Point-to-Hyperplane Nearest Neighbor Search Beyond the Unit Hypersphere

Qiang Huang

School of Computing
National University of Singapore
huangq@comp.nus.edu.sg

Yifan Lei

School of Computing
National University of Singapore
lei yifan@comp.nus.edu.sg

Anthony K. H. Tung

School of Computing
National University of Singapore
atung@comp.nus.edu.sg

ABSTRACT

Point-to-Hyperplane Nearest Neighbor Search (P2HNNS) is a fundamental yet challenging problem, and it has plenty of applications in various fields. Existing hyperplane hashing schemes enjoy sub-linear query time and achieve excellent performance on applications such as large-scale active learning with Support Vector Machines (SVMs). However, they only conditionally deal with this problem with a strong assumption that all of the data objects are normalized, located at the unit hypersphere. Those hyperplane hashing schemes may be arbitrarily bad without this assumption.

In this paper, we introduce a new asymmetric transformation and develop the first two provable hyperplane hashing schemes, Nearest Hyperplane hashing (NH) and Furthest Hyperplane hashing (FH), for high-dimensional P2HNNS beyond the unit hypersphere. With this asymmetric transformation, we demonstrate that the hash functions of NH and FH are locality-sensitive to the hyperplane queries, and both of them enjoy quality guarantee on query results. Moreover, we propose a data-dependent multi-partition strategy to boost the search performance of FH. NH can perform the hyperplane queries in sub-linear time, while FH enjoys a better practical performance. We evaluate NH and FH over five real-life datasets and show that we are around $3 \sim 100\times$ faster than the best competitor in four out of five datasets, especially for the recall in [20%, 80%]. Code is available at <https://github.com/HuangQiang/P2HNNS>.

CCS CONCEPTS

• **Information systems** → **Nearest-neighbor search**; • **Theory of computation** → *Nearest neighbor algorithms*.

KEYWORDS

Locality-Sensitive Hashing; Nearest Neighbor Search; Furthest Neighbor Search; Asymmetric Transformation; Active Learning

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1 INTRODUCTION

Nearest Neighbor Search (NNS) over high-dimensional spaces is of vital importance in many areas, such as databases, data mining, and artificial intelligence. Locality-Sensitive Hashing (LSH) [13, 19] and its variants [1, 3, 8, 11] have been one of the most popular methods for high-dimensional NNS due to their excellent practical performance and robust theoretical guarantees on sub-linear query time and query results. However, most of the existing methods [2, 12, 15, 16, 21–23, 26, 27, 37, 38, 43, 49, 50] mainly deal with the classic point-to-point NNS (or simply NNS), that is, both queries and data objects are points in the same dimension.

In this paper, we study a fundamental yet more difficult problem, named *Point-to-Hyperplane NNS* (P2HNNS), which has plenty of applications in large-scale active learning with SVMs [34, 39, 41], maximum margin clustering [47, 48], and large-margin dimensionality reduction [33, 45]. Given a database $\mathcal{D}$ of n data objects represented as points in a $(d-1)$ -dimensional Euclidean space $\mathbb{R}^{d-1}$ and a hyperplane query $\mathbf{q} = (q_1, \dots, q_d) \in \mathbb{R}^d$ such that $q_d + \sum_{i=1}^{d-1} x_i q_i = 0$ for any point $\mathbf{x} = (x_1, \dots, x_{d-1}) \in \mathbb{R}^{d-1}$ at this hyperplane, P2HNNS aims to find the nearest data point $\mathbf{p}^* \in \mathcal{D}$ of $\mathbf{q}$ such that

$$\mathbf{p}^* = \arg \min_{\mathbf{p} \in \mathcal{D}} d_{P2H}(\mathbf{p}, \mathbf{q}), \quad (1)$$

where $d_{P2H}(\mathbf{p}, \mathbf{q})$ represents the distance from a data point $\mathbf{p}$ to the hyperplane query $\mathbf{q}$. For any $\mathbf{p} = (p_1, \dots, p_{d-1})$ and $\mathbf{q} = (q_1, \dots, q_d)$, $d_{P2H}(\mathbf{p}, \mathbf{q})$ is computed as follows:

$$d_{P2H}(\mathbf{p}, \mathbf{q}) = \frac{|q_d + \sum_{i=1}^{d-1} p_i q_i|}{\sqrt{\sum_{i=1}^{d-1} q_i^2}}. \quad (2)$$

There are many hyperplane hashing schemes [4, 20, 24, 25, 42] to deal with the problem of high-dimensional P2HNNS. Jain et al. [20, 42] introduced the first two provable hashing schemes Angle Hyperplane hashing (AH) and Embedding Hyperplane hashing (EH) with two families of linear hash functions which are locality-sensitive to the angular distance between data objects and the normal vector of the hyperplane query. Later, researchers proposed the Bilinear Hyperplane hashing (BH) [24] and Multilinear Hyperplane hashing (MH) [25], which amplifies the collision probabilities between different angular distances using the bilinear hash functions and multilinear hash functions, respectively. By exploiting the asymmetry hash functions between data objects and hyperplane queries, Aumüller et al. developed a general distance-sensitive hashing [4] beyond LSH. All of these methods use the static (K, L) -bucketing strategy [19] for high-dimensional P2HNNS, where K is the number of concatenated hash functions; L is the number of hash tables.

However, these hyperplane hashing schemes [4, 20, 24, 25, 42] only *conditionally* deal with this problem with a strong assumption:

all of the data are located at the unit hypersphere, i.e., the Euclidean norm $\|\mathbf{p}\| = \sqrt{\sum_{i=1}^{d-1} p_i^2} = 1$ for all $\mathbf{p} \in \mathcal{D} \subset \mathbb{R}^{d-1}$. They work well in large-scale active learning [20, 24, 25]. However, in most applications, such as clustering [47, 48] and dimension reduction [33, 45], data objects are often not normalized. For such applications, these hyperplane hashing schemes may be *arbitrarily bad*. Let $\mathbf{w}_q = (q_1, \dots, q_{d-1})$ be the normal vector of the hyperplane query $\mathbf{q}$. The reason is that, if this assumption fails, the locality-sensitive property to the angular distance $\alpha_{\mathbf{p}, \mathbf{w}_q}$ between $\mathbf{p}$ and $\mathbf{w}_q$ will be no longer valid, and hence the retrieved identities will not follow the order of $\arg \min_{\mathbf{p} \in \mathcal{D}} d_{P2H}(\mathbf{p}, \mathbf{q})$, i.e.,

$$\arg \min_{\mathbf{p} \in \mathcal{D}} \frac{|\sum_{i=1}^{d-1} p_i q_i|}{\|\mathbf{p}\| \|\mathbf{w}_q\|} \not\Rightarrow \arg \min_{\mathbf{p} \in \mathcal{D}} d_{P2H}(\mathbf{p}, \mathbf{q}).$$

To the best of our knowledge, there is no existing hyperplane hashing scheme which can deal with the P2HNNS in sub-linear time without the unit hypersphere assumption of the data objects.

Moreover, ρ is a key performance indicator for the static (K, L) -bucketing setting, where $\rho = \ln(1/p_1)/\ln(1/p_2)$; p_1 and p_2 are the positive and negative collision probabilities, respectively. The hyperplane hashing schemes [4, 20, 24, 25, 42] with this setting require $L = O(n^\rho)$ hash tables to achieve a constant success probability for P2HNNS. The ρ value can be small when $d_{P2H}(\mathbf{p}, \mathbf{q})$ is large. However, as summarized in [25], the p_1 and p_2 values of AH, EH, BH, and MH are less than $\frac{1}{2}$, i.e., $0 < p_2 < p_1 < \frac{1}{2}$. When $d_{P2H}(\mathbf{p}, \mathbf{q}) \rightarrow 0$, $\lim_{d_{P2H} \rightarrow 0} p_1 = \lim_{d_{P2H} \rightarrow 0} p_2 = \frac{1}{2}$, then $\rho = \lim_{d_{P2H} \rightarrow 0} \ln(1/p_1)/\ln(1/p_2) = 1$, leading to $L = O(n)$ hash tables. Therefore, even though they enjoy sub-linear query time, their practical space and query time overhead can be very large.

1.1 Our Contributions

To deal with the high-dimensional P2HNNS beyond the unit hypersphere, we propose a novel asymmetric transformation which converts this challenging problem into the classic NNS problem. With a state-of-the-art LSH scheme based on Longest Circular Co-Substring (LCCS-LSH) [22], we develop the first provably *sub-linear time* method Nearest Hyperplane hashing (NH) for P2HNNS without the data normalization assumption. We show the hash functions of NH enjoy the *locality-sensitive* property to the hyperplane queries: the collision probability between any $\mathbf{p}$ and $\mathbf{q}$ increases as $d_{P2H}(\mathbf{p}, \mathbf{q})$ decreases. NH can deal with the P2HNNS in sub-quadratic space and sub-linear query time.

With a minor modification of this asymmetric transformation, the P2HNNS problem can be transformed to another classic problem of point-to-point Furthest Neighbor Search (or simply FNS). By adopting a state-of-the-art LSH scheme Reverse Query-Aware LSH (RQALSH) [14] for FNS, we develop another provable method Furthest Hyperplane hashing (FH) for P2HNNS. The hash functions of FH also have the *locality-sensitive* property such that the probability with which any $\mathbf{p}$ and $\mathbf{q}$ are separated into different buckets increases as $d_{P2H}(\mathbf{p}, \mathbf{q})$ decreases. Moreover, we propose a data-dependent multi-partition strategy to boost the search performance of FH, which enjoys better practical performance than NH.

We provide a comprehensive comparison of NH and FH with four different methodologies, which includes two state-of-the-art methods BH and MH and two trivial baselines Random-Scan and

Sorted-Scan. The results over five real-life datasets show that we are around $3 \sim 100\times$ faster than the best competitor in four out of five datasets, especially when the recall falls in the range of [20%, 80%].

1.2 Related Work

P2HNNS. A trivial solution for P2HNNS is the exhaustive linear scan, but its query time $O(nd)$ is inefficient when n or d is large. For the data in low to medium dimensions (e.g., $d \leq 10$), this problem can be well-solved by adapting vanilla tree structures (e.g., Ball-Tree [29], Cone-Tree [32], and KD-Tree [6]) with the branch-and-bound space partitioning strategy. However, due to the “curse of dimensionality,” the space and/or query time complexities of these methods are highly dependent on d and their expansion constants, which are not suitable in high-dimensional spaces [7, 44].

