPLEXES THEOREM). *There exists an explicit infinite family of bounded degree cosystolic expanders of every dimension d , where*

- *The number of vertices of a complex in the family (denoted by n) grows to infinity.*
- *The local degrees are upper bounded by a constant $Q > 0$, that is independent of n .*
- *All proper links are spherical buildings.*
- *The i -th cohomology for $i = 1, 2$ do not vanish.*
- *There is a linear lower bound for the length of the i -cosystole, if it exists, for every $i < d$.*
- *The injectivity radius of the complex is $\Theta(\log n)$.*

3 PRODUCTS OF COSYSTOLIC EXPANDERS

We now restate and prove the first main theorem, Theorem 1.4 from the introduction, in greater generality.

THEOREM 3.1. *Let X, Y be complexes of dimensions d, d' , respectively, weighted by the Hamming norm. Assume that*

- $\mathbf{X}$ has cosystolic bounds η_i , $i \leq d-1$.
- $\mathbf{Y}$ has upper localities q'_i , $i \leq d'$.
- $\mathbf{X}$ has cofilling constants μ_i , $i \leq d-1$.

(1) If $\mathbf{Y}$ has cosystolic bounds η'_i , $i \leq d-1$, then there exist constants λ_l , for $l \leq \min\{d, d'\}$, which depend only on

$$(\eta_i)_{i \leq l}, (\eta'_i)_{i \leq l}, (q'_i)_{i \leq l}, (\mu_i)_{i \leq l},$$

such that for all $l < \min\{d, d'\}$, if the l -th cohomology is non zero, then

$$\text{CoSys}^l(\mathbf{X} \otimes \mathbf{Y}) \geq \lambda_l \min_{i \leq l} \{|X_{l-i}| |Y_i|\}.$$

(2) If $\mathbf{Y}$ has the collective cofilling property with constants: $(\mu_i^{\text{coll}})', i \leq d'-1$ then $\mathbf{X} \times \mathbf{Y}$ has the cofilling property, for all $l < \min\{d, d'\}$ with cofilling constants ν_l , which depend only on

$$(\eta_i)_{i \leq l}, (q'_i)_{i \leq l}, (\mu_i)_{i \leq l}, ((\mu_i^{\text{coll}})'), i \leq l.$$

(3) If both $\mathbf{X}$ and $\mathbf{Y}$ have the collective cofilling property with constants μ_i^{coll} , $i \leq d-1$ and $(\mu_i^{\text{coll}})', i \leq d'-1$ respectively, then the product has the collective cofilling property for all $l < \min\{d, d'\}$ with collective cofilling constants ν_l^{coll} , which depend only on

$$(\eta_i)_{i \leq l}, (q'_i)_{i \leq l}, (\mu_i^{\text{coll}})_{i \leq l}, ((\mu_i^{\text{coll}})'), i \leq l.$$

PROOF. Let $n_i = |X_i|$, $n'_i = |Y_i|$. Then for $l \leq \min\{d, d'\}$, $|\mathbf{X} \otimes \mathbf{Y}|_{l,l-i} = n_i n'_{l-i}$. Write

$$\lambda_l = \frac{\prod_{i=0}^{l-1} \eta_i}{\prod_{i=0}^{l-1} q'_i \mu_{l-i-1}} \min_{j \leq l} \{\eta_{l-j} \eta'_j\} \text{ and } N := \min_{i \leq l} n_i n'_{l-i}.$$

Note that $\eta_\bullet, \eta'_\bullet \leq 1$ while $\mu_\bullet, q'_\bullet \geq 1$.

We begin with the first case. Let α be a short cocycle, that is a cocycle of norm

$$|\alpha| < \lambda_l N.$$

We will construct a sequence α_j , $j = 0, \dots, l+1$ with the following properties:

- $\alpha_0 = \alpha$.
- For all j and for all $i < j$, $(\alpha_j)_{l-i,i} = 0$, where $(\alpha_j)_{l-i,i}$ is the $(l-i, i)$ -component of α_j , and for $i > j$, $(\alpha_j)_{l-i,i} = \alpha_{l-i,i}$.
- $|\alpha_j| \leq b_j |\alpha| < b_j \lambda_l |N|$ where

$$b_0 = 1, b_1 = q'_0 \mu_{l-1}, b_{j+1} = \frac{q'_j \mu_{l-j-1}}{\eta_{l-j}} b_j, j \geq 1. \quad (3)$$

- $\alpha_{j+1} - \alpha_j$ is a coboundary.

The existence of this sequence implies, in particular, that α is a coboundary, since $\alpha_{l+1} = 0$ and differs from α by a coboundary. In turn this implies that all l -cocycles of $\mathbf{X} \otimes \mathbf{Y}$ which are not coboundaries are of norm at least $\lambda_l N$, proving the theorem.

Observe that the definition of the bounds b_j , and simple induction, imply

$$b_j \lambda_l \leq \min_{i \leq l} \{\eta_{l-i} \eta'_i\}, \quad (4)$$

indeed, simple induction shows

$$b_j \lambda_l \leq \frac{\prod_{i=0}^{l-j} \eta_i}{\prod_{i=j}^{l-1} q'_i \mu_{l-i-1}} \min_j \{\eta_{l-j} \eta'_j\}.$$

Note also that b_j is independent of the cosystolic bounds of $\mathbf{Y}$.

