The Laplace Microarchitecture for Tracking Data Uncertainty and Its Implementation in a RISC-V Processor

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ABSTRACT

We present Laplace, a microarchitecture for tracking machine representations of probability distributions paired with architectural state. We present two new methods for in-processor distribution representations which are approximations of probability distributions just as floating-point number representations are approximations of real-valued numbers. Laplace executes unmodified RISC-V binaries and can track uncertainty through them. We present two sets of ISA extensions to provide a mechanism to initialize distributional information in the microarchitecture and to allow applications to query statistics of the distributional information without exposing the uncertainty representations above the ISA.

We evaluate the accuracy and performance of Laplace using a suite of 21 benchmarks spanning domains ranging from variational quantum algorithms and sensor data processing to materials properties modeling. Monte Carlo simulation on the benchmarks requires 2076× more instructions on average (and up to 21343× in some cases) to achieve the same accuracy that Laplace can achieve in a single execution. Compared to state-of-the-art alternatives to Monte Carlo, Laplace achieves an average 1.3× accuracy improvement versus PaCAL [22] and more than 4.6× accuracy improvement versus the method used by the NIST Uncertainty Machine [26], quantified using the Wasserstein distance to Monte Carlo.

Unlike existing methods for uncertainty tracking which require software to be rewritten in a domain-specific language or extensive source-level changes, Laplace achieves all of these benefits while requiring no changes to existing binaries in order to track uncertainty through them, with only minimal changes required to get uncertainty information into the microarchitecture. We have deployed an implementation of Laplace as a commercial product in the form of a cloud-accessible virtual machine.

CCS CONCEPTS

• **Computer systems organization** → **Reduced instruction set computing**; *Embedded hardware*; • **Mathematics of computing** → **Probabilistic representations**; • **Hardware** → **Emerging architectures**.

KEYWORDS

uncertainty tracking, distributional representations, arithmetic on distributions, RISC-V

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1 INTRODUCTION

Uncertain data are ubiquitous in computing systems. One common example is sensor measurements: there is always uncertainty, no matter how precise the instrument, between the recorded value (the *measurement*) and the quantity being measured (the *measurand*) [23]. This form of measurement uncertainty can be quantified by using multiple simultaneous measurements or by performing repeated measurements at a rate faster than the measurand can change. It is then possible to quantify the variation across measurements using statistical analysis or by noting the number of significant (i.e., unchanging) digits across the measurements. Such numerically-quantified uncertainty is referred to in the literature as *aleatoric uncertainty* [53]. Uncertainty may also exist when there is insufficient information about a quantity of interest. Such uncertainty in values resulting from incomplete information is often referred to in the literature as *epistemic uncertainty*.

Despite the relevance of uncertainty in computing systems, existing computer architectures neither have support for representing uncertainty, nor for arithmetic on uncertain values. Computer architectures today represent uncertain values with single *particle values* (i.e., data with no associated distribution), usually by taking the mean value as the representation for use in computation. In

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Figure 1: Uncertainty in stiffness (Young’s modulus, E) leads to uncertainty in extension $u(x)$ for a given applied load q_0 .

much the same way that the abstraction of floating-point number representations and floating-point arithmetic improved the ease of implementation of real-valued computations, architectural and microarchitectural support for representations of uncertainty could similarly enable new approaches to computation.

1.1 Example uses of uncertainty tracking

Figure 1(a) shows an engineering problem for which uncertainties in the problem parameters have real-world implications. The figure shows a beam of cross-sectional area A and length L made of material with Young’s modulus E [25], loaded with an applied axial load q_0 per unit length. The beam’s extension $u(x)$ at position x from the cantilever, is

$$u(x) = \frac{q_0}{2AE} (2xL - x^2). \quad (1)$$

Because of material variabilities or variations in atmospheric conditions, the Young’s modulus may be uncertain and this uncertainty can be quantified with a probability distribution for the Young’s modulus rather than using a single number. In the same way that engineers will often want to evaluate a floating-point implementation of Equation 1 rather than one where the parameters are constrained to integer values, it is useful to evaluate Equation 1 with parameters such as the Young’s modulus as distributions representing uncertainty. The implementations of analytic models such as Equation 1 or their finite-element counterparts may be in legacy or third-party libraries, making it attractive to have methods for tracking uncertainty that work on existing program binaries.

Figure 2 shows a second example of how uncertainty in the inputs to a computation can lead to uncertainty in the computation’s result. The figure shows images from the MNIST dataset known to be ambiguous [35], along with distributions of labels for the images, assigned by 50 observers in a user study we conducted for this work. For example, for the first (left-most) image in the sequence (“6”), Figure 2(a) shows the distribution of labels, with peaks at 0 and 6. Figure 2(i) shows the result of statistical sampling and propagation of these distributions for the digits d_i through the expression $\sum_{i=0}^7 d_i 10^i$ to determine the distribution of plausible interpretations of the handwritten 8-digit sequence “6 9 2 5 1 4 5 5”.

The evaluation of uncertainty in Figure 1(b) and Figure 2(i) uses the standard (but expensive) method of statistical sampling, often referred to as the Monte Carlo method [40], with 30 000 runs. In contrast to this expensive approach, Laplace’s methods for tracking uncertainty make tracking uncertainty through arbitrary binaries as straightforward to achieve as implementing these computations using floating-point arithmetic. The results in Section 5 show that

the methods for tracking uncertainty achieve accuracy close to that of the expensive Monte Carlo method.

1.2 Laplace

This article presents Laplace, a microarchitecture for tracking machine representations of probability distributions paired with architectural state. We present and evaluate two different machine-level representations of approximations of probability distributions. Just as fixed-point and floating-point number representations trade the number of bits in their representations for accuracy with respect to the represented real-valued number, the distribution representations we present trade the representation size for accuracy of representing empirical probability distributions. The representations allow computing systems to efficiently represent uncertain quantities even when those quantities have rarely-occurring high-moment values in the tails of the distribution.

Laplace honors traditional program semantics: Laplace allows programs to associate distributional information with registers and by extension all memory words that are used to load and store register values. In addition to maintaining the non-uncertain architectural state in registers, arithmetic instructions in Laplace use the distributional information in the microarchitecture associated with register operands to update the distributional information of the result register. Our microarchitectural implementation tracks distributional information across all operations in the floating-point datapath. In applications that do not ingest distributional information, the distributions are degenerate with point mass. In applications which use Laplace’s extensions to set the distributions associated with data in a program’s input or state, the microarchitecture tracks distributions across all operations in the floating-point datapath.

Laplace is ISA-independent: We implement the Laplace microarchitecture as the execution model for a commercial RISC-V virtual machine product implementing the RISC-V (RV32IMFD) ISA. In this implementation, Laplace represents and tracks uncertainty across computation transparently to the default semantics of the RISC-V ISA. Our implementation provides a mechanism to input uncertainty information into the processor and to query the uncertainty information which Laplace maintains in parallel with the non-uncertain architectural state, in the form of two groups of instruction extensions (Table 1) which together form a new RISC-V “xua” extension. These extensions purposefully do not expose the uncertainty representations to applications. They only allow applications ❶ to provide uncertainty distributions in a canonical form (as samples) and ❷ to query derived properties (such as the n th moment) of the microarchitectural distributions. We extend GCC (version 8.2) and Binutils (version 2.32) to support these RISC-V instruction extensions and we implement high-level language bindings to these extensions for C/C++ but our implementation can already, in principle, support any host language supported by GCC 8.2. We chose to use the 32-bit RISC-V ISA as a starting point for pragmatic implementation reasons but Laplace can in principle be implemented within any instruction set architecture.

Why implement uncertainty tracking below the ISA abstraction: Facilities for computing with uncertainty could be implemented in many ways: As a software-only implementation in a

Figure 2: Eight ambiguous images [35] from the MNIST dataset and the distribution of labels (a–h) assigned to each by 50 observers. Based on these uncertainties in the labels, we can determine the distribution of plausible values of the 8-digit number using standard floating-point arithmetic instructions and can query the uncertainty associated with the result (i).

library [30], as a domain-specific language or new type system embedded in a host language [7], or below the ISA abstraction (whether in hardware or in a virtual machine). We chose a microarchitecture-level abstraction because it is scalable to future improved machine-level representations for non-parametric distributions beyond those we have developed to date. By tracking uncertainty below the ISA, Laplace can track distributional information through any statically- or dynamically-linked library of the running application.