Hyperplane hashing schemes [4, 20, 24, 25, 42] have been proposed recently for high-dimensional P2HNNS. In [20, 42], Jain et al. designed two families of linear hash functions which are locality-sensitive to the angular distance $\alpha_{\mathbf{p}, \mathbf{w}_q} = |\frac{\pi}{2} - \theta_{\mathbf{p}, \mathbf{w}_q}|$, where $\theta_{\mathbf{p}, \mathbf{w}_q}$ is the angle between $\mathbf{p}$ and the normal vector $\mathbf{w}_q$. With these two LSH families, they proposed the first two hyperplane hashing schemes AH and EH for P2HNNS with theoretical guarantees. To enhance the locality-sensitive property with a larger difference of the collision probabilities between different $\alpha_{\mathbf{p}, \mathbf{w}_q}$ values, BH [24] and MH [25] have been developed with the newly designed bilinear hash functions and multilinear hash functions. Moreover, the researchers developed two learning based methods to boost the search performance of BH and MH, respectively.

Recently, Aumüller et al. proposed a new distance-sensitive hashing scheme [4] beyond LSH for P2HNNS, with an asymmetry design of a pair of hash functions for data objects and hyperplane queries. Compared with the linear scan, these schemes enjoy comparable accuracy of retrieved results and sub-linear query time by reducing the number of data objects to check in orders of magnitude.

Asymmetric Transformation. For the problem of Maximum Inner Product Search (MIPS), there exists many asymmetric LSH schemes which design different asymmetric transformations to address the problem of data normalization. L2-ALSH [35] and Sign-ALSH [36] use the asymmetric transformations by extending data objects and queries with several dimensions, but they contain additive error. XBOX [5] develops an exact asymmetric transformation with only one dimension extension. Simple-LSH [28] adopts a transformation which maps all data objects and queries to the same hypersphere to reduce the distortion error. Recently, H2-ALSH [17] and Norm-Ranging LSH [46] integrate the data partition into the asymmetric transformations to further improve the MIPS performance. The key of these schemes is to design asymmetric transformations for data objects and queries, so that the (asymmetric) hash functions could be locality-sensitive to the inner product.

According to Equation 7 which will be illustrated later, the P2HNNS can be regarded as a minimum absolute inner product search problem. Both problems are related to the inner product, but the P2HNNS contains an extra absolute value operator. Thus, existing asymmetric transformations for MIPS cannot be used directly for P2HNNS, but they motivate us to develop new transformations for data points and hyperplane queries to remove the absolute value operator and address the data normalization problem.

2 PRELIMINARIES

In this work, we convert the P2HNNS into the classic point-to-point NNS (or FNS) problem. For those two problems, both data objects and queries are represented as points in the same dimension.

2.1 Locality-Sensitive Hashing

LSH schemes are one of the most popular methods for high dimensional NNS [2, 11, 12, 22]. Given a hash function h , we say two objects $\mathbf{o}$ and $\mathbf{q}$ collide in the same bucket if $h(\mathbf{o}) = h(\mathbf{q})$. Let $\text{Dist}(\mathbf{o}, \mathbf{q})$ be a distance function between any two $\mathbf{o}$ and $\mathbf{q}$. We first define a modified LSH family in a form which will be used later.

Definition 2.1 ((Asymmetric) LSH Family): *Given a search radius R ($R > 0$) and an approximation ratio c , a family of hash functions $\mathcal{H}$, along with two vector transformations $P : \mathbb{R}^d \rightarrow \mathbb{R}^{d'}$ and $Q : \mathbb{R}^d \rightarrow \mathbb{R}^{d'}$, is called (R, cR, p_1, p_2) -sensitive for $\text{Dist}(\cdot, \cdot)$ if, for any $\mathbf{o}, \mathbf{q} \in \mathbb{R}^d$, the hash function h drawn uniformly from $\mathcal{H}$ satisfies the following conditions:*

- $\text{Dist}(\mathbf{o}, \mathbf{q}) \leq R \Rightarrow \Pr[h(P(\mathbf{o})) = h(Q(\mathbf{q}))] \geq p_1$;
- $\text{Dist}(\mathbf{o}, \mathbf{q}) \geq cR \Rightarrow \Pr[h(P(\mathbf{o})) = h(Q(\mathbf{q}))] \leq p_2$;
- $p_1 > p_2$ and $c > 1$.

Note that P is only applied to the data objects, while Q is only applied to the hyperplane queries; when $Q(\mathbf{x}) = P(\mathbf{x}) = \mathbf{x}$, Definition 2.1 degenerates to the vanilla definition of LSH family.

LCCS-LSH. LCCS-LSH [22] is a state-of-the-art LSH scheme for NNS. To deal with the Euclidean distance, LCCS-LSH uses the popular random projection LSH family [11], whose LSH function is defined as follows:

$$h_{a,b}(\mathbf{o}) = \left\lfloor \frac{\langle \mathbf{a}, \mathbf{o} \rangle + b}{w} \right\rfloor, \quad (3)$$

where $\mathbf{a}$ is a random vector with each entry drawn i.i.d. from normal distribution, i.e., $a_i \sim \mathcal{N}(0, 1)$; w is a pre-specified bucket width; b is a random offset chosen uniformly at random from $[0, w)$.

Let $s = \|\mathbf{o} - \mathbf{q}\| = \sqrt{\sum_{i=1}^d (o_i - q_i)^2}$ be the Euclidean distance between any two $\mathbf{o} = (o_1, \dots, o_d)$ and $\mathbf{q} = (q_1, \dots, q_d)$. The collision probability $p(s)$ is computed as follows [11]:

$$\begin{aligned} p(s) &= \Pr[h_{a,b}(\mathbf{o}) = h_{a,b}(\mathbf{q})] \\ &= 1 - 2\Phi(-w/s) - \frac{2}{\sqrt{2\pi}(w/s)}(1 - e^{-(w/s)^2/2}), \end{aligned} \quad (4)$$

where $\Phi(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$ is the Cumulative Distribution Function (CDF) of $\mathcal{N}(0, 1)$.

Let $p_1 = p(1)$ and $p_2 = p(c)$, where $p(\cdot)$ is defined by Equation 4. We have Theorem 2.1 for LCCS-LSH for NNS as follows [22]:

Theorem 2.1: *Setting $t = O(n^\rho)$. Given an approximation ratio c ($c > 1$), LCCS-LSH is a data structure which can find an $\mathbf{o} \in \mathcal{D}$ such that $\|\mathbf{o} - \mathbf{q}\| \leq c \cdot \|\mathbf{o}^* - \mathbf{q}\|$ with probability at least $1/4$ in $O(n^{1+\rho})$ space, $O(n^{1+\rho}(d + \log n))$ indexing time, and $O(n^\rho(d + \log n))$ query time, where $\rho = \ln(1/p_1)/\ln(1/p_2)$ and $\mathbf{o}^* = \arg \min_{\mathbf{o} \in \mathcal{D}} \|\mathbf{o} - \mathbf{q}\|$.*

2.2 Reverse Locality-Sensitive Hashing

As the LSH family designed for NNS, there exists a concept of Reverse LSH (RLSH) family [14] designed for FNS, which is a set of hash functions with the *locality-sensitive* property that dissimilar

objects have a higher probability of being separated into different buckets than the similar ones. Given a hash function h , $\mathbf{o}$ and $\mathbf{q}$ are separated into different buckets if $h(\mathbf{o}) \neq h(\mathbf{q})$. Next, we define a modified RLSH family in a form which will be used later.

Definition 2.2 ((Asymmetric) RLSH Family): *Given a search radius R ($R > 0$) and an approximation ratio c , a family of hash functions $\mathcal{H}$, along with two vector transformations $P : \mathbb{R}^d \rightarrow \mathbb{R}^{d'}$ and $Q : \mathbb{R}^d \rightarrow \mathbb{R}^{d'}$, is called $(R, \frac{R}{c}, \xi_1, \xi_2)$ -sensitive for $\text{Dist}(\cdot, \cdot)$ if, for any $\mathbf{o}, \mathbf{q} \in \mathbb{R}^d$, the hash function h drawn uniformly from $\mathcal{H}$ satisfies the following conditions:*

- $\text{Dist}(\mathbf{o}, \mathbf{q}) \geq R \Rightarrow \Pr[h(P(\mathbf{o})) \neq h(Q(\mathbf{q}))] \geq \xi_1$;
- $\text{Dist}(\mathbf{o}, \mathbf{q}) \leq \frac{R}{c} \Rightarrow \Pr[h(P(\mathbf{o})) \neq h(Q(\mathbf{q}))] \leq \xi_2$;
- $\xi_1 > \xi_2$ and $c > 1$.

Similar to Definition 2.1, P is only applied to data objects, and Q is only applied to hyperplane queries; when $Q(\mathbf{x}) = P(\mathbf{x}) = \mathbf{x}$, Definition 2.2 degenerates to the vanilla definition of RLSH family.

RQALSH. RQALSH [14] is one of the most efficient hashing schemes for high-dimensional FNS [9, 10, 14, 18, 30, 31]. Its family is a typical RLSH family designed for Euclidean distance. Let $\mathbf{a}$ be a random vector with each entry drawn i.i.d. from normal distribution, i.e., $a_i \sim \mathcal{N}(0, 1)$. The RQALSH function is defined as follows [14]:

$$h_{\mathbf{a}}(\mathbf{o}) = \langle \mathbf{a}, \mathbf{o} \rangle. \quad (5)$$

Let w be a pre-specified bucket width. We say $\mathbf{o}$ and $\mathbf{q}$ are *separated* into different buckets under $h_{\mathbf{a}}$ if $|h_{\mathbf{a}}(\mathbf{o}) - h_{\mathbf{a}}(\mathbf{q})| > \frac{w}{2}$. Let $s = \|\mathbf{o} - \mathbf{q}\|$ be the Euclidean distance between any two $\mathbf{o}, \mathbf{q} \in \mathbb{R}^d$. The separation probability $\xi(s)$ is computed as follows [14]:

$$\xi(s) = \Pr[|h_{\mathbf{a}}(\mathbf{o}) - h_{\mathbf{a}}(\mathbf{q})| > \frac{w}{2}] = 2\Phi(-\frac{w}{2s}). \quad (6)$$

Let $\xi_1 = \xi(1)$ and $\xi_2 = \xi(\frac{1}{c})$, where $\xi(\cdot)$ is defined by Equation 6. We have Theorem 2.2 for RQALSH for FNS as follows [14]:

Theorem 2.2: *Let $m = O(\log n)$ and $l = O(\log n)$ ($l \leq m$). Given an approximation ratio c ($c > 1$), RQALSH is a data structure which can find an $\mathbf{o} \in \mathcal{D}$ such that $\|\mathbf{o} - \mathbf{q}\| \geq \|\mathbf{o}^* - \mathbf{q}\|/c$ with constant probability in $O(n \log n)$ space, $O((d + \log n)n \log n)$ indexing time, and $O((d + n) \log n)$ query time, where $\mathbf{o}^* = \arg \max_{\mathbf{o} \in \mathcal{D}} \|\mathbf{o} - \mathbf{q}\|$.*

3 HYPERPLANE HASHING SCHEMES

3.1 Overview

The problem of P2HNNS is difficult because the data is often not normalized and there exists an absolute value operator in Equation 2. Thus, we develop some transformations on data objects and the hyperplane queries to make this problem easier to deal with.