For $j = 0, \dots, l$ let $\bar{e}_i^j$, $i \in [h^j(\mathbf{X})]$ and $\bar{f}_i^j$, $i \in [h^j(\mathbf{Y})]$ be bases for the j^{th} cohomology of $H^j(\mathbf{X})$, $H^j(\mathbf{Y})$ respectively. Let $e_i^j \in Z^j(\mathbf{X})$ be a representative of $\bar{e}_i^j$ and define f_i^j similarly.

By Theorem 2.3 α can be written as

$$\delta\beta + \sum_{(j,i,i') \in I} e_i^{l-j} \otimes f_{i'}^j,$$

for some set $I \subseteq \bigsqcup_{j=0}^l [h^{l-j}(\mathbf{X})] \times [h^j(\mathbf{Y})]$ and a cochain $\beta \in C^{l-1}(\mathbf{X} \otimes \mathbf{Y})$. We consider $e_i^j \otimes f_{i'}^{l-j}$ as a cochain in $C^{l-j,j}(\mathbf{X} \otimes \mathbf{Y}) \hookrightarrow C^l(\mathbf{X} \otimes \mathbf{Y})$. Write $I_j \subseteq I$ for the subset of triples with first entry j .

We begin with constructing α_1 . Since

$$(\delta\alpha)_{l+1,0} = \delta^{\mathbf{X}} \alpha_{l,0} = 0,$$

all columns of $\alpha_{l,0}$ are l -cocycles of $\mathbf{X}$. Moreover,

$$\alpha_{l,0} = \delta^{\mathbf{X}}(\beta_{l-1,0}) + \sum_{(0,i,i') \in I_0} e_i^l \otimes f_{i'}^0. \quad (5)$$

We first show that $I_0 = \emptyset$. We rewrite (5) as

$$\alpha_{l,0} = \delta^{\mathbf{X}}(\beta_{l-1,0}) + \sum_{i \in [h^l(\mathbf{X})]} e_i^l \otimes v_i, \quad (6)$$

where $v_i \in Z^0(\mathbf{Y})$ a linear combination of the $f_{i'}^0$'s. As the elements $f_{i'}^0$ are linearly independent, $I_0 \neq \emptyset$ precisely if at least one v_i is non zero. In this case the columns indexed by

$$S_0 = \bigcup_i \text{supp}(v_i)$$

are $\mathbf{X}$ -cocycles which are not coboundaries. Indeed, the column $t \in S_0$ equals

$$\sum_{i: t \in \text{supp}(v_i)} e_i^l + \delta^{\mathbf{X}} \beta_{l-1,0}^t,$$

where $\beta_{l-1,0}^t$ is the t^{th} -column of $\beta_{l-1,0}$. This combination is a cocycle, but not a coboundary, since e_i^l are linearly independent and the summation over i is non vacuous as $t \in S_0$. Each column in S_0 is of $\mathbf{X}$ -norm at least $\eta_l n_l$. If some $v_i \neq 0$, then since v_i itself is a $0 - \mathbf{Y}$ -cocycle which is not a coboundary then its $\mathbf{Y}$ -norm is at least $\eta'_0 n'_0$, hence S_0 is of size at least $\eta'_0 n'_0$. Putting together,

$$\eta_l \eta'_0 n_l n'_0 \leq |\alpha_{l,0}| \leq |\alpha| < \lambda_l N = b_0 \lambda_l N,$$

which contradicts (4) applied to $b_0 \lambda_l$. Therefore $\alpha_{l,0} = \delta^{\mathbf{X}}(\beta_{l-1,0})$, and in particular all its columns are $\mathbf{X}$ -coboundaries. By the definition of μ_l , we can find a cochain $\gamma_0 \in C^{l-1}(\mathbf{X}) \otimes C^0(\mathbf{Y})$ with

$$\alpha_{l,0} = \delta^{\mathbf{X}} \gamma_0, |\gamma_0| \leq \mu_{l-1} |\alpha_{l,0}|, \quad (7)$$

this is done by taking a short $\delta^{\mathbf{X}}$ -preimage for each column of $\alpha_{l,0}$. Define

$$\alpha_1 = \alpha + \delta^{\mathbf{X} \otimes \mathbf{Y}} \gamma_0 = \alpha - \alpha_{l,0} + \delta^{\mathbf{Y}} \gamma_0.$$

α_1 satisfies the second and forth requirements. For the third one,

$$|\alpha_1| \leq |\alpha - \alpha_{l,0}| + |\delta^{\mathbf{Y}} \gamma_0| = |\alpha| - |\alpha_{l,0}| + |\delta^{\mathbf{Y}} \gamma_0| \leq$$

$$|\alpha| + (q'_0 \mu_{l-1} - 1) |\alpha_{l,0}| \leq q'_0 \mu_{l-1} |\alpha| = b_1 |\alpha|,$$

where we have used the definition of b_1 and the observation

$$|\delta^{\mathbf{Y}} \gamma_0| \leq q'_0 |\gamma_0| \leq q'_0 \mu_{l-1} |\alpha_{l,0}|.$$

Write $\beta_1 = \beta - \gamma_0$.

We will construct α_j , $j > 1$ inductively. We will construct, alongside with α_j an additional class $\beta_j \in C^{l-1}(\mathbf{X} \otimes \mathbf{Y})$, $j \geq 1$ which satisfies properties we now list.