1.3 Contributions

Laplace makes five contributions to the state-of-the-art:

- ❶ **New methods for distribution representation and arithmetic.** We present two finite-dimensional representations for storing and operating on random variables (Section 2.2) and algorithms for performing arithmetic operations on these distribution representations in order to propagate the distributional information (Section 2.3).
- ❷ **Laplace microarchitecture.** We present Laplace, a microarchitecture for computing on data associated with distributional information (Section 3.1).
- ❸ **Insights from new tools to support uncertainty-tracking microarchitectures.** We implement a complete compilation toolchain for the extensions we make to the RISC-V ISA to benefit from the Laplace microarchitecture, based on the GNU GCC compiler. We show how programs can exploit the benefits of Laplace with minimal changes to source code and with the possibility of tracking uncertainty through pre-compiled libraries.
- ❹ **Demonstration of new applications enabled by easily-accessible uncertainty tracking.** We evaluate Laplace using a suite of 21 benchmarks including examples of applications which benefit from uncertainty tracking, ranging from variational quantum algorithms and sensor data processing to materials properties modeling.
- ❺ **Comparison to the state of the art.** We compare the accuracy and performance of Laplace to the standard reference of a Monte Carlo simulation. The results show that Laplace achieves the same accuracy as Monte Carlo simulations, executing 2 076× fewer instructions on average and up to 21 343× fewer instructions when using larger in-processor

representations. Compared to state-of-the-art alternatives to Monte Carlo, Laplace achieves a 1.3× improvement in accuracy versus PaCAL [22] and more than 4.6× accuracy improvement compared to the method used by the NIST Uncertainty Machine [26]. We quantify accuracy improvement using the Wasserstein distance of the uncertainty tracking results compared to Monte Carlo results.

Laplace achieves all of these benefits while requiring no changes to binaries in order to track uncertainty through them, with only minimal changes required in order to get uncertainty information into the microarchitecture.

Laplace is in use commercially, as the execution model for a cloud-based virtual machine, targeted at domains including physical system modeling, autonomous systems, sensor data processing, machine learning, and quantum computing¹.

2 DISTRIBUTION REPRESENTATIONS

We implement two complementary methods for representing uncertainty within the Laplace microarchitecture.

2.1 Terminology

The mean value or expected value of a random variable X with probability mass function $f_X(x)$ is

$$E(X) = \sum_{x: f_X(x) > 0} x f_X(x). \quad (2)$$

We often refer to the mean value of X as μ_X . Let k be a positive integer. The k th moment m_k of X is

$$m_k = E(X^k). \quad (3)$$

The k th centralized moment σ_k of random variable X is

$$\sigma_k = E((X - m_1)^k). \quad (4)$$

We refer to σ_2 as the variance, or $\text{var}(X)$, of the random variable X . The variance provides a measure of how much X deviates from its mean value.

Terminology in the context of Laplace: Laplace operates on machine representations of probability mass functions. In the same way that each value of a floating-point register is an approximate

¹ This instance of the Laplace microarchitecture is available at <https://signaloid.io>.

representation of a real-valued number, each value in Laplace’s uncertainty representation is an approximation of an entire probability mass function (not just of a probability, which is a real-valued number in $[0, 1]$). The distributions that Laplace handles need not be nameable or *parametric* distributions such as uniform, Bernoulli, binomial, Gaussian, Laplacian, and so on. Laplace *does not* assume that the distributions in a computation are all the same, *does not* assume that the distributions involved in arithmetic operations are the same, and *does not* assume that a given item of machine state remains in the same distribution over time (i.e., Laplace does not assume homoscedasticity). Laplace does not expose its internal representation to applications above the ISA but instead allows applications to evaluate derived properties of distributions such as the expectation, centralized moments, and tail probability.

2.2 Definitions of representations

We introduce two microarchitecture-level representations of uncertainty distributions. Because the representations are of finite length, they are an approximation of the probability density function of the original random variable. The methods share an underlying representational primitive which we refer to as the *Dirac mixture representation*.

Dirac mixture representation: Let $\delta(x)$ be the Dirac delta function — a unit impulse at position x . Given a value $x_0 \in \mathbb{R}$, we view $\delta(x - x_0)$ as a *probability mass function* and we call it a *particle value* (or just *particle*). Using this definition, we transform an array of particle values $x_1, \dots, x_M$ into a probability mass function using a weighted sum:

$$f_X(x) = \sum_{n=1}^M p_n \delta(x - x_n), \text{ where } p_n \in [0, 1], \sum_{n=1}^M p_n = 1.$$

In-memory storage of Dirac mixtures: A Dirac mixture requires only a series of positions (x_i) and probability masses (p_i) for its description. For a random variable d , we represent its Dirac mixture of size N as an array of $2N$ floating-point values. For the i th Dirac delta of the mixture, Laplace stores the position ($d_{\text{pos}}[i]$) and probability mass ($d_{\text{mass}}[i]$).

Regularly-quantized representation (RQHR): This representation is conceptually similar to histograms, which uniformly split the representational overhead across the support of the distribution. To create an RQHR from a set of empirical samples, we divide the support of the samples into N bins of equal width L . We define the height of each bin $1 \leq i \leq N$ as p_i and set it equal to the relative frequency of the source particles falling within the i th bin. We store the RQHR as a Dirac mixture of size N , setting $d_{\text{pos}}[i]$ of the Dirac mixture equal to the median of the lower and upper bound of the bin and $d_{\text{mass}}[i]$ equal to p_i of the i th histogram bin of RQHR.

Telescoping torques representation (TTR): The telescoping torques representation (TTR) is based on the intuition that the mean value of a set of samples splits the samples into two sets with equal *probability torque* (i.e., the product between the sample value and its probability). This allows TTR to capture the effects of unlikely, yet high-value samples even at small representational sizes.

Let src_{DM} be a Dirac mixture of particle values of samples from a random variable X . We convert src_{DM} to a TTR of size $N > 0$ using

Figure 3: A Dirac delta (solid vertical arrow) placed at the mean of the Dirac mixture representation of src_{DM} (top) is its 0th-order TTR (Step 0). In Step 1, the algorithm splits the source Dirac mixture using the mean ($\mu = 22.5$) of Step 0 and calculates the 1st-order TTR comprising two Dirac deltas. In Step 2, the algorithm uses the positions of these two Dirac deltas to subdivide the source Dirac mixture and calculates four Dirac deltas which form the 2nd-order TTR of src_{DM} .

a recursive algorithm that takes $\log_2 N$ steps. We call this number of steps the *order* of the TTR. At each step, the algorithm splits the input Dirac mixture on its mean value μ , creating two Dirac mixtures which are the inputs of two new calls to the algorithm. Figure 3 shows an example for $N = 4$, starting from a src_{DM} of 8 particles. In step 0, the method creates the 0th-order TTR of src_{DM} which is a single Dirac delta located at the mean (μ) of src_{DM} (i.e., $d_{\text{pos}}[0] = 22.5$) with associated weight equal to the sum of the probability masses of all Dirac deltas in src_{DM} ($d_{\text{mass}}[0] = 1.0$).

Using μ as a boundary, step 1 splits src_{DM} into two mixtures (each with four Dirac deltas in this example) and repeats the process for each mixture. The 1st-order TTR of src_{DM} is the union of the 0th-order TTRs of each sub-mixture. In the example of Figure 3, this corresponds to two Dirac deltas with probability mass equal to 0.5, each one at the mean value of each sub-mixture. In step 2, the process repeats, further dividing the sub-mixtures of step 1 into two and calculating their 0th-order TTRs. This results in the 2nd-order TTR of src_{DM} , consisting of four Dirac deltas.