First of all, we extend the data objects with one dimension so that both data and queries have the same dimension. Then, we show there is no LSH and RLSH families for $d_{P2H}(\cdot, \cdot)$. Thus, we design an asymmetric transformation to convert the P2HNNS into the classic NNS (or FNS). The asymmetric transformation consists of two steps: object embedding and hypersphere transformation. The step of object embedding can remove the absolute value operator, while the step of hypersphere transformation can address the problem of data normalization. With different designs of the hypersphere transformation, we propose two schemes Nearest Hyperplane hashing (NH) and Furthest Hyperplane hashing (FH) for P2HNNS. An overview of NH and FH is depicted in Figure 1.

Figure 1: An overview of the hyperplane hashing schemes NH and FH

3.2 Object Extending

To simplify Equation 2, we first extend all $\mathbf{p} \in \mathcal{D}$ by one dimension with 1, i.e., $\mathbf{o} = (\mathbf{p}; 1) = (p_1, \dots, p_{d-1}, 1)$, where $(\cdot;)$ represents the concatenation of dimensions. For a certain $\mathbf{q}$, the term $\sqrt{\sum_{i=1}^{d-1} q_i^2}$ is fixed and is not equal to 0. For simplicity, we assume $\sqrt{\sum_{i=1}^{d-1} q_i^2} = 1$; otherwise, we can rescale $\mathbf{q}$ without changing the order of the P2HNNS results as defined by Equation 1. Then, we have:

$$d_{P2H}(\mathbf{p}, \mathbf{q}) = |\langle \mathbf{o}, \mathbf{q} \rangle| = \left| \sum_{i=1}^d o_i q_i \right|, \quad (7)$$

where $\langle \mathbf{o}, \mathbf{q} \rangle$ is the inner product of $\mathbf{o}$ and $\mathbf{q}$. Hereafter, we use $\mathbf{p} \in \mathbb{R}^{d-1}$ to represent the original data object, $\mathbf{o} \in \mathbb{R}^d$ to indicate the data object after extending, and $\mathcal{D}'$ to denote a database of extended data objects, i.e., $\mathcal{D}' = \{\mathbf{o} \mid \mathbf{o} = (\mathbf{p}; 1), \mathbf{p} \in \mathcal{D}\}$.

No LSH and RLSH Families. By converting $d_{P2H}(\mathbf{p}, \mathbf{q})$ into $|\langle \mathbf{o}, \mathbf{q} \rangle|$, we show it is impossible to construct LSH or RLSH family for d_{P2H} .

Theorem 3.1: *There is no LSH and RLSH families for $d_{P2H}(\cdot, \cdot)$.*

PROOF. We will prove by contradiction. Let $\text{Dist}(\mathbf{o}, \mathbf{q}) = |\langle \mathbf{o}, \mathbf{q} \rangle|$. For any object $\mathbf{o}$ to itself, $\text{Dist}(\mathbf{o}, \mathbf{o}) = |\langle \mathbf{o}, \mathbf{o} \rangle| = \|\mathbf{o}\|^2$.

Suppose there exists such an (R, cR, p_1, p_2) -sensitive LSH family $\mathcal{H}$ for $d_{P2H}(\cdot, \cdot)$. We can always construct a data object $\mathbf{o}$ such that $0 \leq \text{Dist}(\mathbf{o}, \mathbf{o}) = \|\mathbf{o}\|^2 \leq R$. Then, we can generate a query $\mathbf{q} = \frac{2cR\mathbf{o}}{\|\mathbf{o}\|^2}$ such that $\text{Dist}(\mathbf{o}, \mathbf{q}) = |\langle \mathbf{o}, \mathbf{q} \rangle| = 2cR \geq cR$ for any $R > 0$ and $c > 1$. For any h drawn uniformly from $\mathcal{H}$, the collision probability between $\mathbf{o}$ and itself is always 1. Furthermore, according to the definition of LSH family, i.e., Definition 2.1 with the setting $Q(\mathbf{x}) = P(\mathbf{x}) = \mathbf{x}$, the event " $h(\mathbf{o}) = h(\mathbf{q})$ " should have a higher probability than the event " $h(\mathbf{o}) = h(\mathbf{o})$ ", i.e., $\Pr[h(\mathbf{o}) = h(\mathbf{q})] > \Pr[h(\mathbf{o}) = h(\mathbf{o})] = 1$. Since any probability cannot be larger than 1, there is no LSH family for $d_{P2H}(\cdot, \cdot)$.

Similarly, we assume there exists an $(R, \frac{R}{c}, \xi_1, \xi_2)$ -RLSH family $\mathcal{H}$ for $d_{P2H}(\cdot, \cdot)$. We can always construct a data object $\mathbf{o}$ such that $\text{Dist}(\mathbf{o}, \mathbf{o}) = \|\mathbf{o}\|^2 \geq R$. Then, we can generate a query $\mathbf{q} = \frac{R\mathbf{o}}{2c\|\mathbf{o}\|^2}$ such that $\text{Dist}(\mathbf{o}, \mathbf{q}) = |\langle \mathbf{o}, \mathbf{q} \rangle| = \frac{R}{2c} \leq \frac{R}{c}$ for any $R > 0$ and $c > 1$. For any h drawn uniformly from $\mathcal{H}$, the separation probability between $\mathbf{o}$ and itself is always 0. Furthermore, according to the definition of RLSH family, i.e., Definition 2.2 with the setting $Q(\mathbf{x}) = P(\mathbf{x}) = \mathbf{x}$, $\Pr[h(\mathbf{o}) \neq h(\mathbf{q})] < \Pr[h(\mathbf{o}) \neq h(\mathbf{o})] = 0$. Since any probability cannot be smaller than 0, there is no RLSH family for $d_{P2H}(\cdot, \cdot)$. Therefore, Theorem 3.1 is proved. $\square$

Since there is no LSH and RLSH families for $d_{P2H}(\cdot, \cdot)$, we follow pioneering works for MIPS and relax the definition of LSH (and RLSH) family by allowing asymmetric transformations (i.e., P and Q) on data objects and queries. With the use of P and Q , the proof of Theorem 3.1 is no longer valid. Thus, there may exist asymmetric LSH and asymmetric RLSH families for $d_{P2H}(\cdot, \cdot)$.

3.3 Asymmetric Transformation

We now introduce an asymmetric transformation which converts the P2HNNS over $\mathbb{R}^d$ into the classic NNS over $\mathbb{R}^{d(d+1)/2+1}$. It has two steps: object embedding and hypersphere transformation.

Object Embedding. We first design two vector transformations $\mathbf{f} : \mathbb{R}^d \rightarrow \mathbb{R}^{d(d+1)/2}$ and $\mathbf{g} : \mathbb{R}^d \rightarrow \mathbb{R}^{d(d+1)/2}$ to embed data objects and queries, respectively. Let $\mathbf{o}'$ and $\mathbf{q}'$ be the data object and query after object embedding. $\mathbf{f}$ and $\mathbf{g}$ are defined as follows:

$$\mathbf{o}' = \mathbf{f}(\mathbf{o}) = (o_1^2, \dots, o_d^2, o_1 o_2, o_1 o_3, \dots, o_{d-1} o_d), \quad (8)$$

$$\mathbf{q}' = \mathbf{g}(\mathbf{q}) = (q_1^2, \dots, q_d^2, 2q_1 q_2, 2q_1 q_3, \dots, 2q_{d-1} q_d). \quad (9)$$

Based on Equations 8 and 9, $\langle \mathbf{o}', \mathbf{q}' \rangle = |\langle \mathbf{o}, \mathbf{q} \rangle|^2$. Thus, with this step, we can remove the absolute value operator appeared in Equation 7.

Hypersphere Transformation. After object embedding, we develop two vector transformations $P : \mathbb{R}^{d(d+1)/2} \rightarrow \mathbb{R}^{d(d+1)/2+1}$ and $Q : \mathbb{R}^{d(d+1)/2} \rightarrow \mathbb{R}^{d(d+1)/2+1}$ for $\mathbf{o}'$ and $\mathbf{q}'$, respectively. Let $M = \max \|\mathbf{o}'\|$. P and Q are defined as follows:

$$P(\mathbf{o}') = P \circ \mathbf{f}(\mathbf{o}) = (\mathbf{o}'; \sqrt{M^2 - \|\mathbf{o}'\|^2}), \quad (10)$$

$$Q(\mathbf{q}') = Q \circ \mathbf{g}(\mathbf{q}) = (-\frac{M}{\|\mathbf{q}'\|} \mathbf{q}'; 0). \quad (11)$$

As $\|P(\mathbf{o}')\| = \|Q(\mathbf{q}')\| = M$, this step maps all $\mathbf{o}'$ and $\mathbf{q}'$ to the hypersphere over $\mathbb{R}^{d(d+1)/2}$ with the same radius M . Thus, this step addresses the data normalization problem, and we call it *hypersphere transformation*. According to Equations 8, 9, 10 and 11,

$$\|P \circ \mathbf{f}(\mathbf{o}) - Q \circ \mathbf{g}(\mathbf{q})\| = \sqrt{2M^2 + 2\langle \mathbf{o}, \mathbf{q} \rangle^2}. \quad (12)$$

Since M is a fixed constant, according to Equations 7 and 12,

$$\arg \min_{\mathbf{o} \in \mathcal{D}'} \|P \circ \mathbf{f}(\mathbf{o}) - Q \circ \mathbf{g}(\mathbf{q})\| \Leftrightarrow \arg \min_{\mathbf{p} \in \mathcal{D}} d_{P2H}(\mathbf{p}, \mathbf{q}).$$

One can develop different formulas of $\mathbf{f}$ and $\mathbf{g}$ in object embedding to complete this transformation. For example, following EH [20], Equation 12 is also satisfied with the formula of $\mathbf{f}(\mathbf{o}) = \mathbf{g}(\mathbf{q}) = \text{vec}(\mathbf{o}\mathbf{o}^T)$, where $\text{vec}(\cdot)$ vectorizes the rank-1 matrix $\mathbf{o}\mathbf{o}^T$ row by row. However, this design will increase the data dimension from d

to d^2 , while our proposed f and g increase it from d to $d(d+1)/2$ only, which can save nearly half of computations.