- $(\beta_j)_{(l-1-s,s)} = 0$, for $s < j-1$.
-

$$\alpha_j = \delta\beta_j + \sum_{(s,i,i') \in I} e_i^{l-s} \otimes f_{i'}^s, \quad (8)$$

with the same set $I \subseteq \bigsqcup_{s=0}^l [h^{l-s}(\mathbf{X})] \times [h^s(\mathbf{Y})]$.

We will also show inductively, in parallel to the construction of α_j, β_j , that $I_s = \emptyset$, for all $s < j$. Note that β_1 satisfies the required properties with respect to α_1 , and that I_0 is indeed empty, so that the basis of induction is satisfied.

Suppose we have constructed α_j, β_j , and that we have shown $I_s = \emptyset$ for $s \leq j-1$. Note that

$$(\delta\beta_j)_{l-j+1,j-1} = (\alpha_j)_{l-j+1,j-1} = 0 \Rightarrow \delta^{\mathbf{X}}((\beta_j)_{(l-j,j-1)}) = 0. \quad (9)$$

We now construct $\alpha_{j+1}, \beta_{j+1}$. By construction,

$$(\alpha_j)_{l-i,i} = 0, \quad i = 0, \dots, j-1.$$

We first show that $I_j = \emptyset$. We can write, by restricting to the $(l-j, j)$ -component of α_j .

$$(\alpha_j)_{l-j,j} = \sum_{i \in [h^{l-j}(\mathbf{X})]} e_i^{l-j} \otimes v_i + \delta^{\mathbf{Y}}\beta_j' + \delta^{\mathbf{X}}\beta_j'', \quad (10)$$

where $\beta_j' = (\beta_j)_{l-j,j-1} \in C^{l-j,j-1}(\mathbf{X} \otimes \mathbf{Y})$, $\beta_j'' = (\beta_j)_{l-j-1,j} \in C^{l-j-1,j}(\mathbf{X} \otimes \mathbf{Y})$ and, by (9),

$$\delta^{\mathbf{X}}\beta_j'' = 0,$$

so that all columns of β_j' are $\mathbf{X}$ -cocycles. As above

$$v_i = \sum_{i': (j,i,i') \in I_j} f_{i'}^j.$$

$I_j \neq \emptyset$ precisely if at least one v_i in the summation is non zero. Any such $v_i \neq 0$ must be a cocycle which is not coboundary. Since $\delta^{\mathbf{X}}\beta_j' = 0$ we can write $\beta_j' = \beta_j^Z + \beta_j^B$ where

$$\beta_j^Z = \sum_{i \in [h^{l-j}(\mathbf{X})]} e_i^{l-j} \otimes u_i, \quad \beta_j^B = \sum_i (\delta^{\mathbf{X}}g_i) \otimes w_i, \quad (11)$$

for arbitrary elements $u_i, w_i \in C^{j-1}(\mathbf{Y})^3$ and $g_i \in C^{l-j-1}(\mathbf{X})$. Write

$$v_i' = v_i + \delta^{\mathbf{Y}}u_i.$$

Then whenever $v_i \neq 0$ also $v_i' \neq 0$ and in this case both are $\mathbf{Y}$ -cocycles which are not coboundaries. Let

$$S_j = \bigcup_i \text{supp}(v_i'). \quad (12)$$

Since we assumed $I_j \neq \emptyset$, then also $S_j \neq \emptyset$. Moreover, the same reasoning we used to lower bound $|S_0|$ shows,

$$\eta_j' n_j' \leq |S_j|.$$

³ u_i depend on j and hence should be written u_i^j , and the same comment holds for v_i and w_i , and g_i . In order to lighten notations we omit the j -dependence for the notations, but we will need to add the superscript j later, when we consider cofilling problems.

The same reasoning also shows that the columns of

$$\sum_{i \in [h^{l-j}(\mathbf{X})]} e_i^{l-j} \otimes v_i + \delta^{\mathbf{Y}}\beta_j^Z = \sum_{i \in [h^{l-j}(\mathbf{X})]} e_i^{l-j} \otimes v_i'$$

indexed by S_j are $\mathbf{X}$ -cocycles which are not coboundaries. All columns of

$$\delta^{\mathbf{X}}\beta_j''$$

are clearly $\mathbf{X}$ -coboundaries. Since

$$\delta^{\mathbf{Y}}\beta_j^B = \delta^{\mathbf{Y}}(\sum_i \delta^{\mathbf{X}}g_i \otimes w_i) = \delta^{\mathbf{X}}(\sum_i g_i \otimes (\delta^{\mathbf{Y}}w_i)), \quad (13)$$

also all columns of $\delta^{\mathbf{Y}}\beta_j^B$ are $\mathbf{X}$ -coboundaries.

Thus, by (10), the columns of $(\alpha_j)_{l-j,j}$ indexed by S_j are all cocycles which are not coboundaries, and therefore are of $\mathbf{X}$ -norm at least $\eta_{l-j} n_{l-j}$. Putting together, we see that if $I_j \neq \emptyset$ then

$$\eta_{l-j} \eta_j' n_{l-j} n_j' \leq |(\alpha_j)_{l-j,j}| \leq |\alpha_j| < b_j \lambda_l N,$$

which contradicts (4).

