Modelling Social Capital: The Structural Hole Connections Game

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Abstract—*Social Capital* is considered as the value that an actor draws from the network. It is measured as an ability to bond with others (*bonding capital*) as well as an ability to form a bridge that connects otherwise disconnected actors or groups in the network (*bridging capital*). In order to model the social capital, the strategic nature of forming links needs to be considered and baked within the network formation models. In this paper, we develop a strategic network formation game named *Structural Hole Connections Game* (SHC) and define an associated allocation function that distributes the total network value to the actors in the network in such a way that captures their bonding and bridging social capital. Our proposed SHC game generalizes the utility function that models both the bonding and bridging capabilities of an actor with high social capital. We first analytically deduce the efficient and stable networks of the SHC game. Finally, we analyse a real-world social network of employees of an IT company and identify individuals with bonding and bridging social profiles.

I. INTRODUCTION

The value that an individual can derive from its position in the network structure forms the basis for individual's *social capital*. The benefits that a node receives from its embeddedness in the community through direct connections is known as *bonding capital* whereas the value obtained for being in the middle of groups forms the *bridging capital*.

The studies on social capital assessment work on already formed networks and fail to provide insights on how individuals with particular type of network structure ended up having certain type of capital. Due to their ability to affect individual's well-being, networks are important to understand in two ways: (i) how networks affect behaviour of individuals and (ii) which network structures are likely to emerge in certain environments. This highlights the importance of studying the network modelling process to understand why the networks form as they do. The study of empirical networks suggests, there are two complementary approaches to modelling the network formation. The first approach assumes the formation

of links by chance (a random process) [1] and second approach considers the link formation process as strategic choice [2], [3]. The example of strategic network is the Florentine Marriage network, analysis of which reveals how strategically linking with powerful families through marriages, Medici family rose to power [4]. By carefully studying the network of marriages, one can ask questions such as was it simply the chance Medici family came to have prestigious position or was it by choice and careful planning? Further, one may ask whether this network was optimal from Medici's perspective, from other families perspective, and from the perspective of well-being of whole society. It is only through the models of network formation that consider costs and benefits during the formation process, these questions can be answered.

A. Our contribution

In this work, we propose a strategic network formation game known as SHC whose payoff function is characterised by parametric combination of bonding and bridging values for a node. The game generalises and combines Connections model [2] (referred here as CONN) and a game for Structural hole [5] (referred here as KSH) from the state-of-the-art to take into account the value accruing to individuals in the network due to both their direct and indirect contacts (i.e. value attained through bonding); and value accruing due to acting as intermediaries between other individuals in the network (i.e. value attained through bridging). This work extends our previous work [6] in which we introduced a centrality (dubbed as *structural hole centrality*) measure to calculate bridging benefits of a node. Here, we formalise the SHC game for both bonding and bridging capital.

Here, first we theoretically derive the efficient and stable networks of the SHC game using pairwise equilibrium as notion of stability. With our analytical proofs we show network topology that emerge from SHC have triangles in addition to standard observed topology (i.e. empty, star, and complete) at pairwise equilibrium. Furthermore, we evaluate utility function of the SHC game using real-world enterprise social network and derive stable and efficient versions of the real network.

II. MODELLING SOCIAL CAPITAL WITH STRATEGIC GAMES

It transpires that the efficient networks of the CONN are either the empty network, the full network or a star network,

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ASONAM '21, November 8-11, 2021, Virtual Event, Netherlands

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ACM ISBN 978-1-4503-9128-3/21/11...\$15.00