Randomized Sampling. Our proposed f and g , however, still increase the data dimension from d to $O(d^2)$. As will be introduced in Sections 3.4 and 3.5, the hash functions we use contain the inner product $\langle \cdot, \cdot \rangle$, where the computational cost is linear to d . Thus, the computation of the hash functions for $P \circ f(o)$ and $Q \circ g(q)$ may be very expensive, especially when d is very large. To alleviate this problem, we follow [20] and use a form of randomized sampling to approximate our proposed asymmetric transformation. This approximation relies on Lemma 3.1 as follows:

Lemma 3.1: Let $\mathbf{o} = (o_1, \dots, o_d) \in \mathbb{R}^d$ and define $p_i = o_i^2 / \|\mathbf{o}\|^2$. Construct $\tilde{\mathbf{o}} \in \mathbb{R}^d$ such that the i^{th} element is o_i with probability p_i and is 0 otherwise. Select λ such elements using sampling with replacement. Then, for any $\mathbf{a} \in \mathbb{R}^d$, $\epsilon > 0$, $\tau \geq 1$, $\lambda \geq \tau/\epsilon^2$,

$$\Pr[|\langle \mathbf{a}, \tilde{\mathbf{o}} \rangle - \langle \mathbf{a}, \mathbf{o} \rangle| \leq \epsilon \|\mathbf{a}\|^2 \|\mathbf{o}\|^2] > 1 - \frac{1}{\tau}.$$

The proof can be referred to the supplementary of [20]. According to Lemma 3.1, when computing the hash value of $P \circ f(o)$ (or $Q \circ g(q)$), we can first randomly sample the index i of $P \circ f(o)$ (or $Q \circ g(q)$) with probability p_i and then compute the hash value through such non-zero indices. This approximation may introduce a small additive error to the hash value, but the computing time can be reduced from $O(d^2)$ to $O(1/\epsilon^2)$ for any $\epsilon > 0$.

Until now, we have presented the asymmetric transformation and discussed its approximation with randomized sampling. Next, we will introduce the details of NH and FH for P2HNNS.

3.4 Nearest Hyperplane Hashing

NH Hash Family. Let $h_{a,b}(\cdot)$ be a random projection LSH function as defined by Equation 3. Given the asymmetric transformation $P \circ f(o)$ and $Q \circ g(q)$, the NH hash family $\mathcal{H}_N$ of hash functions $h_N(\cdot)$ is defined as follows:

$$h_N(x) = \begin{cases} h_{a,b}(P \circ f(x)), & \text{if } x \text{ is a data object } \mathbf{o}, \\ h_{a,b}(Q \circ g(x)), & \text{if } x \text{ is a hyperplane query } \mathbf{q}. \end{cases} \quad (13)$$

According to Equations 7 and 12, the collision probability of $h_N(\cdot)$ for certain $\mathbf{o}, \mathbf{q}$ is computed as follows:

$$\Pr[h_N(\mathbf{o}) = h_N(\mathbf{q})] = p(\sqrt{2M^2 + 2\langle \mathbf{o}, \mathbf{q} \rangle^2}), \quad (14)$$

where $p(\cdot)$ is defined by Equation 4. Let $p_1 = p(\sqrt{2M^2 + 2R^2})$ and $p_2 = p(\sqrt{2M^2 + 2c^2R^2})$. With $P \circ f(o)$ and $Q \circ g(q)$, we show $\mathcal{H}_N$ is an asymmetric LSH family for $d_{P2H}(\cdot, \cdot)$.

Lemma 3.2: Given a search radius R ($R > 0$), an approximation ratio c ($c > 1$), and a $(\sqrt{2M^2 + 2R^2}, \sqrt{2M^2 + 2c^2R^2}, p_1, p_2)$ -sensitive hash family $\mathcal{H}$ for Euclidean distance for NNS, the NH hash family $\mathcal{H}_N$ of hash functions $h_N(\cdot)$ is (R, cR, p_1, p_2) -sensitive for $d_{P2H}(\cdot, \cdot)$.

PROOF. According to Equations 7 and 12, we have:

$$d_{P2H}(\mathbf{p}, \mathbf{q}) \leq R \Leftrightarrow \|P \circ f(\mathbf{o}) - Q \circ g(\mathbf{q})\| \leq \sqrt{2M^2 + 2R^2},$$

$$d_{P2H}(\mathbf{p}, \mathbf{q}) \geq cR \Leftrightarrow \|P \circ f(\mathbf{o}) - Q \circ g(\mathbf{q})\| \geq \sqrt{2M^2 + 2c^2R^2}.$$

Recall that $p_1 = p(\sqrt{2M^2 + 2R^2})$ and $p_2 = p(\sqrt{2M^2 + 2c^2R^2})$. Since $\mathcal{H}$ is a $(\sqrt{2M^2 + 2R^2}, \sqrt{2M^2 + 2c^2R^2}, p_1, p_2)$ -sensitive hash

Algorithm 1: Indexing of NH (Pipeline Computation)

Input: $\mathcal{D} = \{\mathbf{p}_1, \dots, \mathbf{p}_n\}$ and t i.i.d. random projection LSH functions $\{h_{a_1, b_1}, \dots, h_{a_t, b_t}\}$.

Output: None.

```

1  $M = 0;$ 
2 for  $i = 1$  to  $n$  do
3    $\mathbf{o}_i = (\mathbf{p}_i; 1); \mathbf{o}'_i = f(\mathbf{o}_i);$ 
4   if  $\|\mathbf{o}'_i\| > M$  then  $M = \|\mathbf{o}'_i\|;$ 
5   for  $j = 1$  to  $t$  do
6      $h_{N_j}(\mathbf{o}_i) = \langle \mathbf{a}_j, \mathbf{o}'_i \rangle;$ 
7 for  $i = 1$  to  $n$  do
8    $P(\mathbf{o}'_i)_{\text{last}} = \sqrt{M^2 - \|\mathbf{o}'_i\|^2};$ 
9   for  $j = 1$  to  $t$  do
10     $h_{N_j}(\mathbf{o}_i) = \left\lfloor \frac{h_{N_j}(\mathbf{o}_i) + a_{j, \text{last}} \cdot P(\mathbf{o}'_i)_{\text{last}} + b}{w} \right\rfloor;$ 
11    $H(\mathbf{o}_i) = [h_{N_1}(\mathbf{o}_i), \dots, h_{N_t}(\mathbf{o}_i)];$ 
12 Build an index  $\mathcal{T}$  for all  $H(\mathbf{o}_i)$ 's using LCCS-LSH [22];
```

family for Euclidean distance for NNS, based on Expression 14 and Definition 2.1, we infer that

$$d_{P2H}(\mathbf{p}, \mathbf{q}) \leq R \Rightarrow \Pr[h_N(\mathbf{o}) = h_N(\mathbf{q})] \geq p_1,$$

$$d_{P2H}(\mathbf{p}, \mathbf{q}) \geq cR \Rightarrow \Pr[h_N(\mathbf{o}) = h_N(\mathbf{q})] \leq p_2.$$

Since $R > 0$ and $c > 1$, $p_1 > p_2$. Thus, the NH hash family $\mathcal{H}_N$ of hash functions $h_N(\cdot)$ is (R, cR, p_1, p_2) -sensitive for $d_{P2H}(\cdot, \cdot)$. $\square$

NH. With the NH hash family $\mathcal{H}_N$, we describe NH for P2HNNS. NH consists of two phases: indexing phase and query phase.

Indexing Phase. We propose a *pipeline computation* to build index. Specifically, once we extend each $\mathbf{p}$ to $\mathbf{o}$ and embed $\mathbf{o}$ to $f(\mathbf{o})$, we immediately compute the intermediate value of $h_N(\mathbf{o})$ for $f(\mathbf{o})$. According to Equation 10, we require a global M to compute $P(\mathbf{o}')$. However, since $h_N(\mathbf{o})$ contains the inner product $\langle \mathbf{a}, P \circ f(\mathbf{o}) \rangle$ which is a linear operation of cumulative addition, we can postpone the addition of the last element $\sqrt{M^2 - \|\mathbf{o}'\|^2}$ of $P(\mathbf{o}')$ until we determine the M value. We repeat this procedure t times to get t $h_{N_j}(\mathbf{o})$'s. Let $H(\mathbf{o}) = [h_{N_1}(\mathbf{o}), \dots, h_{N_t}(\mathbf{o})]$ be the hash string of $\mathbf{o}$. Finally, NH adopts LCCS-LSH to build an index $\mathcal{T}$ for all $H(\mathbf{o})$'s. The pseudo-code is depicted in Algorithm 1.

This pipeline computation can avoid $O(nd^2)$ space to store the intermediate values $P \circ f(\mathbf{o})$ for all $\mathbf{o}$'s. Moreover, with the linear property of $\langle \cdot, \cdot \rangle$, it will not introduce extra error for $h_N(\mathbf{o})$.

Query Phase. Given a P2HNNS query $\mathbf{q}$, NH first computes its hash string $H(\mathbf{q}) = [h_{N_1}(\mathbf{q}), \dots, h_{N_t}(\mathbf{q})]$. Then, it calls LCCS-LSH for NNS to identify a set of candidates C with $\mathcal{T}$. Finally, NH determines the nearest point $\mathbf{p}_{\min}$ from C as the final answer. The pseudo-code of NH for the P2HNNS is shown in Algorithm 2.