So we see that $I_j = \emptyset$, hence all $v_i = 0$. In this case, using the same notations,

$$(\alpha_j)_{l-j,j} = \delta^{\mathbf{Y}}\beta_j^Z + \delta^{\mathbf{Y}}\beta_j^B + \delta^{\mathbf{X}}\beta_j''. \quad (14)$$

We can still define S_j to be $\bigcup_i \text{supp}(v_i') = \bigcup_i \text{supp}(\delta^{\mathbf{Y}}u_i)$, and again, by the same consideration, all columns of $(\alpha_j)_{l-j,j}$ indexed by S_j are of $\mathbf{X}$ -norm at least $\eta_{l-j} n_{l-j}$. Write

$$\alpha_j' = \alpha_j + \delta\beta_j^Z = \alpha_j + \delta^{\mathbf{Y}}\beta_j^Z. \quad (15)$$

α_j' differs from α_j only in the columns labeled by S_j , hence

$$|\alpha_j'| \leq \frac{1}{\eta_{l-j}} |\alpha_j|. \quad (16)$$

In addition α_j' differs from α_j by a coboundary, and all columns of $(\alpha_j')_{(l-j,j)}$ are $\mathbf{X}$ -coboundaries. In fact,

$$(\alpha_j')_{(l-j,j)} = \delta^{\mathbf{X}}\beta_j'' + \delta^{\mathbf{Y}}\beta_j^B = \delta^{\mathbf{X}}\beta_j'' + \delta^{\mathbf{X}}(\sum_i g_i \otimes (\delta^{\mathbf{Y}}w_i)), \quad (17)$$

where we have used (13). Therefore, as in the first step, we can find a short $\delta^{\mathbf{X}}$ -preimage for each column of $(\alpha_j')_{l-j,j}$ and this way to define

$$\gamma_j \in C^{l-j-1,j}(\mathbf{X} \otimes \mathbf{Y}), \quad \delta^{\mathbf{X}}\gamma_j = (\alpha_j')_{l-j,j}, \quad (18)$$

$$|\gamma_j| \leq \mu_{l-j-1} |(\alpha_j')_{l-j,j}| \leq \mu_{l-j-1} |\alpha_j'|. \quad (19)$$

Put

$$\alpha_{j+1} = \alpha_j' + \delta\gamma_j, \quad (20)$$

and again it satisfies the second and forth requirements. Regarding the third property:

$$\alpha_j = \alpha_j' + \delta^{\mathbf{X}}\gamma_j + \delta^{\mathbf{Y}}\gamma_j = \alpha_j' - (\alpha_j')_{l-j,j} + \delta^{\mathbf{Y}}\gamma_j,$$

thus,

$$\begin{aligned} |\alpha_j| &= |\alpha_j'| - |(\alpha_j')_{l-j,j}| + |\delta^{\mathbf{Y}}\gamma_j| \leq \\ &= |\alpha_j'| - |(\alpha_j')_{l-j,j}| + q_j' \mu_{l-j-1} |(\alpha_j')_{l-j,j}| \leq \\ &= |\alpha_j'| + (q_j' \mu_{l-j-1} - 1) |(\alpha_j')_{l-j,j}| \leq \\ &= \frac{q_j' \mu_{l-j-1}}{\eta_{l-j}} |\alpha_j| \leq b_{j+1} |\alpha_j|, \end{aligned}$$

where we have used (18), (16), (3) and the induction hypothesis.

We also set

$$\beta_{j+1} = \beta_j - \beta'_j - \gamma_j - \sum_i g_i \otimes (\delta^Y w_i). \quad (21)$$

We claim it satisfies the requirements. The first property holds since the $(l-j, j-1)$ -component of β_j is β'_j , and the other corrections belong to $C^{(l-j-1, j)}(X \otimes Y)$. For the second one, note that since $I_j = \emptyset$, it is enough to show

$$\delta(\beta_{j+1} - \beta_j) = \alpha_{j+1} - \alpha_j.$$

From the definitions

$$\alpha_{j+1} - \alpha_j = \delta(\beta_j^Z + \gamma_j).$$

Now, (21) and the definition of β_j^Z , β_j^B give

$$\delta(\beta_{j+1} - \beta_j) = \delta(\beta_j^Z + \beta_j^B + \sum_i g_i \otimes (\delta^Y w_i) + \gamma_j) = \delta(\beta_j^Z + \gamma_j),$$

where we have used (13) to cancel the middle terms in the middle expression. The induction follows.

We now turn to prove the second claim, that the product has the cofilling property if Y has the collective cofilling property, as in the statement of the theorem.

We continue with the above notations, only that this time $\alpha \in B^l(X \times Y)$ is an arbitrary coboundary, without the smallness assumption, and hence also the estimate (3) will not hold. We will find $\tilde{\beta} \in C^{l-1}(X \times Y)$ with $\delta^l \tilde{\beta} = \alpha$. In this case, from Theorem 2.3, $I = \emptyset$.⁴ Using (14) and (11) we can write

$$(\alpha_j)_{l-j, j} = \sum_{i \in [h^{l-j}(X)]} e_i^{l-j} \otimes \delta^Y u_i^j + \delta^Y \beta_j^B + \delta^X \beta_j''.$$

Let $\tilde{u}_i^j \in C^{j-1}(Y)$ be δ^Y -preimages of $\delta^Y(u_i^j)$, $i \in [h^{l-j}(X)]$, which satisfy

$$|\bigcup_i \tilde{u}_i^j| \leq (\mu_j^{\text{coll}})' |\bigcup_i \delta^Y u_i^j|. \quad (22)$$

The existence of such elements is guaranteed by the collective cofilling property of Y . Let $\tilde{\beta} \in C^{l-1}(X \times Y)$ be the $(l-1)$ -cochain

$$\tilde{\beta}_{l-1-j, j} = \gamma_j + \sum_{i \in [h^{l-j}(X)]} e_i^{l-1-j} \otimes \tilde{u}_i^{1+j}, \quad 0 \leq j \leq l-1.$$

We first show that $\delta \tilde{\beta} = \alpha$ and then estimate $|\tilde{\beta}|$.

$$(\delta \tilde{\beta})_{l, 0} = \delta^X \gamma_0 + \sum_{i \in [h^{l-1}(X)]} \delta^X(e_i^{l-1}) \otimes \tilde{u}_i^1 = \delta^X \gamma_0 = \alpha_{l, 0},$$

from the definition of γ_0 and the fact that $e_i^{l-1} \in Z^l(X)$.