2.3 Arithmetic operations on representations

Let Φ be a function such that $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}$ and let X and Y be two discrete random variables represented using our distributional representations. We define the necessary algorithms to compute the distributional representation of the discrete random variable $Z = \Phi(X, Y)$, for Φ as one of addition, multiplication, subtraction, and division. For RQHR and TTR, these basic arithmetic operations on uncorrelated random variables are circular convolutions of their positions and probability masses. Algorithm 1 shows the addition of two uncorrelated random variables represented with Dirac mixtures of size N . We store the results of the operations on the Dirac delta positions and masses of the two source representations in dst_{DM} , which is a Dirac mixture representation of size $N \times N$. We use the algorithms of Section 2.2 to convert dst_{DM} to RQHR or TTR.

Figure 4: For each integer or floating-point ALU instruction, a distributional co-ALU performs the uncertainty tracking, fetching the uncertainty representations for the source operands from the corresponding distributional register file (not visible at the ISA level), performing the distributional arithmetic on the internal distribution representation of size N (Section 2.2), and storing the result back in the distributional register file. An additional on-chip memory holds $AAMax$ ancestor addresses for each distributional register necessary for autocorrelation tracking (Section 3.3). Color legend: Unmodified architectural state is in white ($\square$); microarchitectural structures modified to allow applications to access derived information on uncertainty representations are in green ($\blacksquare$); microarchitectural structures specific to uncertainty-distribution tracking (and not exposed above the ISA) are in grey ($\blacksquare$).

Algorithm 1: Addition of distributional representations based on Dirac mixtures, i.e., RQHR and TTR.

```

1 Function uncertainAdd(src1DM, src2DM)
2   dstDM  $\leftarrow$  new DiracMixture( $N \times N$ );
3   for  $i \leftarrow 1$  to  $N$  do
4     for  $j \leftarrow 1$  to  $N$  do
5        $k \leftarrow (i - 1) * N + j$ ;
6       dstDM.dpos[ $k$ ]  $\leftarrow$  src1DM.dpos[ $i$ ] + src2DM.dpos[ $j$ ];
7       dstDM.dmass[ $k$ ]  $\leftarrow$  src1DM.dmass[ $i$ ] * src2DM.dmass[ $j$ ];
8     end
9   end
10  dstVar  $\leftarrow$  convertDiracMixtureToTTR(dstDM);
11  return dstVar

```

Both representations are of fixed size and do not increase in size as computation progresses.

To multiply two uncorrelated random variables, we replace line 6 of Algorithm 1 with a multiplication. Adding or multiplying a particle value by a Dirac mixture representation variable results in an offset or scaling of the Dirac deltas positions, respectively. We use this property to calculate the subtraction of two random variables. We implement division of two random variables of Dirac mixture representation as a multiplication of the dividend with the reciprocal of the divisor. We construct the latter by calculating the reciprocals of the positions of its Dirac mixture representation. The implementation which we evaluate in this work does not handle the case of arithmetic operations on random variables with arbitrary correlation. The implementation we evaluate only handles the case of autocorrelated arithmetic operations, i.e., expressions where a source operand appears multiple times. We elaborate on this in Section 3.3.

2.4 Summary statistics on representations

Let X be a random variable represented as an RQHR or TTR of size K . Laplace can calculate the following statistical properties of X . **N th centralized moment:** For any integer $N \geq 0$, the N th moment of X is given by $\sigma_N = E((X - E(X))^N)$ where $E(X) = \sum_{i=0}^K d_{\text{pos}}[i]d_{\text{mass}}[i]$. **N th mode/anti-mode:** The N th mode of X is the particle value x at which the probability mass function f_X takes its N th highest value and is calculated as $d_{\text{pos}}[i_N]$ where i_N is the index at which d_{mass} takes the N th highest value. N th anti-mode is calculated similarly but with i_N being the index at which d_{mass} takes the N th lowest value. If N is greater than the size of the distributional representation the statistic evaluates to a NaN. **Distribution's support minimum or maximum value:** Returns the minimum or maximum value of the Dirac mixture positions of the distributional representation of X (i.e., minimum or maximum of d_{pos}). **Tail probability:** Given a cutoff value $x_0 \in \mathbb{R}$ we calculate the tail probability of X as $\Pr(X > x_0) = \sum_{i \in [0, K] \text{ s.t. } d_{\text{pos}}[i] > x_0} d_{\text{mass}}[i]$. We calculate the probability $\Pr(X \leq x_0)$ as $1.0 - \Pr(X > x_0)$.

3 LAPLACE MICROARCHITECTURE

Laplace tracks distributional information within the microarchitecture, transparently to applications. Each instance of the Laplace microarchitecture supports a single distributional representation (Section 2.2), e.g., RQHR of size 8 or TTR of size 4, as a design-time choice. Implementing the Laplace microarchitecture does not introduce new data hazards and does not impact branch predictors and prefetchers of the architecture for which it is implemented (RISC-V RV32IMFD in this work).

3.1 Register files and ALUs

Figure 4 shows the microarchitectural structure of Laplace. Laplace does not modify the number of registers of the architecture within

Figure 5: Bit-level representation of TTR/RQHR.

which it is implemented. Instead, it extends the integer and floating-point register files with additional microarchitecture-level distributional registers which store the bits of the distributional information that correspond to each register of the ISA-visible architecture. Figure 5 shows the bit-level representation of an RQHR or TTR of size N . The lower-order 64 bits store the number of particle samples used to derive the distributional representation. The next N 64-bit values store the support positions of the representation. They are followed by the N 64-bit values of the probability masses.

Let f_0 be a conventional floating-point register of the architecture within which we implement Laplace (in this work, RISC-V RV32IMFD) and let df_0 be the respective microarchitecture-level distributional floating-point register. Laplace performs all arithmetic and logic instructions on both the conventional and distributional registers in parallel. For example, the addition of source registers f_1 and f_2 into destination register f_0 also triggers the addition of the distributional information of registers df_1 and df_2 into the distributional register df_0 . *The semantics of non-distributional register values and operations on them remain unchanged.* Tracking of distributional information happens in parallel to (and not affecting the behavior of) the non-distributional architectural state.

Laplace extends both integer and floating-point arithmetic and logic units (ALUs) with two distributional co-ALUs. The conventional, unchanged ALU operates on the particle values of the source registers. The distributional co-ALU performs the same operation on the distributional representations of the source registers using the algorithms of Section 2.3. The distributional co-ALU can calculate summary statistics (Section 2.4) of the distributional representation of a source register.

3.2 Loading/storing uncertainty information

A load instruction in Laplace loads the distributional representation that corresponds to an address of Laplace's main memory to the distributional part of the destination register. A store instruction stores the distributional information of the source register to the main memory of Laplace. Figure 4 shows how a portion of the physical memory of the processor implementing the Laplace microarchitecture stores the distributional representations. The load/store unit of Laplace maps an address accessed by an application to the address that stores its distributional representation.

Section 4 introduces an additional load instruction that initializes the distributional information of a destination register by creating a distribution representation from an in-memory array of source particle samples.

Figure 6 shows the implementation of the hardware module that converts source samples to TTR. The module comprises multiple levels of conversion units, one level for each step involved in the conversion of an array of samples in the form of a Dirac mixture to a TTR (Section 2.2). A pair of integers identifies each conversion unit. The first integer corresponds to the conversion step and the second is an increasing index, e.g., **Conversion unit [1, 1]** is the second conversion unit of conversion Step 1 (Section 2.2). Because Figure 6 shows an example for the conversion of input samples to TTR of size four, the output level consists of four conversion units which generate the four support positions (**dmPos**) and probability masses (**dmMass**) of the TTR of size four.