Example 3.1: We now use Figure 1 again to illustrate how to conduct the P2HNNS with NH. Suppose $d = 3$, $\mathcal{D} = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$, and there exists a hyperplane query $\mathbf{q} = (1, 1, -1)$.

First of all, we extend each $\mathbf{p}_i$ by one more dimension with 1, e.g., $\mathbf{o}_1 = (\mathbf{p}_1; 1) = (1, 2, 1)$, so that all $\mathbf{o}_i$'s and $\mathbf{q}$ are in the same

Algorithm 2: P2HNNS of NH

Input: a query q , an index $\mathcal{T}$, and t i.i.d. random projection LSH functions $\{h_{a_1, b_1}, \dots, h_{a_t, b_t}\}$;

Output: the nearest point p_{min} ;

```

1 for  $i = 1$  to  $t$  do
2    $h_{N_i}(q) = h_{a_i, b_i}(Q \circ g(q))$ ;
3  $H(q) = [h_{N_1}(q), \dots, h_{N_t}(q)]$ ;
4  $C = \text{LCCS-LSH}(H(q))$  on  $\mathcal{T}$ ;
5 return  $p_{min} = \arg \min_{p \in C} d_{P2H}(p, q)$ ;
```

dimension. Then, we use f and g to embed all o_i and q from $\mathbb{R}^3$ to $\mathbb{R}^6$, e.g., according to Equations 8 and 9, $o'_1 = f(o_1) = (1, 4, 1, 2, 1, 2)$, $q' = g(q) = (1, 1, 1, 2, -2, -2)$, and $\langle o'_1, q' \rangle = |\langle o_1, q \rangle|^2 = 4$. After the step of object embedding, we get $M = \max \|o'_i\| = \sqrt{57}$. Next, we use P and Q to convert all o'_i and q' from $\mathbb{R}^6$ to $\mathbb{R}^7$. For example, according to Equations 10 and 11, $o''_1 = P(o'_1) = (1, 4, 1, 2, 1, 2, \sqrt{30})$, $q''_{NH} = Q(q') = -\sqrt{\frac{57}{15}}(1, 1, 1, 2, -2, -2, 0)$, and we have $\|o''_1\| = \|q''_{NH}\| = \sqrt{57}$. With this asymmetric transformation, the P2HNNS over $\mathbb{R}^3$ are converted into the NNS over $\mathbb{R}^7$. Finally, NH leverages LCCS-LSH to find the nearest neighbor of q''_{NH} . Let $t = 3$. We can see that o''_1 collides with q''_{NH} in 3 times, which is higher than o''_2 and o''_3 . Since o''_1 has the longest common hash strings with q''_{NH} , NH returns p_1 as the answer. $\triangle$

Based on Theorem 2.1 and Lemma 3.2, we show NH can deal with the P2HNNS in *sub-quadratic* space and *sub-linear* query time.

Theorem 3.2: Given a family $\mathcal{H}_N$ of hash functions $h_N(\cdot)$ as defined by Equation 13, setting $t = O(n^\rho)$, NH is a data structure which can find a $p \in \mathcal{D}$ such that $d_{P2H}(p, q) \leq c \cdot d_{P2H}(p^*, q)$ with constant probability in $O(n^{1+\rho})$ space, $O(n^{1+\rho}(d^2 + \log n))$ indexing time, and $O(n^\rho(d^2 + \log n))$ query time, where $\rho = \ln(1/p_1)/\ln(1/p_2)$ and $p^* = \arg \min_{p \in \mathcal{D}} d_{P2H}(p, q)$.

According to Theorem 3.2, NH has an accuracy guarantee on c for P2HNNS, but for the NNS, the accuracy guarantee c' of LCCS-LSH is not equal to c . According to Lemma 3.2, we require a $(\sqrt{2M^2 + 2R^2}, \sqrt{2M^2 + 2c^2R^2}, p_1, p_2)$ -sensitive hash family for NNS. Thus, the accuracy guarantee of LCCS-LSH is $c' = \sqrt{2M^2 + 2c^2R^2}/\sqrt{2M^2 + 2R^2}$. Moreover, from Theorem 3.2, $P \circ f(o)$ and $Q \circ g(q)$ will increase the computational cost of $h_N(\cdot)$ from d to $O(d^2)$, leading to large indexing and query time complexities of NH. In practice, we will adopt the randomized sampling as discussed in Section 3.3 for NH to reduce this computational cost.

3.5 Furthest Hyperplane Hashing

According to Equation 14, there exists a large constant $2M^2$ when we compute the collision probability for the NH hash family. It will diminish the difference between p_1 and p_2 , and hence the close data objects might be hard to be identified from the far-apart ones. Similar to EH, BH, and MH, when $d_{P2H}(p, q) \rightarrow 0$, $\lim_{d_{P2H} \rightarrow 0} p_1 = \lim_{d_{P2H} \rightarrow 0} p_2 = p(\sqrt{2}M)$, the ρ value of NH will approach to 1, i.e., $\rho = \lim_{d_{P2H} \rightarrow 0} \ln(1/p_1)/\ln(1/p_2) = 1$. Thus, we consider to convert the problem of P2HNNS into another classic problem of point-to-point FNS and present FH accordingly.

New Variant of Asymmetric Transformation. The new variant of asymmetric transformation follows the same step of object embedding f and g , but we define a new hypersphere transformation of Q' for q' :

$$Q'(q') = Q' \circ g(q) = (\frac{M}{\|q'\|} q'; 0). \quad (15)$$

According to Equations 8, 9, 10 and 15, we have:

$$\|P \circ f(o) - Q' \circ g(q)\| = \sqrt{2M^2 - 2\langle o, q \rangle^2}. \quad (16)$$

According to Equations 7 and 16, we have:

$$\arg \max_{o \in \mathcal{D}} \|P \circ f(o) - Q' \circ g(q)\| \Leftrightarrow \arg \min_{p \in \mathcal{D}} d_{P2H}(p, q).$$

Thus, with $P \circ f(o)$ and $Q' \circ g(q)$, the P2HNNS over $\mathbb{R}^d$ can be reduced to the classic FNS over $\mathbb{R}^{d(d+1)/2+1}$. As will be shown in Equation 18, by making $(2M^2 - 2\langle o, q \rangle^2)$, this new asymmetric transformation can *amplify* the difference between ξ_1 and ξ_2 .

FH Hash Family. Suppose $h_a(\cdot)$ is an RQALSH function as defined by Equation 5. Given this new variant of asymmetric transformation, we define an FH hash family $\mathcal{H}_F$ of hash functions $h_F(\cdot)$ as follows:

$$h_F(x) = \begin{cases} h_a(P \circ f(x)), & \text{if } x \text{ is a data object } o, \\ h_a(Q' \circ g(x)), & \text{if } x \text{ is a hyperplane query } q. \end{cases} \quad (17)$$

According to Equations 7 and 16, the separation probability of $h_F(\cdot)$ for certain o, q is computed as follows:

$$\Pr[|h_F(o) - h_F(q)| > \frac{w}{2}] = \xi(\sqrt{2M^2 - 2\langle o, q \rangle^2}), \quad (18)$$

where $\xi(\cdot)$ is defined by Equation 6. Let $\xi_1 = \xi(\sqrt{2M^2 - 2R^2/c^2})$ and $\xi_2 = \xi(\sqrt{2M^2 - 2R^2})$. With this new variant of asymmetric transformation, we show $\mathcal{H}_F$ is an asymmetric RLSH family for $d_{P2H}(\cdot, \cdot)$.

Lemma 3.3: Given a search radius R ($R > 0$), an approximation ratio c ($c > 1$), and a $(\sqrt{2M^2 - 2R^2/c^2}, \sqrt{2M^2 - 2R^2}, \xi_1, \xi_2)$ -sensitive hash family for Euclidean distance for FNS, the FH hash family $\mathcal{H}_F$ of hash functions $h_F(\cdot)$ is $(\frac{R}{c}, R, \xi_1, \xi_2)$ -sensitive for $d_{P2H}(\cdot, \cdot)$.

PROOF. The proof of Lemma 3.3 is similar to that of Lemma 3.2. To be concise, we omit the details here. $\square$

Trivial FH. With the FH hash family $\mathcal{H}_F$, we first introduce a trivial version of FH for P2HNNS. In the indexing phase, the trivial FH first generates m i.i.d. RQALSH functions as defined by Equation 6. Then, it extends each p to o and follows the pipeline computation of NH to calculate $h_{F_1}(o), \dots, h_{F_m}(o)$ for each o . Finally, it uses RQALSH to build index with the hash values of n data objects. In the query phase, after computing $h_{F_1}(q), \dots, h_{F_m}(q)$, the trivial FH uses RQALSH to identify a set of candidates C for the FNS of $Q' \circ g(q)$ whose separation countings are more than l times. It returns $p_{min} = \arg \min_{p \in C} d_{P2H}(p, q)$ as the final answer.

We choose RQALSH for FNS based on two considerations: (1) it is a competitive method that has been shown to be superior to other state-of-the-art hashing schemes, such as QDAFN [30] and Drusilla-Select [10]; (2) it is independent of d and enjoys quality guarantee on query results, which helps us establish quality guarantee for FH. Based on Theorem 2.2 and Lemma 3.3, we show that the trivial FH enjoys a theoretical guarantee for P2HNNS.

Theorem 3.3: Given a family $\mathcal{H}_{\mathcal{F}}$ of hash functions $h_{\mathcal{F}}(\cdot)$ as defined by Equation 17, setting $m = O(\log n)$ and $l = O(\log n)$ ($l \leq m$), the trivial FH is a data structure which can find a $\mathbf{p} \in \mathcal{D}$ such that $d_{P2H}(\mathbf{p}, \mathbf{q}) \leq c \cdot d_{P2H}(\mathbf{p}^*, \mathbf{q})$ with constant probability in $O(n \log n)$ space, $O((d^2 + \log n)n \log n)$ indexing time, and $O((d^2 + n) \log n)$ query time, where $\mathbf{p}^* = \arg \min_{\mathbf{p} \in \mathcal{D}} d_{P2H}(\mathbf{p}, \mathbf{q})$.