For $j > 0$ we need to show that $\alpha_{l-j, j}$ equals

$$(\delta \tilde{\beta})_{l-j, j} = \delta^Y \tilde{\beta}_{l-j, j-1} + \delta^X \tilde{\beta}_{l-1-j, j} \quad (23)$$

$$= \delta^Y \gamma_{j-1} + \sum_{i \in [h^{l-j}(X)]} e_i^{l-j} \otimes \delta^Y \tilde{u}_i^j + \delta^X \gamma_j, \quad (24)$$

where we set $\gamma_l = 0$ and used the δ^X -closeness of e_i^{l-1-j} .

⁴In the previous part, in order to prove that $I = \emptyset$, we had to use the cosystolic bounds of Y .

Consider α_{j-1} . Its $(l-j, j)$ -component equals $\alpha_{l-j, j}$, by construction. Similarly, by construction, the $(l-j, j)$ -component of α_{j+1} is 0. Thus,

$$\alpha_{l-j, j} = (\alpha_{j+1} - \alpha_{j-1})_{l-j, j}.$$

On the other hand, using equations (15), (20), we have

$$\alpha_{j+1} - \alpha_{j-1} = \delta \gamma_{j-1} + \delta \beta_{j-1}^Z + \delta \gamma_j + \delta \beta_j^Z.$$

Taking the $(l-j, j)$ -component, using (11) and $\delta^Y \tilde{u}_i^j = \delta^Y u_i^j$ proves (23).

In order to estimate $\tilde{\beta}$, note that

$$\begin{aligned} |\sum_{i \in [h^{l-j}(X)]} e_i^{l-j} \otimes \tilde{u}_i^j| &\leq n_{l-j} |\bigcup_i \tilde{u}_i^j| \leq (\mu_j^{\text{coll}})' n_{l-j} |\delta^Y u_i^j| \\ &= (\mu_j^{\text{coll}})' n_{l-j} |S_j|, \end{aligned}$$

where we have used (22) and the definition of S_j , (12). On the other hand, as we saw above, each column of α_j indexed by S_j , is a non trivial cocycle, hence of length at least $\eta_{l-j} n_{l-j}$. Thus,

$$|\alpha_j| \geq |(\alpha_j)_{l-j, j}| \geq \eta_{l-j} n_{l-j} |S_j| \geq \frac{\eta_{l-j}}{(\mu_j^{\text{coll}})'} |\sum_{i \in [h^{l-j}(X)]} e_i^{l-j} \otimes \tilde{u}_i^j|. \quad (25)$$

Although we gave up on the smallness assumption for α , (7), (16), (18) and $\alpha_j \leq b_j |\alpha|$ still hold, with b_j defined in (3), since deriving them only used $I = \emptyset$, which holds in our case. Therefore

$$|\sum_{i \in [h^{l-j}(X)]} e_i^{l-j} \otimes \tilde{u}_i^j| \leq \frac{(\mu_j^{\text{coll}})'}{\eta_{l-j}} |\alpha_j| \leq \frac{(\mu_j^{\text{coll}})' b_j}{\eta_{l-j}} |\alpha|$$

and

$$|\gamma_0| \leq \mu_{l-1} |\alpha|, \quad |\gamma_j| \leq \frac{\mu_{l-j-1}}{\eta_{l-j}} |\alpha_j| \leq \frac{\mu_{l-j-1} b_j}{\eta_{l-j}} |\alpha|, \quad j > 0.$$

Summing these equations we see that

$$\begin{aligned} |\tilde{\beta}_{l-1, 0}| &\leq (\mu_{l-1} + \frac{(\mu_{1+j}^{\text{coll}})' b_{1+j}}{\eta_{l-1-j}}) |\alpha|, \\ |\tilde{\beta}_{l-j-1, j}| &\leq (\frac{\mu_{l-j-1} b_j}{\eta_{l-j}} + \frac{(\mu_{1+j}^{\text{coll}})' b_{1+j}}{\eta_{l-1-j}}) |\alpha|, \quad j > 0, \end{aligned}$$

and hence

$$|\tilde{\beta}| \leq v_l |\alpha|,$$

for

$$v_l = \mu_{l-1} + \sum_{j=1}^{l-1} \frac{b_j}{\eta_{l-j}} ((\mu_j^{\text{coll}})' + \mu_{l-j-1}). \quad (26)$$

As claimed.