<http://dx.doi.org/10.1145/3487351.3488365>

Fig. 1: Node 7 as a bridge

depending on the relationship between the cost and the path discount (see [2]). In particular, the efficient networks have no community structure. The bridging capital of actors comes from their ability to connect between different communities. For example, in Figure 1, the benefit accruing to node 7 in each of the displayed graphs is the same, as far as the CONN game is concerned. However, in Figure 1b node 7 is forming an important connection between two triads, while this is not the case in Figure 1a. With this distinction in mind, we extend beyond the symmetric CONN game, to consider a nodal benefit that is dependent on the end node, v , so that $\delta_{uv} = \delta^{d_{uv}-1} b_v$, where we define b_v as some benefit or value that is obtained through a connection to v^1 . More specifically, we define $b_v > 0$ to be a *structural* measurement of the neighbourhood of v , that captures v 's value as a gateway to community structure. Note that such a nodal benefit ensures the *anonymity* of the network value function, that is, that the value remains independent of the node labels. To particularise this idea, we define $b_v \propto t_v$ where t_v is the number of triangles that include v as a vertex. $\mathbf{b}$ is normalised so that, whenever $\mathbf{b} \neq 0$, it contains the same total benefit as the symmetric CONN game, where $\forall v, b_v = 1$. To ensure component additivity, we normalise over each connected component. Let $\mathcal{C}(v, g)$ be the connected component containing node v , such that $|\mathcal{C}(v, g)| = n_c$, then

$$b_v = \begin{cases} n_c \frac{t_v}{\sum_{u \in \mathcal{C}(v, g)} t_u} & \text{provided } \max_{u \in \mathcal{C}(v, g)} t_u > 0 \\ 0 & \text{otherwise} \end{cases}$$

With this benefit function, none of the paths in the graph in Figure 1a provide non-zero value, while all paths in Figure 1b that terminate at one of the triangles provide positive value to the network. In Section III, we study the efficient networks of this game.

A. The structural hole connections game (SHC)

In merging the perspectives of KSH and CONN, we consider that a source node on a connecting path retains some fraction $\gamma \leq 1$ of the value of the end-point of the path, while the remainder of the value, $(1 - \gamma)$, goes to the intermediaries. Thus, we allocate $100 \times \gamma\%$ of the value, as the CONN does, to the source of the connection and $100 \times (1 - \gamma)\%$ of the value, as the KSH does, to the intermediaries, where this value

is allocated equally to all intermediaries along all paths of each fixed length ℓ . To summarise, in the SHC game, a node obtains value from the network • by direct connections to other nodes; • by being the source of a length ℓ geodesic path to another node; • by being an intermediary on a length ℓ geodesic path to another node, where $1 < \ell \leq d_{\max}$, and $d_{\max}$ is a configurable maximum path length over which we allow network value to flow. This leads to a utility function of the form:

$$\mu = (A + \gamma S + (1 - \gamma)H)\mathbf{b} - A\mathbf{c}, \quad (1)$$

where A is the adjacency matrix and distributes the nodal benefits to directly connected nodes; $S = \{s_{uv}\}$ distributes nodal benefits to the sources of indirect paths; and $H = \{h_{uv}\}$ distributes the nodal benefits to intermediaries along indirect paths. These matrices are defined as (see [6] for more details),

$$s_{uv} = \sum_{\ell=2}^{d_{\max}} \delta^{\ell-1} \mathbb{1}(m_{\ell uv} > 0); \quad h_{uv} = \sum_{\ell=2}^{d_{\max}} \frac{\delta^{\ell-1}}{\ell-1} \sum_u f_{\ell uv},$$

where $m_{\ell uv}$ is the number of geodesic paths of length ℓ between u and v and $f_{\ell uv}$ is the fraction of all length ℓ geodesic paths between u and v that contain w .

III. EFFICIENT AND STABLE NETWORKS OF THE SHC

We identify the strongly efficient networks for the SHC using the triangular benefit function. The nodal benefit b_u is the normalised count of the number of triangles t_u incident on the node u , such that $\sum_u b_u = n$, where n is the number of nodes in the connected component. Summing over all u , the normalising constant is $n/3T$ where T is the total number of triangles in the component, i.e. $b_u = n \frac{t_u}{3T}$. Recalling that γ only determines the allocation of value to individuals in the network, we can focus on networks that maximise

$$\begin{aligned} \mu(g) &= \sum_{u,v} (a_{uv} + s_{uv}) b_v - c \sum_{u,v} a_{uv} \\ &= \frac{n}{3T} \left(\sum_v d_v t_v + \sum_{uv} s_{uv} t_v \right) - 2mc \end{aligned}$$

where d_v is the degree of node v , such that $\sum d_u = 2m$, and m is the total number of edges in the graph. First we consider networks of maximum value for a given fixed cost, determined by the number of edges m . In such graphs, all nodes not directly connected to a node i can be reached in a single hop and we can write:

$$\begin{aligned} \mu(G) &= (n/3T) \left(\sum_u d_u t_u + \delta \sum_u (n-1-d_u) t_u \right) \\ &= (n/3T) (1 - \delta) \sum_u d_u t_u + \delta n(n-1) \end{aligned}$$