Each conversion unit has three inputs and three outputs. The first input is the memory address of the array of samples that the conversion unit will process. Laplace stores the samples in the main memory as a Dirac mixture, sorted by ascending support position (**dmPos** value). The second and third inputs are two integers that correspond to the starting index (**startInd**) and ending index (**endInd**) of the continuous array of samples that the conversion unit will process. Each conversion unit outputs a support position (**dPos**) value and a probability mass (**dMass**) value. The conversion unit calculates the output probability mass as $dMass = \sum_{startInd}^{endInd} dmMass[i]$. The conversion unit calculates the mean value of the source Dirac mixture as $\mu = \sum_{startInd}^{endInd} dmPos[i] * dmMass[i]$ (Section 2.4). The conversion unit calculates the output support position as $dPos = \frac{\mu}{dMass}$. The third output of the conversion unit is an integer **kPartition** which corresponds to the index below which all sorted support positions of the input Dirac mixture are less than the calculated **dPos**.

In intermediate conversion levels, the conversion units propagate their output **kPartition** to the next conversion level. Depending on the subset of arrays that they must process, the **kPartition** acts as the **startInd** or **endInd** value of the conversion units of the next level. In the final conversion level, the TTR conversion module writes the output distributional information to the distributional destination register and writes the mean value of the source samples to the conventional destination register.

3.3 Correlation between register values

Let x be a variable with distributional information. Section 2.3 introduced arithmetic operations on uncorrelated random variables. The correct calculation of expressions such as $x*x$ is a subset of the broader problem of handling correlated (i.e., non-independent) distributions. The correct execution of such autocorrelated arithmetic operations requires point-wise operations on the support of the

Figure 6: Module for conversion of an in-memory array of particle samples (stored as a Dirac mixture) to TTR of size N (N is 4 for this example). The module comprises multiple levels of conversion units which take as input the address of the in-memory array of samples. Each unit processes the array indices dictated by $startInd$ (lower bound) and $endInd$ (upper bound) and calculates the support position ($dPos$) and probability mass ($dMass$) of the output TTR.

source operands. Figure 7 shows the calculation of $x \times x$ with and without the correct handling of autocorrelation. Figure 7a shows the distribution of x created in Laplace using samples from a zero-mean Gaussian distribution. Figure 7b shows the combined outcome of uncertainty tracking with autocorrelation tracking on or off and the ground truth of exhaustively evaluating $x \times x$ for all samples of x . The evaluation of $x \times x$ when autocorrelation tracking is on is almost identical to the ground truth. Without such handling of arithmetic on autocorrelated random variables there are negative values in the support of the outcome distribution which is incorrect for the expression $x \times x$.

Figure 7: Input and output of expression $x \times x$. Output plots show the result of uncertainty tracking (UT) with autocorrelation tracking on or off and the ground truth of exhaustively evaluating $x \times x$ for all the samples of x . The evaluation of the expression with enabled autocorrelation tracking is almost identical to the ground truth. Evaluation without autocorrelation tracking leads to negative values in the support of the output distribution, which is incorrect.

Ancestor addresses tracking mechanism: Figure 8 shows the C source code and RV32IFMD ISA disassembly of the calculation of the even powers of a variable x , extracted from a function that uses Taylor series expansion to calculate the cosine of x . The compiler uses two different registers ($fa4$ and $fa5$) to calculate $x \times x$. Tracking only the identities of the source registers of instructions is not sufficient to track correlations between registers and by extension, variables with distributional information. The processor loads the values of both registers from the same memory address, which corresponds to the same variable of the source application. This is the core insight behind our autocorrelation tracking mechanism. The value of each floating-point register originates from one or more addresses of the main memory. By keeping track of these *ancestor addresses* we can dynamically identify correlations between any two registers of Laplace.

Each time Laplace executes a load instruction, it adds the source address to the set of ancestor addresses of the destination register rd . We use an on-chip memory of the Laplace microarchitecture to store a fixed number ($AAMax$) of ancestor addresses for each of the architectural registers. We evict addresses from this memory using a least-recently used (LRU) policy. When Laplace stores a register rs to memory, its set of ancestor addresses is spilled to main memory. For each arithmetic operation, if the source registers have at least one common ancestor, Laplace executes the arithmetic operation in an autocorrelation-tracked manner. Laplace also updates the ancestor addresses of the destination register with the union of the ancestor addresses of the source registers.

Example of tracking correlations in Laplace: Figure 8 shows the execution of $xPower *= x \times x$ which involves two kinds of autocorrelated operations: (i) $x \times x$ and (ii) the more challenging case of $xPower * (x \times x)$ where Laplace needs to detect that $xPower$ is x multiplied repeatedly with itself. Each row of Figure 8 corresponds to the execution of an assembly instruction (with increasing instruction index) and shows the set of ancestor addresses of the destination register before and after the execution of the instruction.

Instr. index	Binary disassembly	Destination register ancestors (before)	Destination register ancestors (after)
0	for (int i=2; i<=kNumOfSeriesTerms; i+=2){		
1	8004178: 00200793 li a5,2		
2	800417c: fcf42223 sw a5,-60(s0)		
3	8004180: 0740006f j 80041f4		
4	8004184: fb843707 fld fa4,-72(s0)		{-72(s0)}
5	8004188: fb843707 fld fa5,-72(s0)		{-72(s0)}
6	800418c: 12f777d3 fmul.d fa5,fa4,fa5	{-72(s0)}	{-72(s0)}
7	8004190: fd043707 fld fa4,-48(s0)	{-72(s0)}	{-48(s0)}
8	8004194: 12f777d3 fmul.d fa5,fa4,fa5	{-48(s0), -72(s0)}	{-48(s0), -72(s0)}
9	8004198: fcf43827 fsd fa5,-48(s0)		
10	8004184: fb843707 fld fa4,-72(s0)	{-48(s0)}	{-72(s0)}
11	8004188: fb843707 fld fa5,-72(s0)	{-48(s0), -72(s0)}	{-72(s0)}
12	800418c: 12f777d3 fmul.d fa5,fa4,fa5	{-72(s0)}	{-72(s0)}
13	8004190: fd043707 fld fa4,-48(s0)	{-72(s0)}	{-48(s0), -72(s0)}
14	8004194: 12f777d3 fmul.d fa5,fa4,fa5	{-72(s0)}	{-48(s0), -72(s0)}
15	8004198: fcf43827 fsd fa5,-48(s0)	{-48(s0), -72(s0)}	{-48(s0), -72(s0)}

Figure 8: Excerpt from a RV32IMFD ISA binary that calculates the even powers of an input variable x . The compiler uses different registers for the calculation of $x*x$, which shows that we cannot rely on the register identities to identify correlations of the source registers of the arithmetic instructions. Laplace instead tracks the ancestor main memory addresses that have contributed to calculation of the value of each register. It performs autocorrelation-tracked execution of the `fmul.d` instructions shown in green (instruction indices 5, 11, and 13).

At instructions with indices 3 and 4, Laplace sets the ancestor addresses of registers `fa4` and `fa5` to `-72(s0)`, where `s0` is a saved integer register. Since `fa4` and `fa5` have identical ancestors, Laplace performs an autocorrelation-tracked multiplication at instruction 5 ($x*x$). At instruction 6, Laplace loads `fa4` from an address that overwrites the ancestor addresses of `fa4`. At instruction 7 Laplace creates the set of ancestor addresses of destination register `fa5` as the union of the ancestors of source registers `fa4` and `fa5`. At instruction 8, Laplace stores the ancestors of `fa4` in main memory. Laplace uses a microarchitectural structure, indexed by a main memory address to map the destination address `-48(s0)` to the address of its ancestor set in main memory. During the second loop iteration, the instructions indexed 9–11 have the same effect as the respective instructions in the first loop iteration. At instruction 12, Laplace updates the ancestor set of `fa4` using the ancestor set that corresponds to address `-48(s0)`. At instruction 13, registers `fa4` and `fa5` share a common ancestor (`-72(s0)`). Laplace performs an autocorrelation-tracked multiplication which corresponds to $xPower * (x*x)$.