We now turn to the last item of the theorem, concerning the product of two complexes having the collective cofilling property. This time we begin with coboundaries $\alpha_a \in B^l(X \times Y)$, $a = 1, \dots, m$, and we aim to construct $(l-1)$ -cochains, $\tilde{\beta}_a \in C^{l-1}(X \times Y)$, $a = 1, \dots, m$ such that for all a , $\delta \tilde{\beta}_a = \alpha_a$ and

$$|\bigcup_{a \in [m]} \tilde{\beta}_a| \leq v_l |\bigcup_{a \in [m]} \alpha_a|.$$

The constant v_l will be given by the same expression as in the previous case, only with each μ_i replaced by μ'_i . The proof is very similar to the previous case, so we only describe the necessary

changes. We use the same notations as before, only we add with an additional index a , so that we have classes $(\gamma_a)_j$, $(u_a)_i^j$, $a \in [m]$, etc., playing the analogous role to before, only with respect to α_a . There are precisely two differences in the constructions. The first is that instead of choosing $(\gamma_a)_j$ by taking short δ^X -preimages for the columns of $(\alpha'_a)_j$, we choose the preimages $(\gamma_a)_j$, $a \in [m]$ so that for any $t \in Y_j$, the norm of the union over $a \in [m]$ of columns indexed by t in $(\gamma_a)_j$, is at most $\mu_{l-j-1}^{\text{coll}}$ times the norm of the union over $a \in [m]$ of the columns indexed t in $(\alpha'_a)_j$. This is achieved by using the collective cofilling of X . Similarly, instead of choosing $(\tilde{u}_a)_i^j$ as δ^Y -preimages of $\delta^Y(u_a)_i^j$ satisfying (22) for each a separately, we choose them to satisfy

$$\left| \bigcup_{a,i} (\tilde{u}_a)_i^j \right| \leq (\mu_j^{\text{coll}})' \left| \bigcup_{a,i} \delta^Y(u_a)_i^j \right|.$$

All the analysis remains the same, and the proof follows. $\square$

4 THE COLLECTIVE COFILLING PROPERTY OF RAMANUJAN COMPLEXES

In this section we deal with the collective cofilling for Ramanujan complexes. We will adapt the strategy of [6, 17] to our situation. Our proof will rely strongly on the notion of local minimality for collections of cochains, which generalizes the corresponding notion for single chains, defined in [17]. Due to space limitations, the proof is not given here, and can be found in the full version of this paper [20].

5 TENSORING RAMANUJAN COMPLEXES AND EXPLICIT QUANTUM LDPC CODES OF DISTANCE $\Omega(\sqrt{n} \log^k n)$ FOR ANY k

5.1 Systoles in Tensor Products of Ramanujan Complexes

Theorem 5.11 in [7] proves that Ramanujan complexes of degree d with injectivity radius at least R and non trivial k -th homology have k -systoles at least $c_{d,k} R^k$. The proof uses only the fact that the universal cover of the (underlying topological space of the) complex is a Bruhat-Tits building, hence generalizes to our setting as well, since products of Bruhat-Tits buildings are Bruhat-Tits buildings as well (and the universal cover of a product is the product of universal covers). We therefore immediately obtain:

LEMMA 5.1. *Let $X_1, \dots, X_l$ be Ramanujan complexes of injectivity radius at least R . Let*

$$Y = X_l \otimes X_{l-1} \otimes \dots \otimes X_1,$$

then for every $k \leq \dim Y$,

$$\text{Sys}_k(Y) = \Omega(R^k),$$

the constant inside Ω depends on k and $\dim(Y)$.

Due to space limitations the proof of this lemma is omitted and can be found in the full version of this paper [20].

5.2 Properties of Tensor Products of Ramanujan Complexes

We now prove a more general version of Theorem 1.7.

THEOREM 5.2. *Let $l < d$ be natural number and consider Ramanujan complexes $X_1, \dots, X_l$ of dimensions $d_1, \dots, d_l \geq d$. Assume that there exists m and constants a, c, C, Q such that for each i , the number of vertices of X_i is between cm and Cm , for some m , its locality is at most Q , and its injectivity radius is at least $a \log m$. Write $n = m^l$. Then the tensor product*

$$Y = X_l \otimes X_{l-1} \otimes \dots \otimes X_1,$$

has the following properties:

- Y has $\Theta(n)$ vertices and locality at most lQ . In addition each $|Y_i|$ is $\Theta(n)$. The constants inside Θ depend on l, c, C, Q .
- $H_k(Y), H^k(Y) \neq 0$, $k \leq l$.
- $\text{CoSys}^k(Y) = \Omega(n)$ for all $k \leq l$. The constant inside Ω depends on l, c, C, Q as well as the cofilling constant and cosystolic bounds of the complexes X_i .
- Y has the collective cofilling property (and in particular the cofilling property) with a constant which depends only on l, c, C, Q the cosystolic bounds and collective cofilling constants of the complexes X_i .
- $\text{Sys}_k(Y) = \Omega(\log^k(n))$ for $k \leq l$, the constant inside Ω depends on k, a and the total dimension $\sum d_i$.

PROOF. The first claim is straightforward; for the 'In addition' part we use that for Ramanujan complexes, or more generally pure complexes, of locality at most Q and dimension at most d , the number of k -simplices is upper and lower bounded by two constants which depend on Q, d times the number of vertices.

By Theorem 2.4 the first homology and cohomology of each X_i , with $\mathbb{F}_2$ coefficients, are non trivial. By Theorem 2.3 this implies that for every s the k th cohomology and homology groups of

$$R_s = X_s \otimes X_{s-1} \otimes \dots \otimes X_1,$$

for $k \leq s$ is non zero.