Furthermore, $\sum_v d_v t_v = \sum_{t \in \mathcal{T}} (d_j + d_k + d_\ell)$ where $\mathcal{T}$ is the set of triangles in the network and $t = (j, k, \ell)$ has nodes j, k, ℓ as its vertices. Hence for a maximum value network it is necessary to maximise the average sum of the degrees over all triangles.

¹We normalise so that the value of a direct link is 1.

 Fig. 2: Value vs Number of edges ($n = 20$)

Lemma III.1. *The network $K_{n-k-1} + S_{k+1,r}$ has maximum value over all networks of m edges, where k is the maximum value such that $m < \binom{n}{2} - \binom{k}{2}$. Here $S_{n,r}$ represents a star graph connecting a hub node to r other nodes within a set of n nodes.*

Proof. After $n-1$ edges have been added to the original star, the highest-degree non-hub node is now fully connected to all other nodes and the graph, which may now be written as $K_2 + \bar{K}_{n-2}$, consists of two fully-connected nodes and $n-2$ other nodes, whose only incident edges connect them to the two fully-connected nodes.

In general, the process may be summarised as follows: given m edges, let $M = \binom{n}{2} - m$ be the number of edges to be removed from K_n . find the maximum $k > 0$ such that $\binom{k}{2} < M$ and form the network $K_{n-k-1} + \bar{K}_{k+1}$ by removing a clique K_{k+1} from K_n . Let $r = M - \binom{k+1}{2}$. Note that $r \leq k$. The maximum value network for m edges is $K_{n-k-1} + S_{k+1,r}$.

In Fig 2, the maximum value is plotted against number of edges for a graph of $n = 20$ nodes and three different costs. The peaks in the graph, highlighted by the vertical grid lines, correspond to the graphs $g_{nk} = K_{n-k} + \bar{K}_k$.

Theorem III.2. *The efficient networks of the SHC game are as follows:*

- 1) *The empty network, when $c > \frac{n+3}{6} + \frac{n-3}{3}\delta$;*
- 2) *The star network with an additional edge joining two non-hub nodes, when $\frac{n}{6}(1-\delta) \leq c < \frac{n+3}{6} + \frac{n-3}{3}\delta$;*
- 3) *The network $K_2 + \bar{K}_{n-2}$ when $\frac{n}{3(n-2)}(1-\delta) \leq c < \frac{n}{6}(1-\delta)$;*
- 4) *The full network, K_n , when $c < \frac{n}{3(n-2)}(1-\delta)$.*

 (a) $K_1 + S_{n-1,1}$ (b) $K_2 + \bar{K}_{n-2}$ (c) K_n
 Fig. 3: Non-empty Efficient Networks ($n = 8$)

 Fig. 4: Cost bounds vs k for pairwise stable $g_{nk} = K_{n-k} + \bar{K}_k$ networks. $\delta = 0.9$, $N = 1000$

Proof. The proof follows by computing the value of $g_{nk} = K_{n-k} + \bar{K}_k$. Note from Figure 2, for $k \leq n-2$, $\mu(g_k)$ is either strictly increasing or strictly decreasing, depending on c . Hence $\max_k g_{nk}$ occurs either at $k = n-2$ or $k = 0$. In fact

$$\mu(g_0) - \mu(g_k) = \binom{k}{2} \left(\frac{2n}{n+2k-2}(1-\delta) - 2c \right)$$

and it follows that $\mu(g_{n-2}) > \mu(g_0)$ when $c < \frac{n}{3(n-2)}(1-\delta)$. The remaining conditions follow by computing:

- $\mu(\bar{K}_n) = 0$
- $\mu(K_1 + S_{n-1,1}) = n(\frac{n+3}{3}(1-\delta) + (n-1)\delta - 2c)$
- $\mu(g_{n-2}) = \frac{2n^2}{3}(1-\delta) + n(n-1)\delta - 2(2n-3)c$
- $\mu(K_n) = n(n-1)(1-c)$