Distributional co-ALU: Figure 9 shows the internals of the distributional co-ALU. Using the information stored in the on-chip memory of active per-register ancestors (Figure 4) the co-ALU determines whether the source registers have common ancestor addresses in the autocorrelation detection unit. The unit sets the output signal `srcOperandsAutocorrelationSignal` if it detects autocorrelation. If not set, then the co-ALU executes Algorithm 1 (or a variant according to the instruction). The smaller ALU components perform the intended arithmetic operation on the support positions of the distributional source registers. For all arithmetic operations, the co-ALU multiplies the masses of the distributional source registers. An asserted `srcOperandsAutocorrelationSignal` signal disables non-pointwise units and the co-ALU performs a point-to-point operation on the source positions and masses. The buffered output

Figure 9: The distributional co-ALU takes as input two distributional source registers and their ancestor addresses. The autocorrelation detection unit exhaustively checks for common ancestors and sets signal `srcOperandsAutocorrelationSignal`, if any. The co-ALU executes Algorithm 1 (or the variant for the intended arithmetic operation). The `srcOperandsAutocorrelationSignal` controls which functional units of the co-ALU will operate on the source operands. A unit like that of Figure 6, converts the result Dirac mixture to TTR.

of both autocorrelation-tracked and uncorrelated-operand calculations is a Dirac mixture. A conversion unit like that of Figure 6, converts the Dirac mixture to TTR and forwards it to the distributional register file.

For the correct tracking of uncertainty in applications, Laplace needs to propagate distributional information through all floating-point operations. However, not all operations have the same distributional execution overhead. If one or both of the distributional source registers do not contain distributional data (i.e., are particle values) then the co-ALU performs a scaling of the support position or no operation, respectively.

4 INJECTING AND EXTRACTING UNCERTAINTY INFORMATION

To initialize distributional information in Laplace and to allow applications to query statistics of the distributional information, we design and implement a new (i.e., non-standard) extension to the RISC-V instruction set: the **xua** RISC-V ISA extension². The **xua** extension consists of nine new instructions whose only purpose is to provide a way to initialize (i.e., inject) or query (i.e., extract) uncertainty information without exposing the underlying microarchitecture-level distribution representation to programs. These instructions *are not required* for tracking uncertainty in the Laplace microarchitecture. Table 1 shows the double-precision floating-point variants of our **xua** RISC-V ISA extension instructions. The single-precision floating-point variants follow the same design principles. In the following text, we show the instructions **colored** and unless specified, use floating-point registers.

Injecting distributional information into the microarchitecture: The **xualdfs** instruction creates a distribution representation from in-memory samples. Integer register **rs1** holds the address of the first of N continuous values in the main memory of Laplace. Integer register **rs2** holds N . The instruction stores the distribution representation into the distributional part of the microarchitectural (i.e., hidden) hardware structure associated with **rd** and the mean value of the input samples in the ISA-visible value of **rd**.

Representation-independent access of distributional information: Instructions **xuanthmoment**, **xuanthmode**, and **xuanthantimode** process the distribution representation associated with register **rs1** and store its N th moment, N th mode, or N th anti-mode in register **rd**, respectively. Integer source register **rs2** stores N . Instructions **xuasupportmin** and **xuasupportmax** store the minimum and maximum value of the distribution representation support associated with **rs1** to **rd**. Instruction **xuaprobabilitygt** calculates the probability of the random variable with distributional information associated with **rs1**, being greater than the real value stored in **rs2**, and stores it in **rd**. Instruction **xuasample** samples the distributional information associated with register **rs1** and stores the particle sample in **rd**. For all **xua*** instructions of Table 1 we use the **custom0** opcode reserved for custom instructions [51].

Mixture instruction: Instruction **xuamixture** (R4-type) takes as input two floating-point registers **rs1** and **rs2**, and a ratio particle value from register **rs3** and sets the distributional information of the register **rd** as the mixture of the distributional information of **rs1** and **rs2** computed as $rs3 * rs1 + (1 - rs3) * rs2$. For this instruction we used the **custom1** opcode [51].

Extension of the semantics of RV32IMFD ISA instructions: We extend the semantics of the **fld** instruction to load distribution representation data associated with an address of the main memory of Laplace into the microarchitecture’s non-ISA-visible distribution registers. We extend the semantics of the **fsd** instruction to store the distribution data of the microarchitecture’s non-ISA-visible distribution registers into the the main memory of Laplace.

For non-arithmetic floating-point instructions, such as conversion, sign-injection, and **classify**, Laplace appropriately propagates distributional information from source to destination registers. For

Table 1: Listing of the R-type instructions of the double-precision **xua extension of the 32-bit RISC-V ISA. Laplace uses these instructions only to insert and query uncertainty. Uncertainty-tracking happens transparently without the need for any explicit instructions.**

funct7	rs2	rs1	funct3	rd	opcode	Instruction
00111100	rs2	rs1	010	rd	0001011	xualdfs.d
11111100	rs2	rs1	001	rd	0001011	xuanthmoment.d
11111100	rs2	rs1	010	rd	0001011	xuaprobabilitygt.d
11111100	rs2	rs1	011	rd	0001011	xuanthmode.d
11111100	rs2	rs1	100	rd	0001011	xuanthantimode.d
11111100	rs2	rs1	101	rd	0001011	xuasupportmin.d
11111100	rs2	rs1	110	rd	0001011	xuasupportmax.d
11111100	rs2	rs1	111	rd	0001011	xuasample.d

Table 2: Function definitions included in *libUncertain* library. The function names for double-precision values all have the prefix `libUncertainDouble` truncated to save space.

```
double DistFromSamples(double *samples, size_t n);
double NthMoment(double value, size_t n);
double NthMode(double value, size_t n);
double NthAntiMode(double value, size_t n);
double SupportMin(double value);
double SupportMax(double value);
double ProbabilityGT(double value, double cutoff);
double Sample(double value);
double Mixture(double a, double b, double aScale);
```

example, in the case of a negation using the **fsgnj** instruction, Laplace stores the distributional information that correspond to the ISA-visible source register to the distributional information that correspond to the ISA-visible destination register. Laplace then negates the support position values of the distributional information that correspond to the ISA-visible destination register. For compare instructions Laplace compares the particle values of the source registers to set the particle value of the destination register.

Control flow of binaries in Laplace: Binaries running on Laplace follow the conventional control flow. Applications can make a control flow decision based on distributional information by querying the distributional information of the registers of Laplace using the instructions of Table 1.

Application development: We extend the GNU Compiler Collection (GCC) for application development. Our compilation infrastructure includes: ① GCC 8.2.0 for the RV32IMFD ISA; ② Binutils 2.32 for the RV32IMFD ISA, extended to support **xua** ISA extension; ③ Newlib 2.5.0.20170922 as the standard C library implementation; ④ Library *libUncertain*, which provides C function wrappers for exposing the **xua** ISA extension instructions to applications.

Laplace treats any floating-point C variable as a variable with distributional information, without any changes in the GCC compiler front-end. Developers interface with the **xua** ISA through the *libUncertain* library, which provides C function declarations that we implement using RISC-V **xua** ISA instructions. Table 2 summarizes the functions that operate on double-precision floating-point C variables. We also provide the respective single-precision functions.

² The RISC-V instruction set manual [51] suggests the usage of “X” as the first letter of a non-standard ISA extension.

5 EVALUATION

We evaluate Laplace using a RISC-V microarchitectural emulator [45] modified to implement the Laplace microarchitecture and the xua RISC-V extension. Benchmarks use `libUncertainDoubleDistFromSamples` to initialize variables with distributional information from samples in data files. We use the mean of these samples as the input of the benchmark running on the conventional microarchitecture (an unmodified RV32IMFD RISC-V processor); the baseline of our accuracy and performance evaluations are repeated Monte Carlo iterations of the benchmarks running on this unmodified RV32IMFD RISC-V processor. Each run of the benchmarks uses point values sampled from a distribution (detailed further in Sections 5.1–5.2). We execute Monte Carlo simulations of 30 000 iterations and aggregate the results to obtain the output distribution. We refer to this as the Monte Carlo evaluation of the benchmarks.