By Theorem 2.4 again, R_1 and each X_i have linear cosystoles and has bounded cofilling constants. Since

$$R_{s+1} = X_{s+1} \otimes R_s,$$

by iterating Theorem 3.1 we obtain that $\text{CoSys}^k(Y)$ is linear in $|Y_k|$ for each $k \leq l$, where the constant of proportionality is at least some universal constant which depends on Q , on the cofilling constants and cosystolic bounds of each X_i , and on c, C .

From Theorem 1.5 and Remark 2.2, each X_i also has the collective cofilling property with respect to the Hamming norm. Iterating the third part of Theorem 3.1, shows the collective cofilling property for the product.

The claim about the systoles is just Lemma 5.1. $\square$

5.3 Generating LDPC Quantum Codes of Distance $\Omega(\sqrt{n} \log^k n)$

Finally we arrive to the construction of the claimed explicit LDPC quantum codes of distance $\Omega(\sqrt{n} \log^k n)$. The explicitness of the code stems from the fact that it is obtained by taking a tensor power of the explicitly known Ramanujan complexes. Then applying the

balancing procedure of [7] to the tensored complex. As the balancing procedure of [7] maintains explicitness the resulting code is an explicit LDPC quantum code with the claimed parameters.

PROOF OF COROLLARY 1.9. Take $X_1 = X_2 = \dots = X_k = X$ be Ramanujan complexes of dimension $d > k$ satisfying the properties of Theorem 2.4. Define Y as in Theorem 5.2, extract its k -th homology and cohomology pair, and apply to it the balancing procedure of Theorem 1.8. The resulting code has the prescribed distance and dimension. $\square$

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A STRONG COFILLING FOR BUILDING-LIKE COMPLEXES

In this section we recall the definition of building-like complexes, following [24]. We then prove that these complexes have the collective cofilling property. The proof is a minor adaptation of the proof of [24] for the usual cofilling property, which in turn generalizes Gromov's ideas from [12]. As in Section 4 we shall work solely with the weight function (1), and consider collective cofilling constants with respect to it. Here, unlike in Section 4, the results will not be correct with the Hamming or normalized Hamming weight, since the complexes we consider do not have a bounded locality, see Remark 2.2.

Let X be a simplicial complex of dimension n , G a subgroup of $\text{Aut}(X)$, and S a set on which G acts. For $0 \leq k \leq n-1$, write

$$\mathcal{F}_k = X_k \times S,$$

it is endowed with the G -action given by

$$g(\sigma, s) = (g\sigma, gs).$$

Let

$$\mathcal{B} = \{B_{\sigma,s} : -1 \leq k \leq n-1, (\sigma, s) \in \mathcal{F}_k\}$$

be a collection of subcomplexes of X with the property that for any two simplices $\sigma \subseteq \sigma'$, and any $s \in S$

$$\sigma \in B_{\sigma,s} \subseteq B_{\sigma',s}.$$

DEFINITION A.1. [Definition 1.2 in [24]] A 4-tuple $(X, S, G, \mathcal{B})$ as above is a building like complex if it satisfies the following:

- (1) G acts transitively on X_n .
- (2) $gB_{\sigma,s} = B_{g\sigma,gs}$.
- (3) For all $-1 \leq i \leq k \leq n-1$, and all $(\sigma, s) \in \mathcal{F}_k$, the reduced homology $\tilde{H}_i(B_{\sigma,s}) = 0$.⁵

⁵The reduced homology $\tilde{H}_\bullet$ of a based chain complex $X = (X_0, \dots, X_d)$ is the i -th homology group of the complex obtained from augmenting X by a -1 -level $\mathbb{Z}$ and defining the augmentation map $\partial_0 : C_0(X) \rightarrow \mathbb{Z}$ by $\sum n_i \sigma_i \mapsto \sum n_i$. It is easy to verify that for $i > 0$, $\tilde{H}_i \approx H_i$, while $H_0 \approx \tilde{H}_0 \oplus \mathbb{Z}$.

For $0 \leq k \leq n-1$, let $a_k = a_k(X, S, G, \mathcal{B})$ denote

$$\max\{|G\sigma' \cap (B_{\sigma,s})_{k+1}| : \sigma' \in X_{k+1}, (\sigma, s) \in \mathcal{F}_k\},$$

where $G\sigma'$ is the G -orbit of σ' , and as usual $(B_{\sigma,s})_{k+1}$ is the collection of $(k+1)$ -simplices of $B_{\sigma,s}$.

THEOREM A.2. For a building like complex $(X, S, G, \mathcal{B})$ as above, and $0 \leq k \leq n-1$,

$$\mu_k^{\text{coll}} \leq \binom{n+1}{k+2} a_k.$$

The proof is very similar to the proof of [24, Theorem 1.3], and it should not come as a surprise. The proofs in [12, 24] construct, for a given coboundary, a δ -preimage by using cones, and averaging over them. A light-weight preimage is obtained by finding a cone that is good on average for all simplices in the given coboundary. In a sense this is a local construction. The fact that one can find such cones make it natural to guess that one can do the same for a collection of coboundaries, and this is indeed the case. On Ramanujan complexes, on the other hand, the proof of existence of light-weight preimages is non constructive, and more importantly, it is not evident to be local, hence one needs to adapt it more in order to obtain the result for collections.

PROOF. As mentioned above, a key ingredient is the existence of cones, cf. [24, Proposition 2.1].