Similar to Theorem 3.2, we can find that the trivial FH has an accuracy guarantee on c for P2HNNS, but for the FNS, the accuracy guarantee c'' of RQALSH is not equal to c . According to Lemma 3.3, we need a $(\sqrt{2M^2 - 2R^2}/c^2, \sqrt{2M^2 - 2R^2}, \xi_1, \xi_2)$ -sensitive hash family for FNS. Thus, the accuracy guarantee of RQALSH is $c'' = \sqrt{2M^2 - 2R^2}/c^2 / \sqrt{2M^2 - 2R^2}$.

Remarks. Compared with NH, the trivial FH enjoys less space and indexing time complexities, but its query time complexity is also much worse than that of NH. However, it is analysed in the worst case, in the sense that RQALSH conducts separation counting for *all* n data objects in *all* m hash tables. For the real-life datasets in high-dimensional spaces, data points are sparse and often lie on a low-dimensional manifold. As such, most data points do not need to be checked for all m hash tables. The query time of the trivial FH in practice may be better than the theoretical bound.

However, once $\mathbf{o}$ satisfies $|h_{\mathcal{F}}(\mathbf{o}) - h_{\mathcal{F}}(\mathbf{q})| > \frac{\sigma}{2}$, RQALSH must conduct a separation counting for $\mathbf{o}$, no matter whether $d_{P2H}(\mathbf{o}, \mathbf{q})$ is small or large. Among the m hash tables, RQALSH may require checking all n data objects with at least once. Thus, in average case, the trivial FH still requires $O(n + d^2 \log n)$ query time for P2HNNS.

Figure 2: An example of the data-dependent multi-partition strategy

FH with a Data-Dependent Multi-Partition Strategy. As n increases, the performance of the trivial FH may degrade rapidly, because it requires conducting pointless separation counting for more and more false positives. With the new $P \circ f(\mathbf{o})$ and $Q' \circ g(\mathbf{q})$, we now introduce a data-dependent multi-partition strategy to speed up the trivial FH. Referring to Figure 2, let $\overline{P \circ f(\mathbf{o})}$ be the centroid of all data objects after $P \circ f(\cdot)$, i.e., $\overline{P \circ f(\mathbf{o})} = \frac{1}{n} \sum_{i=1}^n P \circ f(\mathbf{o}_i)$. No matter for $Q' \circ g(\mathbf{q}_1)$ which is close to $\overline{P \circ f(\mathbf{o})}$ or for $Q' \circ g(\mathbf{q}_2)$ which is far from $\overline{P \circ f(\mathbf{o})}$, the data objects further from $\overline{P \circ f(\mathbf{o})}$ are more likely to be the furthest neighbors of $Q' \circ g(\mathbf{q})$.

Indexing Phase. Based on the above observation, in the indexing phase, we first extend each $\mathbf{p}_i$ to $\mathbf{o}_i$. Then, we compute the centroid $\overline{P \circ f(\mathbf{o})}$ of n data objects after $P \circ f(\cdot)$ and sort them in descending order of $\|P \circ f(\mathbf{o}_i) - \overline{P \circ f(\mathbf{o})}\|$. Let σ be the interval ratio ($0 < \sigma < 1$). We follow that order and partition each $\mathbf{o}_i$ into T disjoint subsets $\{S_1, \dots, S_T\}$ such that $\sigma M_j < \|P \circ f(\mathbf{o}_i) - \overline{P \circ f(\mathbf{o})}\| \leq M_j$

Algorithm 3: Indexing of FH

Input: $\mathcal{D} = \{\mathbf{p}_1, \dots, \mathbf{p}_n\}$, an interval ratio σ , and the number of hash tables m .
Output: $T, \{S_1, \dots, S_T\}$, and $\{M_1, \dots, M_T\}$.
1 Compute $\mathbf{o}_i = (\mathbf{p}_i; 1)$ for each $\mathbf{p}_i \in \mathcal{D}$;
2 $\overline{P \circ f(\mathbf{o})} = \frac{1}{n} \sum_{i=1}^n P \circ f(\mathbf{o}_i)$;
3 Sort $\mathbf{o}_i$ in descending order of $\|P \circ f(\mathbf{o}_i) - \overline{P \circ f(\mathbf{o})}\|$;
4 $i = n; j = 0$;
5 **while** $i \geq 1$ **do**
6 $j = j + 1; S_j = \emptyset$;
7 $M_j = \|P \circ f(\mathbf{o}_i) - \overline{P \circ f(\mathbf{o})}\|$;
8 **while** $i \geq 1 \wedge \|P \circ f(\mathbf{o}_i) - \overline{P \circ f(\mathbf{o})}\| > \sigma M_j$ **do**
9 $S_j = S_j \cup \{\mathbf{o}_i\}; i = i - 1$;
10 Build m hash tables for S_j using RQALSH [14];
11 $T = j$;
12 **return** $T, \{S_1, \dots, S_T\}$, and $\{M_1, \dots, M_T\}$;

Algorithm 4: P2HNNS of FH

Input: a query $\mathbf{q}$, $T, \{S_1, \dots, S_T\}, \{M_1, \dots, M_T\}$, and m hash tables for each $S_j, j \in \{1, \dots, T\}$.
Output: the nearest point $\mathbf{p}_{min}$.
1 $C = \emptyset; \varphi = M_1$;
2 **for** $j = 1$ **to** T **do**
3 $Q' \circ g(\mathbf{q}) = (\frac{M_j}{\|Q' \circ g(\mathbf{q})\|} Q' \circ g(\mathbf{q}); 0); lb = \sqrt{2M_j^2 - 2\varphi^2}$;
4 $C_j = \text{RQALSH}(Q' \circ g(\mathbf{q}), lb)$ on S_j with m hash tables;
5 $C = C \cup C_j$;
6 $\varphi = \min_{\mathbf{p} \in C} d_{P2H}(\mathbf{p}, \mathbf{q})$;
7 **return** $\mathbf{p}_{min} = \arg \min_{\mathbf{p} \in C} d_{P2H}(\mathbf{p}, \mathbf{q})$;

for $\mathbf{o}_i \in S_j$, where $M_j = \max_{\mathbf{o}_i \in S_j} \|P \circ f(\mathbf{o}_i) - \overline{P \circ f(\mathbf{o})}\|$ and $j \in \{1, \dots, T\}$. Finally, we follow the trivial FH and use RQALSH to build m hash tables individually for each S_j . The pseudo-code of the indexing of FH is depicted in Algorithm 3.

Note that $M_1, \dots, M_T$ are sorted in descending order. The number T of disjoint subsets is *automatically* determined by the interval ratio σ and the data distribution after $P \circ f(\cdot)$. Thus, we call this strategy as a data-dependent multi-partition strategy.

Query Phase. Given a P2HNNS query $\mathbf{q}$, FH uses RQALSH to identify a set of candidates C for the FNS of $Q' \circ g(\mathbf{q})$ from S_1 to S_T . Let φ be the smallest $d_{P2H}(\mathbf{p}, \mathbf{q})$ we find so far, i.e., $\varphi = \min_{\mathbf{p} \in C} d_{P2H}(\mathbf{p}, \mathbf{q})$. With M_j and φ , we compute the lower bound lb for the FNS, i.e., $lb = \sqrt{2M_j^2 - 2\varphi^2}$. When using RQALSH for FNS on S_j , we can stop early if $\|P \circ f(\mathbf{o}_i) - Q' \circ g(\mathbf{q})\| \leq lb$. Finally, FH determines $\mathbf{p}_{min} = \arg \min_{\mathbf{p} \in C} d_{P2H}(\mathbf{p}, \mathbf{q})$ as the final answer. The pseudo-code of FH for P2HNNS is shown in Algorithm 4.

As we first search the data objects far-away from $\overline{P \circ f(\mathbf{o})}$, which probably contains the furthest neighbors of $Q' \circ g(\mathbf{q})$, the search process might be stopped in the first few rounds of S_j , so we can avoid most unnecessary separation counting for the false positives. Therefore, FH can be much faster than the trivial FH.

Remarks. The effectiveness of this data-dependent multi-partition strategy, however, is highly sensitive to the data distribution. If most data objects after $P \circ f(\cdot)$ are close to $P \circ f(\bar{o})$, i.e., $\|P \circ f(o_i) - P \circ f(\bar{o})\| \ll M_1$, then our strategy can reduce separation counting for a large number of false positives and boost the search efficiency of FH significantly. Nevertheless, if most data objects after $P \circ f(\cdot)$ are far from $P \circ f(\bar{o})$, i.e., $\|P \circ f(o_i) - P \circ f(\bar{o})\| \gg 0$, our strategy is not effective because RQALSH requires conducting separation counting for many false positives. In the worst case, all of the data objects fall in S_1 , i.e., $\sigma M_1 < \|P \circ f(o_i) - P \circ f(\bar{o})\| \leq M_1$ for all o_i 's. Then, FH will decline to its trivial version.

Based on Theorems 2.2 and 3.3, we have Corollary 3.1 as follows:

Corollary 3.1: *FH with the data-dependent multi-partition strategy is a data structure which can find a $p \in \mathcal{D}$ such that $dp_{2H}(p, q) \leq c \cdot dp_{2H}(p^*, q)$ with constant probability in $O(n \log n)$ space, $O((d^2 + \log n)n \log n)$ indexing time, and $O((d^2 + n) \log n)$ query time, where $p^* = \arg \min_{p \in \mathcal{D}} dp_{2H}(p, q)$.*

Similar to NH, the asymmetric transformation will increase the computational cost of FH hash functions from d to $O(d^2)$, leading to large indexing and query time complexities. Thus, we will also adopt the randomized sampling for FH to alleviate this problem.

4 EXPERIMENTS

We study the performance of NH and FH over five real-life datasets for high-dimensional P2HNNs. All methods are implemented in C++ and compiled with g++ 8.3 using -O3 optimization. We conduct all experiments in a single thread on a machine with 8 Intel i7-3820 @ 3.60GHz CPUs and 64 GB RAM, running on Ubuntu 18.04.

4.1 Datasets and Queries

We use five real-life datasets Yelp,¹ Music-100,² GloVe,³ Tiny-1M [40], and Msong⁴ in our experiments, which cover a wide range of data types, including rating, text, image, and audio data. We randomly generate 100 hyperplane queries for each dataset. The details of the hyperplane query generation can be referred to Appendix A. The statistics of datasets and queries are summarized in Table 1.

4.2 Evaluation Metrics

We use the following metrics for performance evaluation.