Accuracy evaluation: Our accuracy metric is the *Wasserstein distance* [39] between the result of a single execution of the benchmark on Laplace using distributional inputs and the output of the Monte Carlo evaluation of the benchmark. To improve the interpretability of these distances, we normalize them by the Wasserstein distance between the conventional microarchitecture output and the Monte Carlo output. The normalized Wasserstein distance conveys the accuracy improvement factor of Laplace over the conventional microarchitecture. A value less than 1.0 means Laplace is more accurate than the conventional system.

Performance evaluation: To evaluate the performance improvement of Laplace compared to the Monte Carlo evaluation of the same benchmarks running over the baseline RV32IMFD RISC-V processor, we examine the number of ISA instructions that each method requires to achieve the same accuracy. For each benchmark, we calculate the number of Monte Carlo executions over the RV32IMFD RISC-V processor that are required to achieve the Wasserstein distance that Laplace achieves for each benchmark in a single execution.

Overview of benchmarks: Table 3 shows the benchmarks we used to evaluate Laplace. The benchmarks fall into three categories; ① microbenchmarks that evaluate primitive arithmetic operations, ② benchmarks from relevant works used for comparative evaluation, and ③ application benchmarks that evaluate the efficacy of Laplace in domains that deal with uncertainty. We used the C programming language to develop the benchmarks and compiled them for the RV32IMFDXua RISC-V ISA extension.

Results overview: Figure 10 shows the normalized Wasserstein distance that Laplace achieves for each benchmark, for each representation type (RQHR, TTR) and representation size (4 to 256). Figure 11 shows the dynamic instruction count improvement factor of Laplace over a Monte Carlo evaluation that achieves the same normalized Wasserstein distance (Section 5.4).

In general, RQHR leads to less accurate results, especially for small representation sizes, due to the use of uniform binning. By contrast, the ability of TTR to prioritize regions of high probability density makes it accurate in most of the benchmarks, even at small representation sizes. Across the 21 benchmarks, TTR achieves a 3.5× accuracy improvement on average compared to the conventional architecture, achieving more than 27× accuracy improvement in the `quantum-avqe` benchmark at a TTR size of 64. Laplace achieves these

Table 3: Overview of benchmarks evaluated on Laplace.

Benchmark Name	Abbreviation
(Microbenchmarks)	
Floating-point addition	<code>micro-add</code>
Floating-point subtraction	<code>micro-sub</code>
Floating-point multiplication	<code>micro-mul</code>
Floating-point multiplication with autocorrelation	<code>micro-ac-mul</code>
Floating-point division	<code>micro-div</code>
(A+B)/(A-B) calculation	<code>micro-schlieren</code>
Floating-point matrix multiplication [30]	<code>micro-matrix-mul</code>
Daisy from Fbench [4, 14]	<code>fbench-daisy</code>
Herbie from FPbench [14, 36]	<code>fbench-herbie</code>
FPTaylor-real2float from FPbench [14, 44]	<code>fbench-real2float</code>
(Comparisons)	
Product/quotient of Cauchy dist. (PaCAL [22])	<code>pacal-cauchy</code>
Products of Gaussian distributions (PaCAL [22])	<code>pacal-gp</code>
Thermal expansion coefficient (PaCAL [22])	<code>pacal-tc</code>
Thermal expansion coefficient (NIST-UM [27])	<code>nistum-tc</code>
Dynamic viscosity of solution (NIST-UM [27])	<code>nistum-dv</code>
(Applications)	
Speed estimation using GPS data [7]	<code>gps-speed</code>
Bosch BME680 conversion routines	<code>sensors-bme680</code>
MNIST [8] to positional number	<code>mnist-number</code>
Dislocation model from Brown and Ham [9]	<code>brown-ham</code>
Finite elements method 1D [10]	<code>fem-1d</code>
Accelerated variational quantum eigensolver [47]	<code>quantum-avqe</code>

accuracy improvements with a single execution of the application whereas the Monte Carlo simulation requires 30 000 executions.

5.1 Microbenchmarks accuracy evaluation

Arithmetic benchmarks: We use `micro-add`, `micro-sub`, `micro-mul`, and `micro-div` to assess the accuracy of Laplace for simple arithmetic operations. The inputs are samples from two Gaussian distributions with mean values 1.0 and 5.0 and variance of 1.0. For these simple arithmetic operations Laplace achieves an average accuracy improvement of 4.6× compared to the conventional architecture.

We use the expression $\frac{a+b}{a-b}$ (`micro-schlieren`) and a repeated multiplication (`micro-ac-mul`) to evaluate the effect of autocorrelation tracking in Laplace. The input samples for a and b follow Gaussian distributions with mean values 5.0 and -5.0, respectively, and variance 10.0. Benchmark `micro-ac-mul` uses the same input as `micro-mul`. These benchmarks show average accuracy improvement of 1.08× (`micro-schlieren`) and 4.7× (`micro-ac-mul`), up to 98× for RQHR-256 in `micro-ac-mul`. We use `micro-matrix-mul`, a naive multiplication of two 8×8 matrices filled with uncertain data derived from a Gaussian distribution with mean value 1.0 and variance 1.0, for an intensive assessment of the above properties. Laplace leads to improved accuracy of up to 87× for TTR-256.

FPBench: We use the FPbench [14] suite to evaluate Laplace on complicated floating-point arithmetic operations. Benchmarks `fbench-daisy` [4], `fbench-herbie` [36], and `fbench-real2float` [44] contain numerical formulas for common mathematical functions, linear-algebraic operations, physics equations, and complex functions. For each one of the formulas we generate input samples by sampling a Gaussian distribution. To calculate the parameters of each source Gaussian distribution we take into account the input

Figure 10: Laplace using TTR achieves up to 119× improvement in accuracy compared to the conventional system (3.5× on average). The vertical axis gives the normalized Wasserstein distance, quantifying how close the result of using distributional representations with Laplace is, to the Monte Carlo output. A value less than 1.0 (below the red line) indicates Laplace is better than the conventional architecture. We used RQHR and TTR representations with sizes ranging from 4 to 256.

Figure 11: Dynamic instruction count improvement factor (vertical axis) of Laplace over a Monte Carlo simulation that achieves the same Wasserstein distance, for each benchmark and representation (horizontal axis). The red line at 1 indicates the baseline (no improvement). Laplace achieves average dynamic instruction count reduction of 2076× and a maximum reduction of 21343×.

preconditions of each formula [14]. In *daisy*, Laplace with representation sizes greater than 16 achieves 1.5× (RQHR-64) to 2.5× (TTR-256) better accuracy compared to the baseline. In *herbie* and *real2float*, TTR achieves 2.4–5× better accuracy.

5.2 Accuracy comparison to state of the art

PaCAL [22]: We use three benchmarks from the PaCAL work: Cauchy distribution generation (*pacal-cauchy*), product of Gaussians (*pacal-gp*), and calculation of thermal expansion coefficient (*pacal-tc*). We calculate the thermal expansion coefficient of a rod using its initial length L_a , final lengths L_b and the temperature difference ΔT . We assume that L_a is uniformly distributed on $[9, 10]$, L_b is uniformly distributed on $[11, 12]$ and ΔT follows a Gaussian distribution with mean value 2.0 and variance 1.0. In *pacal-cauchy*, TTR manages up to 1.5× better Wasserstein distance, while RQHR offers no improvements. In *pacal-gp* RQHR and TTR show significant improvements over the baseline, averaging 2.8× accuracy improvement, up to more than 98× accuracy improvement for TTR-256. In *pacal-tc* RQHR shows 1.25× accuracy improvement on average, and TTR achieves on average 1.5× improved accuracy (up to 2.9× for TTR-256).

NIST Uncertainty Machine [27]: Using the NIST Uncertainty Machine web interface³, we execute two applications: calculation of thermal expansion coefficient and dynamic viscosity. We compare 1 000 000 Monte Carlo samples of the Uncertainty Machine output against the Laplace output. *Thermal Expansion Coefficient:* Like *pacal-tc*, but models the input parameters as Student's-t random variables [38] with 3 degrees of freedom [27]. Laplace achieves up to 4.6× better accuracy in this benchmark. *Dynamic Viscosity:* Calculates the dynamic viscosity of a sodium hydroxide solution in water by modeling the inputs as independent Gaussian variables [27]. Laplace achieves up to 119× better accuracy.