PROPOSITION A.3. There exists a collection of cones

$$C = \{c_{\sigma,s} \subseteq X_{k+1} : (\sigma, s) \in \mathcal{F}_k, -1 \leq k \leq n-1\},$$

such that for all $(\sigma, s) \in \mathcal{F}_k$

$$\partial c_{\sigma,s} = \sigma + \sum_{i=0}^k c_{(\sigma \setminus \{i\}),s},$$

where $(\sigma \setminus \{i\})$ is the $(k-1)$ -simplex obtained from σ by omitting its i -th vertex.

The cones are used to construct contraction operations:

$\iota_s : C^{k+1}(X) \rightarrow C^k(X)$, for $s \in S$, i.e. operations which satisfy, for a cochain $\beta \in C^{k+1}(X)$, and a k -simplex σ

$$(\iota_s \beta)(\sigma) = \beta(c_{\sigma,s}).$$

[24, Claim 2.2] shows that ι_s satisfies

$$\delta^k \circ \iota_s + \iota_s \circ \delta^{k+1} = \text{id}_{C^{k+1}(X)}. \quad (27)$$

Let $\beta_1, \dots, \beta_m$ be a collection of $(k+1)$ -coboundaries. Then (27) yields

$$\delta^k \iota_s \beta_i = \beta_i.$$

The proof will follow if we could upper bound

$$\min_{s \in S} \left\| \bigcup_{i \in [m]} \iota_s \beta_i \right\| \leq \binom{n+1}{k+2} a_k \left\| \bigcup_{i \in [m]} \beta_i \right\|.$$

We will prove this inequality with average instead of minimum. For this again we mimic [24] and set

$$\theta_k = \max_{\sigma \in X_{k+1}} \left(\frac{1}{|S|w(\sigma)} \sum_{(\eta,s) \in \mathcal{F}_k : \sigma \in c_{\eta,s}} w(\eta) \right).$$

We will show

$$\frac{1}{|S|} \sum_s \left\| \bigcup_{i \in [m]} \iota_s \beta_i \right\| \leq \theta_k \left\| \bigcup_{i \in [m]} \beta_i \right\|. \quad (28)$$

This equation, together with [24, Claim 2.4] which states

$$\theta_k \leq \binom{n+1}{k+2} a_k,$$

finishes the proof. It remains to establish inequality (28). The proof is analogous to the proof of [24, Claim 2.3], but for collections of coboundaries.

$$\begin{aligned} & \frac{1}{|S|} \sum_s \left\| \bigcup_{i \in [m]} \iota_s \beta_i \right\| \\ &= \frac{1}{|S|} \sum_s \sum_{\bigcup_{i \in [m]} \{\sigma \in X_k : \iota_s \beta_i(\sigma) \neq 0\}} w(\sigma) \\ &= \frac{1}{|S|} \sum_s \sum_{\bigcup_{i \in [m]} \{\sigma \in X_k : \beta_i(c_{\sigma,s}) \neq 0\}} w(\sigma) \\ &\leq \frac{1}{|S|} \sum_s \sum_{\bigcup_{i \in [m]} \{\sigma \in X_k : \text{supp} \beta_i \cap \text{supp}(c_{\sigma,s}) \neq \emptyset\}} w(\sigma) \\ &\leq \frac{1}{|S|} \sum_{\tau \in \bigcup_{i \in [m]} \text{supp}(\beta_i)} \sum_s \sum_{\tau \in \text{supp}(c_{\sigma,s})} w(\sigma) \\ &= \frac{1}{|S|} \sum_{\tau \in \bigcup_{i \in [m]} \text{supp}(\beta_i)} \sum_{\{(\sigma,s) \in \mathcal{F}_k : \tau \in \text{supp}(c_{\sigma,s})\}} w(\sigma) \\ &\leq \theta_k \sum_{\tau \in \bigcup_{i \in [m]} \text{supp}(\beta_i)} w(\tau) \\ &= \theta_k \left\| \bigcup_{i \in [m]} \beta_i \right\|. \end{aligned}$$

The first inequality follows from the observation that if $\beta_i(c_{\sigma,s}) \neq 0$ then $\text{supp}(\beta_i) \cap \text{supp}(c_{\sigma,s}) \neq \emptyset$. In the second inequality we pass from summing over $w(\sigma)$ once for all s such that $\bigcup \text{supp}(\beta_i)$ intersects $c_{\sigma,s}$, to summing, for all σ, s , the term $w(\sigma) |(\bigcup \text{supp}(\beta_i)) \cap c_{\sigma,s}|$. The last inequality follows from the definition of θ_k . The equalities are straight forward. $\square$

PROOF OF THEOREM 1.6. In [24, Subsection 3.2], it is shown that spherical buildings are building like complexes. In their notations, for the spherical building $\Delta = \Delta(G; B, N)$, which is associated to the rank $n+1$ BN pair (B, N) , the group G is $\langle B, N \rangle$, the set S is the collection of n -simplices of Δ , and for a k -simplex σ and an n -simplex $\theta \in S$, $B_{\sigma,\theta}$ is the intersection of the apartments which contain both σ, θ . The first two properties of spherical buildings are well known, the third one is proven in [24, Claim 3.5]. It is shown that

$$a_k \leq \binom{n+1}{k+2} |W|,$$

where W is the associated Weyl group, and therefore if ω_n is the size of the largest Weyl group of rank $n+1$, then by Theorem A.2 we immediately get that

$$\mu_k^{\text{coll}} \leq \mu^{\text{coll}}(n, k) = \binom{n+1}{k+2} \omega_n.$$

As needed. $\square$

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