A. Pairwise Stable Networks

An efficient network is not necessarily pairwise stable. In fact, while efficiency is independent of the allocation function, pairwise stability is affected by how value is allocated. It is interesting to look at the stability of $g_{nk} = K_{n-k} + \bar{K}_k$, as these are networks of locally optimal total value, as a function of the total cost. That is, adding or removing an edge to g_{nk} results in a drop in the total value and hence is detrimental for the network as a whole. However, individual nodes may improve their own utility by modifying their local structure. We ask in this section, when is a g_{nk} network stable to such edge addition or removal. In appendix A, stability conditions for the graph $g_{nk} = K_{n-k} + \bar{K}_k$ are derived which can be summarised in the following theorem:

Theorem III.3. Pairwise Stability of g_{nk} :

The network $g_{nk} = K_{n-k} + \bar{K}_k$ is stable in the cost range $c \in [c_{\text{lower}}, c_{\text{upper}}]$ where $c_{\text{lower}}, c_{\text{upper}_1}, c_{\text{upper}_2}, c_{\text{upper}_3}$ are derived in appendices A, B, C respectively; and

$$c_{\text{upper}} = \min(c_{\text{upper}_1}, c_{\text{upper}_2}, c_{\text{upper}_3})$$

In Figure 4, the cost bounds for pairwise stability of g_{nk} are plotted as a function of k , for different settings of γ , which determines the proportion of indirect utility that is assigned to intermediary nodes. We note some features of this plot. Firstly, stability can be achieved for a higher range of cost, when k is large. That is, many nodes connected to a small common core is a stable structure in the SHC game. Furthermore, the range of costs narrows as the allocation of utility to intermediate nodes is increased.

Taking $k = \alpha n$ and limits as $n \rightarrow \infty$, with α fixed, we get,

$$\begin{aligned} c_{\text{lower}} &\rightarrow \frac{1 - \delta\lambda}{1 + 2\alpha} \\ c_{\text{upper}_1} &\rightarrow \frac{1 + \alpha - (1 - \alpha)\delta\lambda}{(1 - \alpha)(1 + 2\alpha)} \\ c_{\text{upper}_2} &\rightarrow \frac{\alpha\delta(2 - \lambda) - \alpha + (1 - \delta\lambda)}{(1 - \alpha)(1 + 2\alpha)} \\ c_{\text{upper}_3} &\rightarrow \frac{(1 + \alpha)(1 - \delta\lambda)}{(1 - \alpha)(1 + 2\alpha)} \end{aligned}$$

It may be checked that $c_{\text{upper}} = \min(c_{\text{upper}_1}, c_{\text{upper}_2}, c_{\text{upper}_3})$ is attained at $c_{\text{upper}} = c_{\text{upper}_2}$. In the limit,

$$c_{\text{upper}} - c_{\text{lower}} = 2 \frac{\alpha\delta(1 - \lambda)}{(1 - \alpha)(1 + 2\alpha)}$$

which implies that, for large n , this network is never stable, when $\gamma = 1$. Bridging capital is necessary in order to achieve stability on these networks. Furthermore, the greater the allocation of value to intermediary nodes, the larger the range of costs for which the network is stable.

IV. SHC APPLICATION : EMPLOYEES SOCIAL CAPITAL ASSESSMENT

We will illustrate an application of SHC game in the assessment of social capital for individuals in an enterprise social network. This network was generated in a study of workplace social capital of employees in an IT organization's software campus residing approx. 700 employees. To collect the ground truth social capital values individuals were asked to complete a questionnaire based on well known set of questions related to bonding and bridging capital in the study of [7]. The resulting ground truth social network had 114 (75% of 152 consenters) nodes and 426 directed edges. There were 38 consenters who did not complete social network questionnaire. For the social capital questionnaire, 54 individuals out of 114 in the social network also provided the responses to the social capital questions. From the responses to questions, an average of bonding and bridging social capital was determined by taking sum and average of responses to bonding and bridging questions (resp), i.e. $AvgBn = \frac{\sum(R_n, Q_{Bn})}{|Q_{Bn}|}$, and $AvgBr = \frac{\sum(R_n, Q_{Br})}{|Q_{Br}|}$. From the ground truth employee network, we take largest component which consists of 107 nodes and 361 edges. For each node in the network, we compute the payoff by executing the SHC game for $d = 5$, $\delta = 0.9$, and γ values from 0 to 1. Thus, we allow long paths up to length of 5 to impact on the node's payoff and discount