5.3 Accuracy evaluation for complete applications

Speed estimation using GPS data [7]: In human activity tracking, it is common to use GPS coordinates to estimate a person's movement speed. For this benchmark we use a public dataset of human activity with timestamped GPS coordinates and GPS accuracy values [21]. We adopt the methodology of Bornholt et al. [7] to derive a distribution of the GPS coordinates around a pair of GPS longitude and latitude values of the human activity dataset. We use the distributions of GPS coordinates to calculate the distribution of

³<https://uncertainty.nist.gov/>

Figure 12: Sampling the BME680 ADC values results in distributions even for small time intervals (Figure 12a). Laplace can track these distributions through the manufacturer’s conversion routines and estimate the distribution of temperature (Figure 12b). The average estimate of the conventional architecture output falls in a zero-probability range.

the estimated speed between two GPS points (*gps-speed*). Compared to the conventional microarchitecture, the average improvement of the TTR in terms of Wasserstein distance from the Monte Carlo evaluation output is 1.9×

Bosch BME680 sensor conversion routines: The BME680 by Bosch is a temperature, pressure, and humidity sensor. Bosch provides routines for converting raw ADC values to meaningful measurements using 20 sensor-specific calibration constants. The *sensors-bme680* benchmark evaluates the effect of noise in the ADC measurements and uncertainty in the calibration constants on the calibrated temperature, pressure, and humidity outputs of official commercial calibration firmware code provided by Bosch [43]. Figure 12a shows the noisy temperature ADC measurements and Figure 12b their effect on the converted temperature. In this application, aleatoric uncertainty leads to a bimodal distribution of the output result. The conventional architecture output falls in a zero-probability range, meaning that average-filtering the noisy ADC leads to a wrong result. Compared to the conventional microarchitecture, Laplace achieves on average 5.7× (up to 10× with TTR-64) smaller Wasserstein distance from the Monte Carlo output.

MNIST to positional number: Section 1.1 describes the *mnist-number* benchmark which converts 8-digit sequences of MNIST images to plausible interpretations as decimal numbers. Figure 2(a) shows the distribution of labels assigned by 50 observers in a user study we conducted for this work. Laplace’s accuracy improvement compared to the conventional microarchitecture averages 17.6× (up to 41× with TTR-256).

Brown-Ham dislocation model: This benchmark calculates the cutting stress of an alloy precipitate using the Brown-Ham dislocation model [1, 9]. Anderson et al. [1] provide empirical value ranges for the inputs of the dislocation model. We assume that the inputs of the dislocation model follow a uniform distribution across these ranges. On average, in comparison to the conventional microarchitecture Laplace achieves 3.83× (up to 9.3×) higher accuracy with respect to the Monte Carlo simulation.

One-dimensional finite element model: Section 1.1 describes an one-dimensional finite element model to calculate the

Figure 13: Laplace is able to accelerate the QPE optimization of AVQE without using sampling to find better estimates of the posterior distribution of ϕ (using TTR-64).

extension of a beam when the model parameters have epistemic uncertainty. Benchmark *fem-1d* implements that model. We assume a uniform distribution of the Young’s modulus input parameter of the benchmark and fixed particle values for the rest of the input parameters. Laplace achieves an average accuracy improvement of 2× compared to the conventional microarchitecture.

Accelerated variational quantum eigensolver: This quantum algorithm calculates ground states of a quantum system Hamiltonian, H [13, 47]. The *quantum-avqe* benchmark tries to find the quantum state $\psi(k)$ that minimizes $\langle \psi(k) | H | \psi(k) \rangle$. Typically, this uses rejection sampling to calculate $P(\phi|E)$ from measurements of $P(E|\phi)$ from a quantum phase estimation (QPE) circuit [52]. With Laplace we can explicitly calculate the posterior from $P(E|\phi)$ and calculate $P(\phi)$. Over multiple iterations on Laplace, the algorithm can calculate better estimates of $P(\phi|E)$. Figure 13a shows the input of the *quantum-avqe* benchmark which we generate using samples from a uniform distribution. Figures 13b and 13c show the decrease in the posterior variance $P(\phi)$ after two and five iterations of the benchmark, respectively. Laplace achieves an accuracy improvement of 6.23× compared to the conventional microarchitecture (up to 41.3× with TTR-256).

5.4 Performance improvement over Monte Carlo

Figure 11 shows the dynamic instruction count ratio between the Monte Carlo and Laplace for all examined representation types and sizes. Except for RQHR-4, Laplace performs at least one order of magnitude better. On average, the accuracy that Laplace achieves in a single execution requires 2076× fewer instructions compared to the number of instructions that repeated Monte Carlo simulations require to achieve the same accuracy. The number of instructions that Laplace requires for the execution of a benchmark is independent of the size of the distributional representation. As the representation size increases, the accuracy of a single execution of Laplace increases and a greater number of Monte Carlo iterations are necessary to achieve the same level of accuracy. Thus, at higher representation sizes Laplace achieves even higher performance compared to Monte Carlo simulations, reaching up to 21 343× improvement.

Table 4: Cardinality of the largest set of ancestor addresses required by Laplace to evaluate each benchmark of Table 3.

Benchmark	Ancestors	Benchmark	Ancestors
micro-add	0	pacal-tc	9
micro-sub	0	nistum-tc	4
micro-mul	0	nistum-dv	6
micro-ac-mul	5	gps-speed	5
micro-div	0	sensors-bme680	25
micro-schlieren	6	mnist-number	2
fpbench-daisy	14	brown-ham-model	4
fpbench-herbie	4	fem-1d	23
fpbench-real2float	13	quantum-avqe	5
pacal-cauchy	9	matrix-multiplication	20
pacal-gp	9		

5.5 Overhead of autocorrelation tracking

To quantify the hardware overhead of our correlation tracking mechanism we measure the largest set of ancestor addresses that Laplace stores during the execution of each benchmark. Table 4 shows that the maximum number of required ancestors per benchmark is bounded and unaffected by the type or the size of the distributional representation. Thus, an on-chip memory of 32 ancestors per architectural register is large enough for all the benchmarks.

6 RELATED RESEARCH

Existing software approaches for representation and arithmetic operations of random variables frequently rely on Monte Carlo or first-order approximations. The NIST Uncertainty Machine [27] is a software tool for math operations on random variables that follow parametric distributions. It uses the first two centralized moments of the distribution [27] or Monte Carlo simulations. Manousakis et al. [30] represent random variables using their first two moments (mean value and variance) and their covariances. They use first-order Differential Analysis to approximate continuous and differentiable non-linear operations and use Monte Carlo simulation as fallback. PaCAL [22] carries out arithmetic operations on independent real-valued random variables by approximating their probability density functions with Chebyshev polynomials. This limits the density classes it can track through computation. In contrast to Laplace, PaCAL cannot handle empirical distributions.

Uncertain<T> [6, 7], a framework that represents random variables using user-specified sampling functions, and ProbZelus [3], a language for reactive probabilistic programming, use Bayesian networks to reduce the amount of dynamically-evaluated expressions, an optimization that Laplace does not implement yet. To the best of our knowledge, Laplace is the first microarchitecture for uncertainty tracking below the ISA abstraction. This allows applications to track and reason about uncertainty using unmodified programming languages and binaries.

Other hardware-based approaches adapted fuzzy logic [54] into microprocessor designs [12, 34, 41, 42, 48–50]. The fundamental limitation of this approach is that it uses only Bernoulli variables [18, §4], and, thus, is not suitable for the types of problems that arise when dealing with uncertain data [5, §2.4].

Attempts in approximate computing, like Truffle [16], Parrot [17], Green [2], and Flikker [29], introduce uncertainty to the computation, by using approximations, to gain performance or to save energy. Laplace does not aim to approximate input data or the outcome of computations. Laplace introduces new microarchitectural units for robust and accurate uncertainty tracking.