- **Index Size and Indexing Time.** We use the index size and indexing time to evaluate the indexing overhead of a method. The index size is defined by the memory usage for a method to build index. Similarly, the indexing time is defined as the wall-clock time for a method to build index.
- **Recall.** We use recall to measure the accuracy of a method. For the k -P2HNNs, it is defined as the fraction of the total amount data objects returned by a method that are appeared in the exact k nearest objects to the hyperplane query.
- **Query Time.** We consider the query time to evaluate the efficiency of a method. It is defined as the wall-clock time of a method to conduct the k -P2HNNs.

¹Round 7 of Yelp Data Challenge: http://www.yelp.com/dataset_challenge.

²Music-100: <https://github.com/stanis-morozov/ip-nsw>.

³GloVe: <https://nlp.stanford.edu/projects/glove/>.

⁴Msong: <http://www.ifs.tuwien.ac.at/mir/msd/download.html>.

Table 1: Statistics of datasets and queries

Datasets	n	d	#Queries	Data Size	Type
Yelp	77,079	50	100	14.7 MB	Rating
Music-100	1,000,000	100	100	381.5 MB	Rating
GloVe	1,183,514	100	100	451.5 MB	Text
Tiny-1M	1,000,000	384	100	1.43 GB	Image
Msong	992,272	420	100	1.55 GB	Audio

- **Fraction.** We also use the fraction to access the efficiency of a method. It is defined as the fraction of the number of candidates verified by a method over the n data objects.

We report the average recall and fraction over all queries, and we run each method for each experiment five times to report its average query time and indexing overhead.

4.3 Benchmark Methods

Both NH and FH are designed for P2HNNs beyond the unit hypersphere, while the state-of-the-art hyperplane hashing schemes, such as BH [24] and MH [25], only conditionally deal with this problem with the assumption of data normalization. They may be arbitrarily bad without this assumption. Thus, there is no direct competitor for NH and FH.

To make a fair comparison with NH and FH, we implement two heuristic linear scan methods and adapt BH and MH as baselines. Specifically, we evaluate the methods described as follows.

- **NH, FH, and FH⁻.** We implement NH and FH as described in Sections 3.4 and 3.5 with the randomized sampling for the asymmetric transformation to reduce the computational cost of hash functions. To measure the effectiveness of the data-dependent multi-partition strategy, we also implement the trivial FH (or simply FH⁻) as a competitor.
- **NH-wo-S, FH-wo-S, and FH⁻-wo-S.** To evaluate the effectiveness of the randomized sampling for the asymmetric transformation, we also implement NH, FH, and FH⁻ without the randomized sampling. We call them NH-wo-S, FH-wo-S, and FH⁻-wo-S, respectively.
- **BH and MH.** To deal with the P2HNNs beyond the unit hypersphere, we adapt BH [24] and MH [25] as follows: (1) In the indexing phase, we first apply the data-dependent multi-partition strategy to partition data objects into disjoint subsets; then for each subset, we normalize the data to satisfy the unit hypersphere assumption and apply BH and MH to build index accordingly. (2) In the query phase, we apply BH and MH for k -P2HNNs on each subset and check the number of candidates based on the fraction of data on each subset. Note that this adaptation cannot avoid the error introduced by the data normalization, but it may be more fair to compare with BH and MH. For simplicity, we still call them BH and MH. In addition, we consider the P2HNNs on normalized data to make a systematic comparison with BH and MH.
- **Random-Scan.** A trivial method for k -P2HNNs is to randomly select and verify data objects to determine the k nearest ones. We call it Random-Scan and use it as a competitor.
- **Sorted-Scan.** Since P2HNNs is essentially a minimum absolute inner product search problem, a heuristic baseline beyond

Figure 3: Query time-recall curves (top) and fraction-recall curves (bottom) of retrieving top-10 results.

Figure 4: The heat-map of $|\cos \theta|$ and $\|o\|$ and their histograms of the five real-life datasets. The histograms of $|\cos \theta|$ and $\|o\|$ are depicted on the right and the top of the heat-map, respectively.

the Random-Scan is to sort the data objects in ascending order of their Euclidean norm and then scan them in that order. We call it Sorted-Scan and use it as a competitor either.

Since BH and MH have been proved to outperform EH [20], we do not choose EH as a competitor. Additionally, there are learning-based BH and MH methods that enjoy better performance than their original data-independent versions. The primary reason is that their hash functions are carefully learned and selected by some well-designed objective functions. Since the hash functions we consider in this paper are drawn i.i.d. from their hash family without learning, we do not compare with them.

For the k -P2HNNS, we consider $k \in \{1, 5, 10, 20, 50, 100\}$. To make a fair comparison to different methods, we limit the maximum number of hash tables to 256 for all methods. Specifically, we set $t \in \{8, 16, 32, 64, 128, 256\}$ for NH and set $m \in \{8, 16, 32, 64, 128, 256\}$ and $l \in \{2, 4, 6, 8, 10\}$ for both FH and FH⁻. Moreover, we set $\lambda \in \{d, 2d, 4d, 8d\}$ for NH, FH, and FH⁻. For NH-wo-S, FH-wo-S, and FH⁻-wo-S, we follow the same settings of NH, FH, and FH⁻ without λ . BH [24] and MH [25] apply the static (K, L) -bucketing strategy [8] for k -P2HNNS; we set $K \in \{2, 4, 6, 8, 10\}$ and $L \in \{8, 16, 32, 64, 128, 256\}$. Besides, MH needs to determine the number of random projection vectors (m_0) for each hash function; we follow [25] and set $m_0 \in \{4, 8, 16\}$. NH, FH, and FH⁻ are not sensitive to w , so we simply set $w = 1.0$. In addition, we observe that the effectiveness of the data-dependent multi-partition strategy mainly depends on the data distribution instead of σ . Thus, we set $\sigma = 0.9$ for FH, BH, and MH to achieve a good performance.

4.4 Results and Analysis

We study the performance of NH and FH in terms of six aspects: the query performance, the indexing performance, the sensitivity to k , the impact of the randomized sampling, the performance on normalized data, and the impact of parameters.

Query Performance. We first study the query performance of NH and FH. Different number of candidates are used for different methods to achieve different recalls. To remove the impact of parameters for each method, we choose the lowest query time of a method for a certain recall from all combinations of its parameters' settings, and plot the query time-recall curves of each method in Figure 3.

NH and FH are around $3 \sim 100\times$ faster than the best competitor in four out of five datasets (except Msong), especially when the recall falls in the range of $[20\%, 80\%]$. The primary reason is that BH and MH only conditionally deal with this problem. Even though we adopt the data-dependent multi-partition strategy to reduce the error caused by the data normalization, the performance of BH and MH are not arbitrarily bad but still degrade to Random-Scan. Another reason is that their p_1 and p_2 are less than $1/2$ [24, 25]. For the case $d_{P2H}(\mathbf{p}, \mathbf{q}) \rightarrow 0$, they need to check a large number of candidates for P2HNNS, which leads to long query time.

The patterns of all methods over Msong and Tiny-1M are different from those over the other three datasets. Referring to Equation 7, $d_{P2H}(\mathbf{p}, \mathbf{q}) = |\langle \mathbf{o}, \mathbf{q} \rangle| = \|\mathbf{o}\| \|\mathbf{q}\| |\cos \theta|$, where θ is the angle between $\mathbf{o}$ and $\mathbf{q}$. Since the query results are impacted by $|\cos \theta|$ and the Euclidean norm $\|\mathbf{o}\|$, we plot the heat-map of $|\cos \theta|$ and $\|\mathbf{o}\|$ and their histograms of the five datasets in Figure 4 for illustration.

Figure 5: Query time-index size curves of retrieving top-10 results (top), query time-indexing time curves of retrieving top-10 results (middle), and query time- k curves (bottom) at about 80% recall.

From Figure 3, Sorted-Scan over Msong is faster than NH and FH. Moreover, Sorted-Scan over Tiny-1M is also more efficient than BH, MH, and Random-Scan. However, the performance of Sorted-Scan is not stable. For Yelp, Music-100, and GloVe, its performance is the worst among all methods. The reason might be the histogram of $|\cos \theta|$ of Msong (and Tiny-1M) is more average than that of Yelp, Music-100, and GloVe. Thus, Sorted-Scan that only considers the Euclidean norm but neglects the angle can perform better. Moreover, the query time of BH and MH over Msong is less than that of Random-Scan, because the Euclidean norm of Msong is more skewed than that of other datasets, which is close to the normalized case. In addition, the advantage of FH and NH over BH and MH is much more apparent. These benefits could be due to our asymmetric transformation which is more effective in distinguishing close data objects from the far-apart ones.

Compared with FH^- , FH achieves about $10\times$ acceleration, which verifies the effectiveness of the data-dependent multi-partition strategy in reducing false positives. Moreover, FH is much faster than NH. This is because the values of p_1 and p_2 of NH differ little due to a large constant $2M^2$, leading to a large t and long query time. Besides, the data-dependent multi-partition strategy can speed up FH. Thus, even though NH enjoy a sub-linear query time bound, FH has a better practical performance.

To further evaluate the query performance without the impact of implementation, we also plot their fraction-recall curves in Figure 3. The fraction-recall curves show similar trends to the query time-recall curves, which indicates FH and NH (especially FH) can check much less candidates than BH and MH for the same recalls.

Indexing Performance. We then study the indexing performance of NH and FH. We plot the query time-index size (and indexing time) curves of all methods at about 80% recall in Figure 5. Each point in

those curves refers to the query time and index size (or indexing time) of a method with a certain combination of their parameters whose recall is about 80%. Similar trends can be observed from other recalls.

From Figure 5, NH and FH enjoy better trade-off between query time and space than BH and MH. As we discussed before, BH and MH suffer from the low collision probability, so they require much more space and query time than NH and FH to achieve a certain recall. Compared with BH and MH, the asymmetric transformation we proposed will increase the data dimension from d to $O(d^2)$, but we use the randomized sampling to reduce the computational cost of hash functions. Thus, the indexing time of NH and FH can be lower than that of BH and MH. Moreover, MH requires longer indexing time than BH, because it requires computing m_0 ($m_0 \geq 2$) random projection vectors to determine a hash value. In addition, for some methods such as FH, increasing hash tables cannot always reduce the query time. It is probably because certain indexing overhead is good enough for FH to achieve 80% recall.