Fig. 5: Employee payoff's vs gamma

according to path length relatively slowly. The payoffs over $\gamma = [0, 1]$ is shown in Figure 5

It can be observed from the payoff plot that nodes fall into two categories. First group with payoffs decreasing, and the other with payoffs increasing w.r.t. increase in γ . To label nodes in our ground truth data, according to their respective payoff profile, we train a number of clustering algorithms (k -means, Agglomerative and Gaussian Mixture) and best performing method is used to group nodes into two groups. In particular, payoffs of nodes were calculated for 100 values of γ between 0 and 1. The Principal Component Analysis (PCA) was performed on payoffs feature vectors for dimensionality reduction. The output from the PCA was two components that can explain the 95% variance in payoffs and these were used to perform k -means clustering with ELBOW method to select optimal k with highest *Silhouette score* of 0.84 for $k = 2$. The clustering methods were evaluated using ground truth (that was manually annotated considering the responses to bonding and bridging questionnaire). Individuals with high $AvgBn$ than the $AvgBr$ were labelled as individuals with 'bonding' profile and vice versa. We use Fowlkes-Mallows index [8] - which is the geometric mean between of the precision and recall, for evaluation of different clustering methods which results k -means as best performing with score of 0.72 whereas Agglomerative and Gaussian Mixture methods scored 0.58 and 0.49 respectively. In Figure 6, profiles of two individuals are compared with ground-truth, where Ego2 is correctly classified as bridging and Ego5 as classified as bonding by the clustering method. The payoffs as a function of γ for both are opposite to each other. Ego2 has decreasing payoff and ego5 has increasing payoff as γ increases from 0 to 1. This aligns with theoretical analysis of the payoff function of our SHC as defined in equation 1 such that with lower bounds of γ give importance to bridging connections or indirect ties whereas higher values of γ weigh bonding connections or direct ties higher.

V. CONCLUSION

The paper has modelled the *binding* and *bridging* capital with a game-theoretic model. In particular, the paper presented

Fig. 6: SHCE profiles and the corresponding ego-networks for classifed nodes as Bonding (Ego=2) and Bridging Bridging(Ego=2)

the SHC game that generalises and combines the CONN and KSH network formation games. The SHC game is more generic in that it behaves like CONN game for certain combination of parameters and for another set of parameters, SHC game can be tuned like the KSH game. We have derived the conditions for efficient and stable networks for the SHC game. Furthermore, stable networks are produced through systematic simulations and interestingly in contrary to previous studies, the emerging stable networks of the SHC game have similar structure to the efficient networks. We finally evaluated the SHC payoff function for real-world enterprise network and observed that game achieves a based clustering achieves Fowlkes-Mallows index score of 0.72 in identifying individuals with bonding and bridging roles. As a future work, we aim to extend our SHC game for directed and weighted networks and analyse more real-world networks to benchmark SHC against existing models in the social and economic networks literature.

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APPENDIX

Given the symmetry of $g_{nk} = K_{n-k} + \bar{K}_k$, only the following three update cases need to be considered. 1) The addition of an edge joining nodes $v \in \bar{K}_k$ and $w \in \bar{K}_k$; 2) The removal of an edge joining nodes $v \in K_{n-k}$ and $w \in K_{n-k}$; 3) The removal of an edge joining nodes $v \in K_{n-k}$ and $w \in \bar{K}_k$.