Particle filters [15, 28] track uncertainty but are more compute-intensive than Laplace due to much larger uncertainty representations. Analogous to the concept of our microarchitectural approach, previous work has investigated accelerating particle filters in the microarchitecture [19, 20, 37] or with SIMD operations [11, 31–33]. The unscented transformation [24, 46] achieves uncertainty tracking with low resource usage and can in principle be an alternative technique for representing and tracking uncertainty inside the Laplace microarchitecture.

7 CONCLUSIONS

This article presents Laplace, a microarchitecture for tracking uncertainty using machine representations of probability distributions, and two new methods for such representations. Laplace uses these methods for compact, in-processor storage of distributional information as well as propagation of distributional information through arithmetic operations. We show how the Laplace RISC-V ISA extension, *xua*, allows programs to inject uncertainty information into a microarchitecture and to extract it, without the microarchitecture revealing the underlying microarchitectural distribution representation. We present an overview of the Laplace microarchitecture and show the necessary microarchitectural components for Laplace to achieve uncertainty tracking below the ISA boundary. These include modules for initializing distributional representations using samples stored in the main memory of Laplace, units that track correlations between representations of random variables, and units that perform arithmetic operations on distribution representations.

We implement Laplace as the microarchitecture for a RISC-V RV32IMFD processor and evaluate its performance using a suite of 21 benchmarks. The results lead to two important insights. First, using the Monte Carlo simulation as the golden standard for uncertainty tracking, Laplace improves the uncertainty tracking accuracy of conventional computing systems and outperforms relevant state-of-the-art software methods. Second, the accuracy of uncertainty tracking that Laplace achieves in a single run requires more than three orders of magnitude fewer instructions compared to the instructions that Monte Carlo simulation needs to achieve the same level of accuracy. These insights validate designing uncertainty tracking as a microarchitecture feature, abstracted through conventional ISA semantics. Challenging research directions open up, including improving the in-processor distribution representations, correlation tracking between random variables in computation, and efficient hardware implementations of Laplace.

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Table 5: Directory path of each benchmark in the provided artifact, relative to artifacts-benchmarks-source-code/src/.

Benchmark Abbreviation	Benchmark subdirectory
micro-add	micro-benchmarks/double-add/
micro-sub	micro-benchmarks/double-subtract/
micro-mul	micro-benchmarks/double-multiply/
micro-ac-mul	micro-benchmarks/double-multiply-autocorrelation/
micro-div	micro-benchmarks/double-divide/
micro-schlieren	micro-benchmarks/schlieren/
micro-matrix-mul	linear-algebra/matrix-multiplication/
fpbench-daisy	numerical-methods/FPbench-benchmarks/daisy/
fpbench-herbie	numerical-methods/FPbench-benchmarks/herbie/
fpbench-real2float	numerical-methods/FPbench-benchmarks/fptaylor-real2float/
pacal-cauchy	comparative-evaluations/UPP-vs-PaCal/
pacal-gp	comparative-evaluations/UPP-vs-PaCal/
pacal-tc	comparative-evaluations/UPP-vs-PaCal/
nistum-tc	comparative-evaluations/UPP-vs-NISTUM/benchmark/thermal-expansion-coefficient/
nistum-dv	comparative-evaluations/UPP-vs-NISTUM/benchmark/dynamic-viscosity-solution/
gps-speed	comparative-evaluations/Speed-Estimation-using-GPS-data/
sensors-bme680	sensor-interfacing/BME680-conversion-routines/
mnist-number	numerical-methods/MNISTtoPositionalNumber/
brown-ham	numerical-methods/Brown-and-Ham-materials-precipitate-cutting-dislocation-model/
fem-1d	numerical-methods/FEM1D/
quantum-avqe	quantum-computing/accelerated-variational-quantum-eigensolver/

A ARTIFACT APPENDIX

We provide the source code of our evaluated benchmarks as publicly available artifacts in Zenodo. For the artifact evaluation we provided a remote virtual machine with all necessary software dependencies, including the Laplace SDK, where reviewers reproduced the evaluation results of Section 5. The provided artifact includes scripts that enabled the reviewers to automate the evaluation of benchmarks. The cloud-accessible version of the Laplace microarchitecture is available at <https://signaloid.io>.

A.1 Artifact check-list (meta-information)

- **Program:** We provide the source code for all the benchmarks detailed in Section 5. Their compressed size is 29.1 MB and their uncompressed size is 61 MB.
- **Compilation:** We make use of ① the GCC compiler 8.2.0 for the RV32IMFD ISA; ② Binutils 2.32 for the RV32IMFD ISA, extended to support xua ISA extension; ③ Newlib 2.5.0.20170922 as the standard C library implementation. We provided this cross-compilation infrastructure as part of the provided remote machine for the evaluation of artifacts.
- **Binary:** We do not provide the binaries of the examined benchmarks. Reviewers were able to build all benchmark binaries using the cross-compilation infrastructure on the remote machine that we provided. The artifacts include a script that automated the building and evaluation of all benchmarks (`exec-MICRO-2021-Artifact-Evaluation-benchmarks.sh`).
- **Data set:** For each benchmark we have created different input datasets. We provide these datasets as part of the archived benchmarks source code. Each benchmark directory contains an `inputs/` directory where we store the input dataset of the benchmark.
- **Hardware:** We do not require any specific hardware.

- **Run-time state:** The Laplace emulator creates a standard initial state for the execution of our benchmarks. In general, the evaluation of our artifacts is not sensitive to run-time state.
- **Execution:** There are no specific execution conditions for the execution of our benchmarks. The evaluation of all our benchmarks for tracking of distributional information requires approximately one hour. The execution time of the Monte Carlo evaluation of our benchmarks depends on the user-defined number of Monte Carlo iterations. With the default number of Monte Carlo iterations in the provided execution scripts the execution takes 10–12 days.
- **Output:** Each execution of a benchmark on the Laplace emulator can generate two kinds of output: (1) The standard output of the benchmark. This can be the outcome of standard C standard library calls, e.g., `printf`. It can also be the output of the distributional representation of a variable using `libUncertainDoublePrint`. This library prints the distributional representation of its source variable in `stdout`. This print includes a vertical listing of the support positions and probability masses of the source variable with distributional information. It also contains a graphical representation of the probability masses magnitude; (2) The emulator can use the DWARF debugging information of its input binary to trace the evolution of values of a benchmark source code expression during the execution of the benchmark.
- **Metrics:** The Laplace emulator stores multiple pieces of information about the execution of a benchmark. Relevant to the evaluation of Section 5 are (i) the number of dynamic instruction which were executed for the evaluation of a benchmark (both for uncertainty tracking and Monte Carlo simulation) and (ii) the maximum number of ancestors stored for any register during the execution of an experiment.
- **Amount of time needed to prepare workflow (approximately):** The time needed by the artifact evaluators to prepare

the workflow was low because we provided access to a remote machine that included all the necessary software dependencies.

- **Publicly available and archived:** Our artifacts are publicly available in Zenodo at <https://doi.org/10.5281/zenodo.5150148>.
- **Code licenses:** We provide our source code under the MIT license.
- **Data licenses:** We provide our input benchmark data under the MIT license.
- **Workflow framework used:** We do not use a workflow framework. We provide bash scripts that automate the evaluation process.

A.2 Artifact description

At the top directory level (`artifacts-benchmarks-source-code`) the artifact contains two directories: `scripts/` and `src/`. The `scripts/` directory contains the scripts that automated the Laplace evaluation process for the reviewers. Under the `src/` directory we provide the C source code of the benchmarks that we detail in Table 3. Table 5 shows the path to each benchmark relative to the `artifacts-benchmarks-source-code/src/` directory. Each one of these subdirectories contains two directories: `benchmark/` and `inputs/`. The `benchmark/` directory contains the C source code of the respective benchmark. It also includes a `Makefile` for building and running the benchmark, as well as Laplace SDK configuration files (`m` files). The `inputs/` directory contains the input files of the respective benchmark. These inputs contain artificially-generated or empirical samples that we use to load the distributional representations for benchmarks running over the Laplace SDK. We randomly sample these inputs when we run the Monte Carlo simulation of each benchmark.

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