Sensitivity to k . We consider $k \in \{1, 5, 10, 20, 50, 100\}$ and study the sensitivity to k for each method. We plot the query time- k curves of all methods at about 80% recall in Figure 5. Similar trends can be observed from other recalls.

From Figure 5, NH and FH is much faster than other competitors. However, the slope of NH and FH is slightly larger than that of BH and MH when k is small, e.g., $k < 20$, because BH and MH are inefficient and hence they are not sensitive to k . As k increases, e.g., $k \geq 50$, the slope of NH and FH becomes as small as that of other competitors. This observation indicates that the performance of NH and FH becomes more and more stable when k increases.

Impact of Randomized Sampling. Next, we study the impact of the randomized sampling for the asymmetric transformations of

Figure 6: Query time-recall curves of retrieving top-10 results (top) and query time-indexing time curves of retrieving top-10 results at about 80% recall (bottom).

NH and FH. Figure 6 shows the query time-recall curves and the query time-indexing time curves at about 80% recall of NH, FH, FH^- , and their variants without randomized sampling.

The query performance of FH, FH^- , and NH is comparable to that of FH^-wo-S , FH^-wo-S , and $NH-wo-S$, which demonstrates the effectiveness of the randomized sampling, in the sense that the data objects after the asymmetric transformation can be effectively reconstructed by the randomized sampling. Moreover, since the randomized sampling reduces the data dimension from $O(d^2)$ to $O(1/\epsilon^2)$, the query time of NH, FH, and FH^- may be lower than that of FH^-wo-S , FH^-wo-S , and $NH-wo-S$. For example, the query time of FH over Msong is much lower than that of FH^-wo-S .

Due to the use of pipeline computation, we can avoid to store the coordinates of $P \circ f(o_i)$. Thus, the index sizes of FH, FH^- , and NH are identical to those of FH^-wo-S , FH^-wo-S , and $NH-wo-S$. However, their indexing time is quite different. Since the randomized sampling reduces the data dimension from $O(d^2)$ to $O(1/\epsilon^2)$, FH, FH^- , and NH enjoy much lower indexing time than FH^-wo-S , FH^-wo-S , and $NH-wo-S$, respectively.

Performance on Normalized Data. To make a fair comparison with BH and MH, we further normalize the datasets to the unit hypersphere and study the performance of NH and FH. Figure 7 shows the query time-recall curves and query time-indexing time curves at about 80% recall of NH, FH, BH, and MH.

Compared to Figure 3, the query performance of NH and FH degrade, especially for NH. This might be because after data normalization, it is hard to distinguish the close data objects from the far-apart ones. Moreover, the query performance of BH and MH do not improve very much. The primary reason is that when $d_{P2H}(p, q) \rightarrow 0$, $\rho = \lim_{d_{P2H} \rightarrow 0} \ln(1/p_1)/\ln(1/p_2) = 1$, leading to $L = O(n)$ hash tables. Due to their limited improvements, NH and FH are still better than BH and MH. The patterns of the four methods over Msong and Tiny-1M are different from those over the other three datasets, because after data normalization, the results of P2HNNS are only impacted by $|\cos \theta|$ and the histograms of $|\cos \theta|$ of Msong and Tiny-1M are different from those of other three datasets. The query time-indexing time curves of NH, FH, BH,

and MH in Figure 7 are similar to that in Figure 5. The reason is that the data normalization does not impact the indexing time.

Impacts of Parameters. Finally, we study the impacts of the parameters of NH and FH. We only show results over GloVe. Similar trends can be observed from other datasets.

Impacts of t and λ for NH. We first study the impact of the number of hash tables t for NH. We fix $\lambda = 2d$. Figure 8 shows the query time-recall curves of NH for $t \in \{8, 16, 32, 64, 128, 256\}$. NH achieves the lowest query time under different recalls with different t 's. Thus, the optimal t value for NH under different recalls is different. In general, a larger t leads to lower query time of NH especially when the recalls are high.

We then study the impact of λ for NH. We fix $t = 256$. Figure 8 shows the query time-recall curves of NH for $\lambda \in \{1d, 2d, 4d, 8d\}$. From Figure 8, the query performance of NH under different settings of λ is close to each other. Thus, the query performance is not sensitive to λ . The reason may be that with a larger value of λ , the data objects may be reconstructed better, but it also introduces longer query time to compute hash values. Since a larger value of λ will lead to larger indexing time, setting a small λ value (e.g., $1d$ or $2d$) is suitable to NH.

Impacts of m , l , and λ for FH. We then study the impacts of m , l , and λ for FH. We first fix $\lambda = 2d$ and $l = 4$. Figure 8 shows the query time-recall curves of FH for $m \in \{8, 16, 32, 64, 128, 256\}$. The query performance of FH is not very sensitive to m . For the low recalls, i.e., the recall value is less than 25%, FH with small m 's is slightly better than that with large m 's. Since a larger m will lead to a larger indexing overhead, setting $m = 16$ or $m = 32$ is suitable to FH.

Next, we fix $\lambda = 2d$ and $m = 16$. Figure 8 shows the query time-recall curves of FH for $l \in \{2, 4, 6, 8, 10\}$. For the high recalls, except for $l = 10$, the query time of FH under other settings of l is close to each other, while for the low recalls, i.e., the recall is less than 25%, the query time of FH with small l 's is slightly lower than that with large l 's. Thus, setting a small l value (e.g., 2 or 4) is satisfied for FH. In addition, the impact of λ for FH is similar to that for NH, and the query performance of FH is not sensitive to λ .

Figure 7: Results on normalized datasets: query time-recall curves of retrieving top-10 results (top) and query time-indexing curves of retrieving top-10 results at about 80% recall (bottom).

Figure 8: Impacts of the parameters of NH and FH over GloVe.

5 CONCLUSION

In this paper, we investigate a very challenging problem of P2HNNS. We design a novel asymmetric transformation and propose the first two provable schemes NH and FH for high-dimensional P2HNNS beyond the unit hypersphere. Like a coin has two sides, this transformation is not perfect, leading to more space and computational cost. We develop a pipeline computation to avoid $O(nd^2)$ space to store the intermediate values of data objects; moreover, we propose to use the randomized sampling to approximate this transformation that can reduce the computational cost of hash functions from $O(d^2)$ to $O(1/\epsilon^2)$. With this transformation, the hash functions of NH and FH are locality-sensitive to the hyperplane queries. We further show that NH can deal with the P2HNNS in sub-linear time, while FH enjoys the better practical performance. Extensive experiments over five real-life datasets confirm their superior performance.

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A HYPERPLANE QUERY GENERATION

Suppose $[l_1, r_1], \dots, [l_{d-1}, r_{d-1}]$ are the ranges of coordinates of data points such that $l_i \leq p_i \leq r_i$ for all $\mathbf{p} = (p_1, \dots, p_{d-1}) \in \mathcal{D} \subset \mathbb{R}^{d-1}$, where $i \in \{1, \dots, d-1\}$. Without loss of generality, we assume the datasets satisfy the following two conditions: (1) the

range of coordinates for all $\mathbf{p} \in \mathcal{D}$ is symmetric, i.e., $l_i = -r_i$ for each $i \in \{1, \dots, d-1\}$; (2) all $\mathbf{p} \in \mathcal{D}$ are bounded inside the unit hypersphere, i.e., $\|\mathbf{p}\| \leq 1$ for all $\mathbf{p} \in \mathcal{D}$.

We can shift and rescale the data points if the real-life datasets do not satisfy these two conditions, i.e., $\mathbf{p}' = (\mathbf{p} - \bar{\mathbf{p}})/\Omega$ for all $\mathbf{p} \in \mathcal{D}$, where $\Omega = \max_{\mathbf{p} \in \mathcal{D}} \|\mathbf{p} - \bar{\mathbf{p}}\|$ and $\bar{\mathbf{p}} = (\frac{l_1+r_1}{2}, \dots, \frac{l_{d-1}+r_{d-1}}{2})$ is the center of all data points. Notably, we do not assume all $\mathbf{p} \in \mathcal{D}$ are normalized or change the distortion between the data points, i.e., $\frac{\|\mathbf{p}-\mathbf{u}\|}{\|\mathbf{p}-\mathbf{v}\|} = \frac{\|\mathbf{p}'-\mathbf{u}'\|}{\|\mathbf{p}'-\mathbf{v}'\|}$ for any three $\mathbf{p}, \mathbf{u}, \mathbf{v} \in \mathcal{D}$. Then, we generate a random hyperplane query $\mathbf{q} \in \mathbb{R}^d$ as follows:

- (1) We generate a random vector $\mathbf{w}_q = (q_1, \dots, q_{d-1})$ as a normal vector to the query hyperplane, i.e., we generate q_i uniformly at random from $[l_i, r_i]$ for each $i \in \{1, \dots, d-1\}$;
- (2) We generate a random point $\mathbf{x} = (x_1, \dots, x_{d-1})$ as a pivot which is confined to be in the query hyperplane, i.e., we generate each x_i i.i.d. from $[l_i, r_i]$, where $i \in \{1, \dots, d-1\}$;
- (3) We construct $\mathbf{q} = (\mathbf{w}_q; q_d) = (q_1, \dots, q_{d-1}, q_d)$ based on $\mathbf{w}_q$ and $\mathbf{x}$, where $q_d = -\sum_{i=1}^{d-1} x_i q_i$.

Since each x_i satisfies the inequality $l_i \leq x_i \leq r_i$, the hyperplane query $\mathbf{q}$ is very likely to divide the dataset into two parts, which helps us simulate the scenarios for the real-life applications, e.g., large-scale active learning with SVMs. Moreover, since we assume the datasets are bounded inside the unit hypersphere, q_d is restricted in a small range. Thus, when we apply $P \circ f(\cdot)$ and $Q \circ g(\cdot)$ (or $Q' \circ g(\cdot)$) to convert the P2HNNS into the classic NNS (or FNS), the distortion error between data points can be largely reduced, i.e., $|\frac{\|\mathbf{p}-\mathbf{u}\|}{\|\mathbf{p}-\mathbf{v}\|} - \frac{\|P \circ f(\mathbf{p}) - P \circ f(\mathbf{u})\|}{\|P \circ f(\mathbf{p}) - P \circ f(\mathbf{v})\|}| \rightarrow 0$ for any three $\mathbf{p}, \mathbf{u}, \mathbf{v} \in \mathcal{D}$.

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