Prior to an update, the following identities hold:

- $T = \frac{1}{6}(n-k)(n-k-1)(n+2k-2)$, $t_K = \frac{1}{2}(n-k-1)(n+k-2)$, $t_{\bar{K}} = \frac{1}{2}(n-k)(n-k-1)$
- $b_K = \frac{n(n+k-2)}{(n-k)(n+2k-2)}$, $b_{\bar{K}} = \frac{n}{n+2k-2}$
- $\mu_K = \frac{n(n^2-2k^2+kn+k-3n+2)}{(n-k)(n+2k-2)} + \delta(1-\lambda)\frac{n(k-1)k}{(n-k)(n+2k-2)}$, $\mu_{\bar{K}} = \frac{n(n+k-2)}{n+2k-2} + \delta\lambda\frac{n(k-1)}{n+2k-2}$

A. Update case 1

The addition of an edge joining $v \in \bar{K}_k$ to $w \in \bar{K}_k$ creates $(n-k)$ new triangles joining v and w to each of the $(n-k)$ nodes in K_{n-k} . Each node in K_{n-k} is consequently incident on one extra triangle, while v and w are each incident on all the newly formed triangles. Hence: • $T' = T + (n-k)$, $t'_K = t_K + 1$, $t'_{\bar{K}} = t_{\bar{K}}$, $t'_v = t'_w = t_{\bar{K}} + (n-k)$; • $\mu'_v = \mu'_w = (n-k)b'_K + b'_w + \lambda\delta(k-2)b'_{\bar{K}}$. The network g_{nk} is stable to edge addition between nodes in $\bar{K}_k$ whenever $0 > \mu'_v - c - \mu_{\bar{K}}$. Substituting for the utilities and simplifying, we obtain: $c_{\text{lower}} = \frac{n(n-3)(n-2)+k(n+2)-2k^2-\delta\lambda(-2k^2+k(n+6)+(n-4)(n+1))}{(n+2k-2)((n-k)(n+2k)-3n+8)}$

B. Update case 2

The removal of an edge connecting $v, w \in K_{n-k}$ results in the removal of the $(n-k-2)$ triangles that joined v and w to each other nodes in K_{n-k} , as well as the removal of the k triangles that joined v and w to each node in $\bar{K}_k$. Hence: • $T' = T - (n-2)$, $t'_K = t_K - 1$, $t'_{\bar{K}} = t_{\bar{K}} - 1$, $t'_v = t'_w = t_{\bar{K}} - (n-2)$; • $\mu'_v = \mu'_w = (n-k-2)b'_K + kb'_{\bar{K}} + \delta\lambda b'_w + \delta(1-\lambda)\frac{k(k-1)}{n-k}b'_{\bar{K}}$. The network g_{nk} is stable to edge removal between two nodes in K_{n-k} whenever $0 > \mu'_v - \mu_{\bar{K}} + c$. Substituting for the utilities and simplifying, we obtain: $c_{\text{upper}_1} = \frac{n(k^3-2(k+2)n^2+k((4-3k)k-4)n+2k((k-1)k+4)-1)+14n-12}{(n-k)(n+2k-2)(n(n-2)-2k^2+k(n+2)-6)} - \delta(1-\lambda)\frac{n6k(k-1)}{(n-k)(n+2k-2)(n(n-2)-2k^2+k(n+2)-6)} - \delta\lambda\frac{n(n+k-4)}{n(n-2)-2k^2+k(n+2)-6}$

C. Update case 3

The removal of an edge between $v \in K_{n-k}$ and $w \in \bar{K}_k$, results in the removal of the triangles that joined v and w to each of the $(n-k-1)$ other nodes in K_{n-k} . It does not affect the triangles incident on the other nodes in $\bar{K}_k$. Hence: • $T' = T - (n-k-1)$, $t'_K = t_K - 1$, $t'_{\bar{K}} = t_{\bar{K}}$, $t'_v = t_K - (n-k-1)$, $t'_w = t_{\bar{K}} - (n-k-1)$; • $\mu'_v = (n-k-1)b'_K + (k-1)b'_{\bar{K}} + \delta\lambda b'_w + \delta(1-\lambda)\frac{(k-1)(k-2)}{n-k}b'_{\bar{K}}$; $\mu'_w = (n-k-1)b'_K + \lambda\delta((k-1)b'_{\bar{K}} + b'_v)$. The network g_{nk} is stable to edge removal between a node in K_{n-k} and a node in $\bar{K}_k$ whenever $0 > \mu'_v - \mu_K + c$ and $0 > \mu'_w - \mu_{\bar{K}} + c$.